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# MULTIVALUED DIFFERENTIAL EQUATIONS ON MANIFOLDS WITH APPLICATION TO CONTROL THEORY[1]

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## 0. Introduction

In control theory the comparison between attainable sets obtained through different solutions has been widely studied. The same problem, for multivalued differential equations on $R^n$, was introduced by Lobry [5]. In his paper he showed that if $F$ is a locally Lipschitz multivalued function with the property of Lipschitz selection, then the closure of the attainable set by piecewise smooth solutions coincides with the closure of the attainable set using absolutely continuous solutions.

In general, the Lipschitz continuity of $F$ with respect to the Hausdorff distance does not imply the Lipschitz selection property. In [9] it can be seen that this property, for multivalued, convex, compact, non-empty functions is a consequence of the Lipschitz continuity with respect to a suitable definition of distance.

As in most applications, the natural environment for a multivalued differential equation is a manifold $M$, and, in particular, a Lie group $G$, and it seems natural in this environment to introduce a qualitative study of such functions. To this aim we shall study the properties of a multivalued field, that is, of a function $F\colon M \to \bigcup_{x \in M} (2^{T_xM} \backslash \{\theta\})$ such that $F(x) \subset T_x M$. As $\bigcup_{x \in M} (2^{T_xM} \backslash \{\theta\})$ cannot be endowed with a reasonable vector bundle structure, we limit ourselves to the case where

$$F(x) \subset \mathcal{O}(T_x M) = \{A \subset T_x M, \text{ convex, compact, non-empty}\}.$$

In this case we can embed $\mathcal{O}(T_x M)$ in the normed space $\tilde{\mathcal{O}}(T_x M)$ [9]. In general, with the natural projection on $M$, $\tilde{\mathcal{O}}(TM) = \bigcup_{x \in M} \tilde{\mathcal{O}}(T_x M)$ does not admit a natural (not even $C^0$) vector bundle structure. However, we can still give a Lipschitz continuity definition for $F$ which depends only upon the differential structure of $M$. We can therefore extend in a natural way the results of [5]. Moreover, in the case of a Lie group $G$, we show that the set $\tilde{\mathcal{O}}(TG)$ can be endowed with $C^\infty$ vector bundle structures, $\tilde{\mathcal{O}}_i(TG)$, with $i = 0, 1, \ldots,$

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$n = \dim G$. Therefore, we prove that the multivalued fields $F$, which are Lipschitz sections of $\tilde{\mathcal{O}}_n(TG)$, have the Lipschitz selection property, and that they are also the convex hull of extremal selections.

Finally, we present examples of applications of our results to control theory. In particular, we show that the attained results apply to the case of control problems linear in the control.

## 1. Convex sets in $n$-Euclidean vector spaces

In the sequel $(E, \cdot)$ and $(F, \cdot)$ will be Euclidean vector spaces of dimension $n$ and $m$ respectively. Hom $(E, F)$ will denote the space of linear maps from $E$ to $F$ with the usual norm given by $\|f\| = \sup_{\|x\|=1} \|f(x)\|$. The $i$-tuple of orthonormal vectors in $E$ is denoted by $\Omega_i(E)$, or by $\Omega_i$ if there is no confusion. The collection of convex, compact, non-empty sets of $E$ will be denoted by $\mathcal{O}(E)$, or by $\mathcal{O}$. If $A \in \mathcal{O}(E)$, the support function of $A$ is defined by $q(A; x) = \max_{a \in A} a \cdot x$ for each $x \in E$. If $A \in \mathcal{O}(E)$, for each $i = 1, \ldots, n$, the $i$-face function is unductively defined by

$$V_1(A; x) = \{a \in A: a \cdot x = q(A; x)\} \quad \text{for each } x \in E,$$

$$V_i(A; x_1, \ldots, x_i) = V_1(V_{i-1}(A; x_1, \ldots, x_{i-1}); x_i) \quad \text{for each } (x_1, \ldots, x_i) \in E^i.$$

It is known that the support function is sub-additive and positively homogeneous and that the 1-face multifunction has the property $V_1(A; rx) = V_1(A; x)$ for each $r \in R^+$. It is also known that the Hausdorff distance can be defined in $\mathcal{O}(E)$ by

$$h^0(A, B) = \max_{x \in \Omega_1} |q(A; x) - q(B; x)|.$$

In [9] it is proved that, for each $i = 1, \ldots, n$, $h^i: \mathcal{O}(E) \times \mathcal{O}(E) \to R$ defined by

$$h^i(A, B) = \sup_{(x_1, \ldots, x_i) \in \Omega_i} h^0(V_i(A; x_1, \ldots, x_i), V_i(B; x_1, \ldots, x_i))$$

