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NOTE ON HIGHER TORSION IN THE HOMOTOPY GROUPS OF SINGLE SUSPENSIONS[1]

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0. Introduction

In a previous paper [2] we showed that higher torsion exists in the homotopy groups of a Moore space. More precisely, we showed that, if p is a prime greater than 3 and $m \geq 4$, the homotopy groups of the Moore space $S^{m-1} \bigcup_{p^r} e^m$, denoted by $P^m(p^r)$, contain infinitely many $Z/p^{r+1}Z$ summands. Since Moore spaces are universal examples for homotopy groups with coefficients, this result gave as an easy corollary the existence of infinitely many summands of higher torsion in the homotopy groups of many double suspensions $\Sigma^2 X$. The restriction to double suspensions was made solely for the technical reason that $H_*(\Omega\Sigma^2 X; Z/pZ)$ is a primitively generated Hopf algebra while $H_*(\Omega\Sigma X; Z/pZ)$ may not be. The restriction to primes greater than 3 was made for the reason that Samelson products do not make the mod 3 homotopy Bockstein spectral sequence into a spectral sequence of Lie algebras.

In the present note we show that our result on higher torsion remains true if a double suspension is replaced by the single suspension ΣX of a simply connected space X which has a suitable vanishing condition on two cohomology products. The method of proof is that of [2, Theorem 14.1] with the addition of a little information on primitive generation of certain auxiliary Hopf algebras.

Samelson products give the mod 3 homotopy Bockstein spectral sequence E^r an anti-symmetric bilinear pairing. Since the writing of [2], we have observed that the Jacobi identity is valid if $r > 1$. This means that the results of [2] on higher torsion of order p^{r+1} are valid if $p = 3$ and $r > 1$. Accordingly, we include 3-primary information in the theorem of this note. The result on the mod 3 homotopy Bockstein spectral sequence will appear in [4]. While we do not discuss it here, this implies that the methods in [2] suffice for the construction of a map $\pi\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ of spaces localized at 3 and $n > 1$ so that the composition

$$\pi\Sigma^2\colon S^{2n-1} \to \Omega^2 S^{2n+1} \to S^{2n-1}$$

has degree 9.

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1. Statement of the theorem

Throughout this paper let p be an odd prime and let X be a simply connected space with $\pi_*(X)_{(p)}$ and $H_*(X)_{(p)}$ of finite type.

Denote by $E^r_\pi(X)$, $E^r_H(X)$, and $E^H_r(X)$ the mod p homotopy, homology, and cohomology Bockstein spectral sequences of X, respectively. The Bockstein differentials are denoted by $\beta^{(r)}$, $\beta^{(r)}$, and $\beta_{(r)}$, respectively. There is a Hurewicz map

$$E^r(\phi)\colon E^r_\pi(X) \to H^r_H(X)$$

and a dual pairing $\langle \ , \ \rangle\colon E^H_r(X) \otimes E^r_H(X) \to Z/pZ$. Both of these are compatible with the differentials [1], [2], [4].

Our first hypothesis is the following.

Hypothesis 1. Let ε generate a Z/p^rZ summand in $\pi_{m-1}(X)$ with $m \geq 4$, $r > 0$, and suppose that the Hurewicz image $\phi(\varepsilon)$ generates a Z/p^rZ summand in $H_{m-1}(X)$.

If $\varepsilon\colon S^{m-1} \to X$ is a homotopy class as above, then ε has an extension to a homotopy class $\delta\colon P^m(p^r) \to X$ in $\pi_m(X; Z/p^rZ)$. Let μ be the image of ε under the mod p reduction map

$$\pi_{m-1}(X) \to \pi_{m-1}(X; Z/pZ)$$

and let ν be the image of δ under the mod p reduction map

$$\pi_m(X; Z/p^rZ) \to \pi_m(X; Z/pZ).$$

Then μ and ν survive to nonzero elements in the *rth* term $E^r_\pi(X)$ of the mod p homotopy Bockstein spectral sequence and $\beta^{(r)}\nu = \mu$.

Similar remarks apply to generators of Z/p^rZ summands in $H_*(X)$. Let u and v denote the Hurewicz images $\phi(\mu)$ and $\varphi(\nu)$, respectively. It follows that u and v survive to nonzero elements in the *rth* term $E^r_H(X)$ of the mod p homology Bockstein spectral sequence and $\beta^{(r)}v = u$.