is a translation invariant distance and also $h^0 \leq h^1 \leq \cdots \leq h^n$. In [9] it is also proved that $(\mathcal{O}, h^i)$ can be embedded in a natural way in a normed vector space $\tilde{\mathcal{O}}_i$ defined as follows: $\tilde{\mathcal{O}}_i = \mathcal{O} \times \mathcal{O}/\rho$ where $\rho$ is given by $(A, B)\rho(C, D)$ if and only if $A + D = B + C$, and $\|[(A, B)]\|_i = h^i(A, B)$. We recall here the vector space operations

$$[(A, B)] + [(C, D)] = [(A + C, B + D)],$$

$$\lambda[(A, B)] = \begin{cases} [(\lambda A, \lambda B)] & \text{if } \lambda \geq 0, \\ [(-\lambda B, -\lambda A)] & \text{if } \lambda \leq 0. \end{cases}$$

In the sequel the equivalence class $[(A, B)]$ will be denoted by $(A, B)$.

LEMMA 1.1. *Let* $(x_1, \ldots, x_i) \in E^i$ *be such that* $(x_1, \ldots, x_i) \neq (0, \ldots, 0)$. *Then there exists an orthonormal set of vectors* $y_1, \ldots, y_j$ *in* $E$ *such that*

$$\operatorname{span}\{x_1, \ldots, x_i\} = \operatorname{span}\{y_1, \ldots, y_j\}$$

*and*

$$V_i(A; x_1, \ldots, x_i) = V_j(A; y_1, \ldots, y_j) \quad \textit{for every } A \in \mathcal{O}.$$

*Proof.* If $x_1 = 0$ then $V_1(A; x_1) = A$. Let $x_k$ be the first vector different from 0 in $(x_1, \ldots, x_i)$. Then $V_k(A; x_1, \ldots, x_k) = V_1(A; x_k) = V_1(A; x_k/\|x_k\|) = V_1(A; y_1)$. The proof follows by induction, taking into account that, for all $A \in \mathcal{O}$ and all $x \in E$, $V_1(A; x) = V_1(A, x')$, where $x'$ is the projection of $x$ on subspace parallel to the minimal affine sub-variety containing $A$. ∎

If $f$ belongs to Hom $(E, F)$ we denote the transposed map of $f$ by $f^*$; that is, $f(x) \cdot y = f^*(y) \cdot x$, for each $x \in E$ and $y \in F$.

LEMMA 1.2. *For each $f$ belonging to* Hom $(E, F)$ *we have:*

(i) $q(f(A); y) = q(A; f^*(y))$, *for all* $A \in \mathcal{O}(E)$ *and all* $y \in F$;

(ii) $$V_i(f(A); y_1, \ldots, y_i) = f(V_i(A; f^*(y_1), \ldots, f^*(y_i))),$$

*for all* $A \in \mathcal{O}(E)$ *and all* $(y_1, \ldots, y_i) \in F^i$.

*Proof.* (i) $q(f(A); y) = \max_{a \in A} f(a) \cdot y = \max_{a \in A} a \cdot f^*(y) = q(A; f^*(y))$.

(ii) $$\begin{aligned} V_1(f(A); y) &= \{f(a) : a \in A, f(A) \cdot y = q(f(A); y)\} \\ &= f\{a \in A : a \cdot f^*(y) = q(A; f^*(y))\} \\ &= f(V_1(A; f^*(y))). \end{aligned}$$

By induction it is easy to see that $V_i(f(A); y_1, \ldots, y_i) = f(V_i(A; f^*(y_1), \ldots, f^*(y_i)))$ for each $i = 1, \ldots, n$. ∎

PROPOSITION 1.1. *For every* $A, B \in \mathcal{O}(E)$ *and for each* $f \in$ Hom $(E, F)$ *we have*

$$h^i(f(A), f(B)) \leq \|f\| h^j(A, B), \quad i = 0, 1, \ldots, m; j = \min(i, n).$$

*Proof.* Taking into account Lemma 1.1, Lemma 1.2(ii) and using the property $h^i \leq h^{i+1}$, we get

$$\begin{aligned} h^0(f(A), f(B)) &= \max_{y \in \Omega_1(F)} |q(f(A); y) - q(f(B); y)| \\ &\leq \|f^*\| \max_{x \in \Omega_1(E)} |q(A; x) - q(B; x)| \\ &= \|f\| h^0(A, B), \end{aligned}$$

and

$$h^i(f(A), f(B)) = \sup_{(y_1, \ldots, y_i) \in \Omega_i(F)} h^0(V_i(f(A); y_1, \ldots, y_i),$$

$$V_i(f(B), y_1, \ldots, y_i)) \leq \|f^*\| \sup_{(y_1, \ldots, y_i) \in \Omega_i(F)} h^0(V_i(A; f^*(y_1), \ldots, f^*(y_i)),$$

$$V_i(B; f^*(y_1), \ldots, f^*(y_i))) \leq \|f\| h^j(A, B).$$ ∎

COROLLARY 1.1. *Let g and g′ be scalar products on E. Then the induced norms* $\|\cdot\|_i$, *and* $\|\cdot\|_i'$ *are equivalent for each* $i = 0, 1, \ldots, n$.

The proof follows applying the previous Proposition to the identity map. ∎

PROPOSITION 1.2. *Let f belong to* Hom $(E, F)$. *For each* $i = 0, 1, \ldots, n$, *the map*

$$\bar{f}: \bar{\mathcal{O}}_i(E) \to \bar{\mathcal{O}}_j(F), \quad j = \min(i, m),$$

*given by* $(A, B) \mapsto (f(A), f(B))$ *is linear and continuous.*

*Proof.* As $f(A + C) = f(A) + f(C)$, for each $A, C \in \mathcal{O}(E)$, it follows that $\bar{f}$ is well defined and additive. $\bar{f}$ is clearly positively homogeneous. If $\lambda \in R^-$, then we get

$$\bar{f}(\lambda(A, B)) = \bar{f}(-\lambda B, -\lambda A) = (-\lambda f(B), -\lambda f(A)) = \lambda(f(A), f(B)).$$

Proposition 1.1 completes the proof. ∎

PROPOSITION 1.3. *For every* $f, g \in$ Hom $(E, F)$ *and for each* $A \in \mathcal{O}(E)$ *we get*

$$\|f(A), g(A)\|_0 \leq \|f - g\| \, \|A\|_0 \quad \textit{where } \|A\|_0 = h^0(A, 0).$$

*Proof.* If $(f - g)^*(x) \neq 0$, from the subadditivity of $q$, we have

$$\begin{aligned} q(f(A); x) - q(g(A); x) &= q(A; f^*(x)) - q(A; g^*(x)) \\ &\leq \|(f - g)^*(x)\| q\left(A; \frac{(f - g)^*(x)}{\|(f - g)^*(x)\|}\right) \\ &\leq \|f - g\| \|A\|_0, \end{aligned}$$

and

$$q(f(A); x) - q(g(A), x) = q(A, f^*(x)) - q(A, g^*(x)) \geq -\|g - f\| \, \|A\|_0.$$

If $(f - g)^*(x) = 0$ we get trivially $q(f(A); x) - q(g(A); x) = 0$. Otherwise, we obtain

$$\|f(A), g(A)\|_0 = \max_{x \in \Omega_1} |q(f(A); x) - q(g(A); x)| \leq \|g - f\| \|A\|_0.$$ ∎

PROPOSITION 1.4. *If* $n \geq 2$, *the map* $\sim$: Hom $(E, E) \to$ Hom $(\tilde{\mathcal{O}}_i(E), \tilde{\mathcal{O}}_i(E))$, *defined by* $f \to \tilde{f}$ *is not continuous at any* $f$ *which is an isomorphism, for each* $i = 0, 1, \ldots, n$.

*Proof.* Since $h^0 \leq h^1 \leq \cdots \leq h^n$ it is sufficient to prove the statement for $i = 0$. Let $f$, $g \in$ Hom $(E, E)$ and let $f$ be an isomorphism. We have $\|\tilde{f} - \tilde{g}\| \leq \|\tilde{f}\| \, \|\mathrm{id} - (f^{-1}g)^{\sim}\|$; therefore it is sufficient to show that the map $\sim$: Hom $(E, E) \to$ Hom $(\tilde{\mathcal{O}}_0, \tilde{\mathcal{O}}_0)$ is not continuous at the identity map.

Let $x$, $y$ be orthonormal vectors in $E$. For each $\varepsilon \in (0, \pi/2)$ we define $g_\varepsilon$: $E \to E$ as the rotation by the angle $\varepsilon$ in span $\{x, y\}$, that is

$$g_\varepsilon(z) = z \cdot x(x \cos \varepsilon + y \sin \varepsilon) + z \cdot y(-x \sin \varepsilon + y \cos \varepsilon) \\ + z - (z \cdot xx + z \cdot yy).$$

We see that $g_\varepsilon$ tends to the identity map as $\varepsilon$ tends to zero. We want to calculate

$$\|\tilde{g}_\varepsilon - \mathrm{id}\|_0 = \sup_{\|(A,B)\|_0 = 1} \|g_\varepsilon(A) + B, g_\varepsilon(B) + A\|_0.$$