Choose classes u^* and v^* in the *rth* term $E^H_r(X)$ of the mod p cohomology Bockstein spectral sequence such that $\langle u^*, u\rangle = 1$ and $v^* = \beta_{(r)}u^*$. Note that $\langle v^*, v\rangle = (-1)^{\deg u}$.

Our second hypothesis is the following.

Hypothesis 2. The elements u^* and v^* can be chosen so that in the *rth* term of the cohomology Bockstein spectral sequence

(1) $(v^*)^p = u^*(v^*)^{p-1} = 0$ if m is even,

and

(2) $(u^*)^p = (u^*)^{p-1}v^* = 0$ if m is odd.

THEOREM. *Suppose there is an element ε of order p^r in $\pi_{m-1}(X)$ which satisfies Hypothesis 1 and 2 above. If p is equal to 3, require either that $r > 1$ or restrict to $k = 1$ in the conclusion below.*

(1) *If m is even,* $\pi_{p^k m-1}(\Sigma X)$ *contains a* $Z/p^{r+1}Z$ *summand for each* $k \geq 1$.
(2) *If m is odd,* $\pi_{2p^k m-1}(\Sigma X)$ *contains a* $Z/p^{r+1}Z$ *summand for each* $k \geq 1$.

2. Proof of the theorem

If x is an even degree element in a differential Lie algebra over Z/pZ, then, for $k \geq 1$, the elements $\tau_k(x)$ and $\sigma_k(x)$ are defined as in [2] by

$$\tau_k(x) = ad^{p^k-1}(x)(dx)$$

and

$$\sigma_k(x) = \frac{1}{2} \sum_{j=1}^{p^k-1} p^{-1}(j, p^k - j)[ad^{j-1}(x)(dx), ad^{p^k-j-1}(x)(dx)],$$

where $ad^1(x)(y) = [x, y]$ and $ad^{n+1}(x)(y) = ad^1(x)ad^n(x)(y)$. Recall that $d\tau_k(x) = d\sigma_k(x) = 0$ by [2, Corollary 4.6].

If p is a prime greater than 3, the mod p homotopy Bockstein spectral sequence $E^r_\pi(\Omega\Sigma X)$ is a spectral sequence of Lie algebras [2], [4]. Hence, $\beta^{(r)}\tau_k(x) = \beta^{(r)}\sigma_k(x) = 0$ for even degree elements x in $E^r_\pi(\Omega\Sigma X)$.

If p equals 3 and $r > 1$, the Samelson product on $E^r_\pi(\Omega\Sigma X)$ is an anti-symmetric bilinear pairing which satisfies the Jacobi identity [4]. The other identity for a Lie algebra, $[y, [y, y]] = 0$ for odd degree y, is not needed in the proof of $d\tau_k(x) = 0$ for even degree x. Hence, $\beta^{(r)}\tau_k(x) = 0$ for even degree elements x in $E^r_\pi(\Omega\Sigma X)$ with $p = 3$ and $r > 1$.

If p equals 3 and $r = 1$, the Samelson product is an anti-symmetric bilinear pairing on $E^1_\pi(\Omega\Sigma X)$ $(= \pi_*(\Omega\Sigma X; Z/3Z))$ which satisfies the Jacobi identity provided that any one of the three elements involved is the reduction of an integral homotopy class. Since $\beta^{(1)}x$ is the reduction of an integral class, a simple computation gives that $\beta^{(1)}\tau_1(x) = 0$ for even degree elements x in $E^1_\pi(\Omega\Sigma X)$ mod 3.

The statements of the above three paragraphs are also valid for the relative Samelson products used in [2]. Hence, if we impose the appropriate restrictions on r and k when p equals 3, we can use some results from [2] which are stated only for primes greater than 3.

The natural map $X \to \Omega\Sigma X$ induces a map $E^r_\pi(X) \to E^r_\pi(\Omega\Sigma X)$ and we denote by μ' and ν' the respective images of μ and ν under this map. Note that $\beta^{(r)}\nu' = \mu'$.

Suppose that the degree m of ν' is even. Then $\beta^{(r)}\tau_k(\nu') = 0$, so that $\tau_k(\nu')$ survives to $E^{r+1}_\pi(\Omega\Sigma X)$. We claim that $\beta^{(r+1)}\tau_k(\nu') \neq 0$. If so, then standard properties of the Bockstein spectral sequence imply that $\pi_{p^k m-2}(\Omega\Sigma X)$ contains a $Z/p^{r+1}Z$ summand, which is equivalent to the theorem.