For each $r \in R^+$ let $(A_r, B_r)$ be defined by

$$A_r = \{ay: a \in [0, r]\}, \quad B_r = \{b(x + ry): b \in [0, 1]\}.$$

We get $\|(A_r, B_r)\|_0 = 1$. Moreover if $r = \cot g\ \varepsilon$, then $g_\varepsilon(B_r) = \{(b/\sin \varepsilon)y: b \in [0, 1]\}$. Therefore

$$A_r + g_\varepsilon(B_r) = \{cy: c \in [0, (\cos \varepsilon + 1)/\sin \varepsilon]\}.$$

On the other hand $(x + ry) \in B_r + g_\varepsilon(A_r)$ and so

$$\|(A_r + g_\varepsilon(B_r), B_r + g_\varepsilon(A_r))\|_0 \geq \inf_{z \in A_r + g_\varepsilon(B_r)} \|x + ry - z\| = 1.$$

Finally $\|g_\varepsilon - \mathrm{id}\|_0 \geq 1$ for every $\varepsilon \in (0, \pi/2)$. This proves the statement. ∎

## 2. Multivalued vector fields on a manifold

Let $M$ be a paracompact connected $C^\infty$ $n$-dimensional manifold without boundary. We denote the tangent bundle of $M$ by $\pi$: $TM \to M$, and we denote the tangent map of a map $f$ by $f_*$. Most of the following results are still valid in the case of $C^r$ manifolds, $r \geq 2$.

DEFINITION 2.1. A multivalued vector field (m.v.f.) on $M$ is a map $F$: $M \to \bigcup_{x \in M} 2^{T_x M} \backslash \{\theta\}$ such that $F(x) \subset T_x M$, for all $x \in M$. Let $\mathscr{E}$ and $\mathscr{F}$ be $C^\infty$ vector bundles on $M$. Let the pair of morphisms

$$\alpha: \mathscr{E} \to \mathscr{F} \quad \text{and} \quad \alpha_0: M \to M$$

be a $C^\infty$ vector bundle morphism, (see [4 p. 43]). We denote, for each $x \in M$, the induced map on the fiber by $\alpha_x: \mathscr{E}_x \to \mathscr{F}_{\alpha_0(x)}$.

In the following, $F(x)$ will be compact, convex, and, identifying $F(x)$ with $(F(x), 0)$, one can assume that $f$ takes its values on $\tilde{\mathcal{O}}(TM) = \bigcup_{x \in M} \tilde{\mathcal{O}}(T_x M)$. Let $\{(U_i, \tau_i)\}$ be a trivialising covering for $TM$; that is, an open covering $\{U_i\}$ of $M$ and a set $\{\tau_i\}$ of bijections $\tau_i: \pi^{-1}(U_i) \to U_i \times R^n$ such that, for each $x \in U_i$, the induced map on the fiber $\tau_{i,x}: \pi^{-1}(x) \to R^n$ is an isomorphism (see [4 p. 42]). Denoting by $\tilde{\pi}: \tilde{\mathcal{O}}(TM) \to M$ the projection defined by $\tilde{\pi}: \tilde{\mathcal{O}}(T_x M) \mapsto x$, a natural trivialising covering for $\tilde{\mathcal{O}}(TM)$ would seem to be $\{(U_i, \tilde{\tau}_i)\}$, where

$$\tilde{\tau}_i: \tilde{\pi}^{-1}(U_i) \to U_i \times \tilde{\mathcal{O}}(R^n)$$

is defined on each fiber by $\tilde{\tau}_{i,x}(A, B) = (\tau_{i,x}(A), \tau_{i,x}(B))$, for each $(A, B) \in \tilde{\mathcal{O}}(T_x M)$. For each pair $(i, j)$ and $x \in U_i \cap U_j$ the map

$$(\tilde{\tau}_j \circ \tilde{\tau}_i^{-1})_x = (\tau_j \circ \tau_i^{-1})_x^{\sim}: \tilde{\mathcal{O}}(R^n) \to \tilde{\mathcal{O}}(R^n)$$

is a linear isomorphism. But, since $\tilde{\mathcal{O}}(R^n)$ is an infinite dimensional space, we need also the condition that the map from $U_i \cap U_j$ to Hom $(\tilde{\mathcal{O}}(R^n), \tilde{\mathcal{O}}(R^n))$, given by

$$x \mapsto (\tilde{\tau}_j \circ \tilde{\tau}_i^{-1})_x = (\tau_j \circ \tau_i^{-1})_x^{\sim},$$

is $C^p$ (see [4 p. 43]). Now, the previous map results from the composition

$$U_i \cap U_j \to \text{Hom}\,(R^n, R^n) \to \text{Hom}\,(\tilde{\mathcal{O}}(R^n), \tilde{\mathcal{O}}(R^n))$$

given by: $x \mapsto (\tau_j \circ \tau_i^{-1})_x \mapsto (\tau_j \circ \tau_i^{-1})_x^{\sim}$. Since the map

$$\sim: \text{Hom}\,(R^n, R^n) \to \text{Hom}\,(\tilde{\mathcal{O}}(R^n), \tilde{\mathcal{O}}(R^n))$$

is not continuous at any isomorphism one cannot be sure that the previous map is continuous, unless $x \mapsto (\tau_j \circ \tau_i^{-1})_x$ is constant for each pair $(i, j)$.

But, even if $TM$ is trivial, two different trivialisations can give different topological structures on $\tilde{\mathcal{O}}(TM)$, so it seems reasonable to consider $\tilde{\mathcal{O}}(TM)$ as a vector bundle only if a canonical trivialisation exists on $TM$, as in the case of Lie groups. Nevertheless, the following proposition allows us to give a reasonable definition of local Lipschitz continuity for an m.v.f.

We note that every map defined on $TM$ induces a map on $\tilde{\mathcal{O}}(TM)$. For simplicity the two maps will be denoted by the same symbol.

PROPOSITION 2.1. *Let $\tau: TU \to U \times R^n$ and $\alpha: TU \to U \times R^n$ be two local trivialisations of $TM$. Let $\pi_2: U \times R^n \to R^n$ be the canonical projection. If $\pi_2 \circ \tau \circ F: U \to \tilde{\mathcal{O}}_0(R^n)$ is a locally Lipschitz map, then $\pi_2 \circ \alpha \circ F: U \to \tilde{\mathcal{O}}_0(R^n)$ is also a locally Lipschitz map.*

*Proof.* Let $z \in U$, and let $V \subset U$ be a compact neighborhood of $z$ such that $\pi_2 \circ \tau \circ F_{|V}$ is a Lipschitz map with respect to a distance in $U$ induced by any chart. Let $\alpha_x \circ \tau_x^{-1}\colon R^n \to R^n$ be denoted by $\gamma_x$. If $x, y \in V$, then

$$\begin{aligned} h^0(\pi_2 \circ \alpha \circ F(x), \pi_2 \circ \alpha \circ F(y)) &= h^0(\alpha_x \circ \tau_x^{-1} \circ \tau_x \circ F(x), \alpha_y \circ \tau_y^{-1} \circ \tau_y \circ F(y)) \\ &= h^0(\gamma_x \circ \tau_x \circ F(x), \gamma_y \circ \tau_y \circ F(y)) \\ &\leq h^0(\gamma_x \circ \tau_x \circ F(x), \gamma_y \circ \tau_x \circ F(x)) \\ &\quad + h^0(\gamma_y \circ \tau_x \circ F(x), \gamma_y \circ \tau_y \circ F(y)) \\ &\leq \|\gamma_x - \gamma_y\| \, \|\tau_x \circ F(x)\|_0 \\ &\quad + \|\gamma_y\| h^0(\tau_x \circ F(x), \tau_y \circ F(y)). \end{aligned}$$

Since $\pi_2 \circ \tau \circ F$ and $x \mapsto \gamma_x$ are continuous maps, there exist

$$k_1 = \max_{x \in V} \|\tau_x \circ F(x)\|_0 \quad \text{and} \quad k_2 = \max_{x \in V} \|\gamma_x\|.$$

Then

$$h^0(\pi_2 \circ \alpha \circ F(x), \pi_2 \circ \alpha \circ F(y)) \leq k_1 \|\gamma_x - \gamma_y\| + k_2 h^0(\tau_x \circ F(x), \tau_y \circ F(y)).$$

The statement follows from Lipschitz continuity of $\pi_2 \circ \tau \circ F$ and of $x \mapsto \gamma_x$. ∎

The previous proposition implies that the following definition is coordinate free.

DEFINITION 2.2. A multivalued vector field is said to be locally Lipschitz if for each $x \in M$ there is an open neighborhood $U_x$ and a trivialisation $\tau\colon TU \to U \times R^n$ such that $\pi_2 \circ \tau \circ F\colon U \to \mathcal{C}_0(R^n)$ is a locally Lipschitz map.

DEFINITION 2.3. Let $x \in C \subset M$. $w \in T_x M$ is said to be tangent to $C$ in $x$ if $w$ is tangent to any $n$-dimensional submanifold (possibly with boundary) $N \subset M$, containing $C$.

(For the definitions of submanifold, boundary and related topics we follow [4 pages 25, 40]).

DEFINITION 2.4. A m.v.f. $F$ is said to be tangent to the set $C \subset M$ if for each $x \in C$ and $w \in F(x)$, $w$ is tangent to $C$ in $x$.

In the case $M = R^n$ the previous definitions are the same as those given by Lobry in [5]. That is, let $C \subset R^n$ be a set; $w \in R^n$ is tangent to $C$ in $x$ if $w \cdot v \leq 0$, for all outward normals $v$ to $C$ in $x$. (We recall that a unit vector $v$ is an outward normal to $C$ in $x$ if there exists $r \in R^+$ such that $d(x + rv, C) = \inf_{y \in C} \|x + rv - y\| = r$).