If m were odd, a similar argument with $\tau_k([\nu', \nu'])$ replacing $\tau_k(\nu')$ would prove the theorem. Hence, for the remainder of this proof, we will assume that m is even.

Recall that $X \to \Omega\Sigma X$ induces an injection $E^r_H(X) \to E^r_H(\Omega\Sigma X)$ and $E^r_H(\Omega\Sigma X)$ is the enveloping algebra of the differential coalgebra $E^r_H(X)$. That is, $E^r_H(\Omega\Sigma X)$ is the Hopf algebra which is the differential tensor algebra $T(\bar{E}^r_H(X))$ with diagonal given on generators by the diagonal of $E^r_H(X)$. We will identify $E^r_H(X)$ with its image in $E^r_H(\Omega\Sigma X)$.

The image of the Hurewicz map is primitive. Hence, we may write $E^r(\phi)$: $E^r_\pi(\Omega\Sigma X) \to PE^r_H(\Omega\Sigma X)$. This is a map of differential Lie algebras, where $PE^r_H(\Omega\Sigma X)$ is given the usual Lie bracket defined by $[z, w] = zw - (-1)^{\deg z \deg 2} wz$. Consider the induced homology map $HE^r(\phi)$: $E^{r+1}_\pi(\Omega\Sigma X) \to HPE^r_H(\Omega\Sigma X)$. By [2, Corollary 10.5], $HE^r(\phi)(\beta^{(r+1)}\tau_k(v'))$ is the homology class of $\lambda\sigma_k(v)$, where $\lambda \neq 0$. If $\sigma_k(v)$ represents a nonzero homology class in $HPE^r_H(\Omega\Sigma X)$, then we are done. If $E^r_H(X)$ were a coalgebra with a trivial diagonal, then [2, Corollary 4.11] asserts that $\sigma_k(v)$ represents a nonzero class. We shall show that there is a differential Hopf algebra morphism g: $E^r_H(\Omega\Sigma X) \to T(V)$ where $H_* V = 0$, V has a trivial diagonal, and $gv \neq 0 \neq gu$. Then [2, Corollary 4.11], applied to $T(V)$, finishes the proof.

Let S be the free commutative differential Z/pZ algebra generated by two elements x and dx of degrees $m - 1$ and m, respectively. Let B be the quotient of S by the ideal generated by $(dx)^p$ and $(dx)^{p-1}x$. (If m were odd, the ideal would be generated by x^p and $x^{p-1}\ dx$.) Hypothesis 2 implies that there is a differential algebra morphism f: $B \to E^H_r(X)$ such that $f(x) = u^*$ and $f(dx) = \beta^{(r)}u^*$. Passing to dual coalgebras gives a map f^*: $E^r_H(X) \to B^*$ and hence a map of enveloping algebras f^*: $E^r_H(\Omega\Sigma X) \to T(\bar{B}^*)$. The next lemma completes the proof.

LEMMA. *$T(\bar{B}^*)$ is isomorphic to a differential tensor Hopf algebra $T(V)$ where V is primitive and $H_* V = 0$.*

Proof. If A is a Z/pZ Hopf algebra with commutative multiplication, then [3, Proposition 4.21] implies that $PA \to QA$ is a monomorphism except in degrees of the form $2pn$ with $A_{2n} \neq 0$. Since $T(\bar{B}^*)$ has a commutative diagonal, $QT(\bar{B}^*) \cong \bar{B}^*$, and $\bar{B}^*_n = 0$ if $n < m - 1$ or $n \geq p(m - 1)$, the dual statement implies that $PT(\bar{B}^*) \to QT(\bar{B}^*)$ is an epimorphism.

Since $HQT(\bar{B}^*) = 0$, $QT(\bar{B}^*)$ is a projective in the category of differential vector spaces and differential morphisms. Hence, the map $PT(\bar{B}^*) \to QT(\bar{B}^*)$ has a right inverse g and we take $V =$ the image of g. ∎

Remark. The proof remains valid if v' and μ' are replaced by any nonzero pair of elements γ and $d\gamma$ in the free differential Lie algebra generated by v' and $\mu' = \beta^{(r)}v'$. See [2, Theorem 14.3] for details. Hence, one gets infinitely many more $Z/p^{r+1}Z$ summands in $\pi_*(\Sigma X)$, even if $p = 3$ and $r = 1$.

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