## 3. Attainable sets

DEFINITION 3.1. A solution of the equation $\dot{\gamma} \in F$, $\gamma(0) = x_0 \in M$, is an absolutely continuous curve $\gamma$: $[0, b) \to M$ such that $\dot{\gamma}(t) \in F(\gamma(t))$ a.e. and $\gamma(0) = x_0$.

In the sequel we need of the following Lobry Lemma [5].

LEMMA 3.1. *Let $F$ be a locally Lipschitz m.v.f. on $R^n$, and let $C$ be a closed set. If $F$ is tangent to $C$, then every solution of $\dot{\gamma} \in F$, $\gamma(0) = x_0 \in C$, belongs to $C$.*

The proof of the lemma follows by estimating the distance between $\gamma(t)$ and $C$ and using the Gronwall Lemma. ∎

LEMMA 3.2. *Let $C \subset M$ be a closed set, and let $F$ be a locally Lypschitz m.v.f. tangent to $C$. Then every solution of $\dot{\gamma} \in F$, $\gamma(0) = x_0 \in C$, belongs to $C$.*

*Proof.* Suppose not. Then, possibly changing the initial point $x_0$, we can assume that $\gamma(t) \notin C$ for all $t > 0$. Let $\phi$: $U \to R^n$ be a chart at $x_0$ such that $\phi(U) = R^n$. $C' = \phi(C \cap U)$ is a closed set. From Definition 2.2 it is easy to see that $F' = \pi_2 \circ \phi_* \circ F \circ \phi^{-1}$: $R^n \to R^n$ is tangent to $C'$. On the other hand

$$\pi_2 \circ (\phi \circ \gamma) = \pi_2 \circ \phi_* \circ \dot{\gamma} \in \pi_2 \circ \phi_* \circ F \circ \gamma = F' \circ (\phi \circ \gamma),$$

$\phi \circ \gamma(0) \in C'$, and $\phi \circ \gamma(t) \notin C'$ for all $t > 0$, a contradiction to Lemma 3.1. ∎

LEMMA 3.3. *Let $i$: $N \to M$ be an immersion* [4 p. 25], *and let $F$ be a locally Lipschitz m.v.f. tangent to $N$ (that is $F \circ i(x) \subset i_* T_x N$ for each $x \in N$). Then every solution of $\dot{\gamma} \in F$, $\gamma(0) = x_0 \in i(N)$, belongs to $i(N)$.*

*Proof.* Let $x \in N$. There is a neighborhood $V_x$ of $x$ in $N$ and a neighborhood $U$ of $i(x)$ in $M$, such that $i(V_x)$ is a closed set of $U$. On the other hand, $F|_U$ is a m.v.f. on $U$ tangent to $i(V_x)$. The statement follows from Lemma 3.2 applied to $F|_U$ and $i(V_x)$. ∎

In the following $\mathscr{A}(x_0, T, F)$ will denote the set which is attainable at time $T \geq 0$ by means of solutions of the equation $\dot{\gamma} \in F$, $\gamma(0) = x_0$, and $\mathscr{A}(x_0, F)$ will denote $\bigcup_{T \geq 0} \mathscr{A}(x_0, T, F)$. If $D$ is a family of Lipschitz vector fields locally defined on $M$ let

$$A(x_0, T, D) = \left\{ x = X^1_{t_1} \circ \cdots \circ X^r_{t_r}(x_0) \colon r \in N,\ X^i \in D,\ t_i \in R^+,\ \sum_{i=1}^r t_i = T \right\},$$

where $(x, t) \mapsto X^i_t(x)$ denotes the one parameter local group generated by the vector field $X^i$. Moreover let $A(x_0, D) = \bigcup_{T \geq 0} A(x_0, T, D)$.

DEFINITION 3.2 (see [5]). A m.v.f. $F$ is said to have the Lipschitz ($C^l$) selection property if, for each $x \in M$ and for each $w \in F(x)$, there exists a Lipschitz ($C^l$) local selection $f$ of $F$ such that $f(x) = w$.

LEMMA 3.4. *Let $F$ be a locally Lipschitz m.v.f. If there is a family $D$ of local Lipschitz selections such that* $\operatorname{co} D(x) = F(x)$ *for each $x \in M$, then $F$ is tangent to* $\operatorname{cl} A(x_0, D)$.

*Proof.* Suppose there exists $x \in \operatorname{cl} A(x_0, D)$, an $n$-dimensional submanifold with boundary $N \supset \operatorname{cl} A(x_0, D)$ and a $w \in F(x)$ such that $w \notin T_x N$. Since $\operatorname{co} D(x) = F(x)$, there is a $\mathfrak{z} \in D$ such that $\mathfrak{z}(x) \notin T_x N$. Now, $x$ must belong to $\partial N$, so $N_1 = (M - N) \cup N$ is a $n$-submanifold with boundary and $\mathfrak{z}(x)$ belongs to the interior of $T_x N_1$. So there is a $t > 0$ such that $\mathfrak{z}_t(x)$ belongs to the interior of $N_1$. Let $V$ be an open set such that $\mathfrak{z}_t(x) \in V \subset \operatorname{int} N_1$. We have that $\mathfrak{z}_{-t}(V)$ is an open neighborhood of $x = \mathfrak{z}_{-t} \circ \mathfrak{z}_t(x)$. Then there is $z \in \mathfrak{z}_{-t}(V) \cap A(x_0, D)$ and we get

$$\mathfrak{z}_t(z) \in V \cap \mathfrak{z}_t(A(x_0, D)).$$

Since $\mathfrak{z}_t(A(x_0, D)) \subset A(x_0, D)$, then $\mathfrak{z}_t(z) \in V \cap A(x_0, D)$, a contradiction because $A(x_0, D) \subset N$ and $V \subset \operatorname{int} N_1$.

THEOREM 3.1. *Let $F$ be as in Lemma* 3.4; *then* $\operatorname{cl} A(x_0, D) = \operatorname{cl} \mathscr{A}(x_0, F)$.

*Proof.* The proof follows from Lemma 3.2 and 3.4. ∎

COROLLARY 3.1. *Let $F$ be as in Lemma* 3.1. *Then* $\operatorname{cl} A(x_0, T, D) = \operatorname{cl} \mathscr{A}(x_0, T, F)$ for all $T > 0$.

*Proof.* The proof follows from Theorem 3.1 applied to the m.v.f. $F\colon M \times R \to TM \times (R \times R)$ defined by $F(x, t) = (F(x), (t, 1))$.

THEOREM 3.2. *Let $F$ be a locally Lipschitz m.v.f. and let $D$ be a family of local Lipschitz vector fields such that $F(x) = \operatorname{co} D(x)$ for each $x \in M$. If* $\mathrm{id}\colon A(x_0, D) \to M$ *is a $C^l$ immersion, $l \geq 1$, then $A(x_0, D) = \mathscr{A}(x_0, F)$.*

*Proof.* Since $\mathrm{id}\colon A(x_0, D) \to M$ is an immersion, we easily get $F$ tangent to $A(x_0, D)$. The theorem follows by applying Lemma 3.3. ∎

COROLLARY 3.2. *Let $F$ and $D$ be as in Theorem* 3.2. *If $D$ is a symmetric family of $C^l$, $l \geq 1$, vector fields, then $A(x_0, D) = \mathscr{A}(x_0, F)$.*

*Proof.* The proof follows from Theorem 3.2, taking into account that in such a case $A(x_0, D)$ is a $C^l$ immersion (see [11]). ∎

COROLLARY 3.3. *Let $F$ be a locally Lipschitz m.v.f. with the $C^l$, $l \geq 1$, selection property such that the null vector of $T_x M$ belongs to the interior of $F(x)$ relative to the minimal linear subvariety of $T_x M$ containing $F(x)$. If $D$ denotes the set of all local $C^l$ selections, then $A(x_0, D) = \mathscr{A}(x_0, F)$.*

*Proof.* From the hypothesis it follows that for each $\mathfrak{z} \in -D$ and each $x \in M$, there is $\varepsilon > 0$, a neighborhood $U$ of $x$, and $\eta \in D$, such that $\mathfrak{z} = \varepsilon\eta$ on $U$. So it is easy to see that $A(x_0, D) = A(x_0, D \cup \{-D\})$. Now Corollary 3.2 can be applied. ∎

The following propositions give some examples of classes of locally Lipschitz m.v.f. with the Lipschitz selection property.

PROPOSITION 3.1. *Let $A \in \mathcal{O}(R^m)$ and let $f\colon M \to \operatorname{Hom}(M \times R^m, TM)$ be a locally Lipschitz $(C^l)$ section. Then the m.v.f. $F$ defined by $F(x) = f(x)(A)$ is locally Lipschitz.*

*Proof.* Let $x \in M$ and let $\phi\colon U \to U'$ be a chart in $x$. If $y, z \in U'$, then, from Proposition 1.3,

$$h^0(\phi_{*,y} \circ F \circ \phi^{-1}(y), \phi_{*,z} \circ F \circ \phi^{-1}(z)) \leq \|\phi_{*,x} \circ f(\phi^{-1}(y)) - \phi_{*,z} \circ f(\phi^{-1}(z))\| \, \|A\|_0.$$

The proof follows because $f$ is a locally Lipschitz map and $\phi$, $\phi_*$ are $C^\infty$ maps.

PROPOSITION 3.2. *Let $A$ be an index set and let $\mathfrak{z}_u\colon M \to TM$ be a locally Lipschitz $(C^l)$ vector field for each $u \in A$. The m.v.f. $F$ defined by $F(x) =$ cl co $\{\mathfrak{z}_u(x)\colon u \in A\}$ is locally Lipschitz continuous if the following conditions are satisfied:*

(i) *$\{\mathfrak{z}_u(x)\colon u \in A\}$ is bounded for each $x \in M$.*
(ii) *For every $x \in M$ there exists a chart $\phi\colon U \to U'$ at $x$, and $k \in R^+$ such that $k$ is a Lipschitz constant for every map $\pi_2 \circ \phi_* \circ \mathfrak{z}_u \circ \phi^{-1}$, $u \in A$.*

*Proof.* Let $x$, $\phi$ be as in (ii). For every $y, z \in U'$ we have

$$\begin{aligned} &h^0(\pi_2 \circ \phi_* \circ F \circ \phi^{-1}(y), \pi_2 \circ \phi_* \circ F \circ \phi^{-1}(z)) \\ &\quad = h^0(\phi_{*,y}(\text{cl co}\{\mathfrak{z}_u(\phi^{-1}(y))\colon u \in A\}), \phi_{*,z}(\text{cl co}\{\mathfrak{z}_u(\phi^{-1}(z))\colon u \in A\})) \\ &\quad = h^0(\text{co}\, \phi_{*,y}\{\mathfrak{z}_u(\phi^{-1}(y))\colon u \in A\}, \text{co}\, \phi_{*,z}\{\mathfrak{z}_u(\phi^{-1}(z))\colon u \in A\}) \\ &\quad \leq h^0(\phi_{*,y}\{\mathfrak{z}_u(\phi^{-1}(y))\colon u \in A\}, \phi_{*,z}\{\mathfrak{z}_u(\phi^{-1}(z))\colon u \in A\}. \end{aligned}$$

On the other hand we have

$$\sup_{u \in A} \inf_{v \in A} \|\phi_{*,y} \circ \mathfrak{z}_u(\phi^{-1}(y)) - \phi_{*,z} \circ \mathfrak{z}_v(\phi^{-1}(z))\| \leq \sup_{u \in A} \|\phi_{*,y} \circ \mathfrak{z}_u(\phi^{-1}(y)) - \phi_{*,z} \circ \mathfrak{z}_u(\phi^{-1}(z))\| \leq k\|y - z\|,$$

and

$$\sup_{u \in A} \inf_{v \in A} \|\phi_{*,z} \circ \mathfrak{z}_u(\phi^{-1}(z)) - \phi_{*,y} \circ \mathfrak{z}_v(\phi^{-1}(y))\| \leq k\|y - z\|. \qquad \blacksquare$$

## 4. Further results on Lie groups

Let $G$ be an $n$-dimensional $C^\infty$ Lie group; that is an $n$-dimensional $C^\infty$ manifold $G$ with a group structure such that the group and the inverse operations are $C^\infty$ maps. We denote the group operation by $(x, y) \mapsto xy$. Then, for each $x \in G$, the left translation $\tau^x\colon G \mapsto G$, given by $\tau^x(y) = xy$, is a $C^\infty$ map. Let

$e$ be the unit element of $G$. It is known that we can obtain a trivialisation of $TG$ by the vector bundle isomorphism $\mathscr{T}: G \times T_e G \to TG$ defined by $\mathscr{T}(x, v) = \tau_*^x(v)$. A vector field $\mathfrak{z}$ over $G$ is called left invariant if $\tau_*^x \circ \mathfrak{z} = \mathfrak{z} \circ \tau^x$, for all $x \in G$. We denote by $lG$ the Lie algebra of $G$; that is, the set of all left invariant vector fields on $G$ with the usual addition and bracket operation. Now, for each $v \in T_e G$, let $\mathscr{T}_v: G \to TG$ be the map defined by $\mathscr{T}_v(x) = \mathscr{T}(x, v)$. The map $v \mapsto \mathscr{T}_v$ is a vector space isomorphism from $T_e G$ into $lG$. Let $g$ be a left invariant riemanian structure on $G$; that is,

$$(*) \qquad \eta(x) \cdot \mathfrak{z}(x) = \eta(e) \cdot \mathfrak{z}(e) \quad \text{for } x \in G,\ \eta,\ \mathfrak{z} \in lG.$$

We observe that every metric structure on $T_e G$ induces such a riemanian structure on $G$. Let

$$\Omega_i = \{\omega = (\mathfrak{z}_1, \ldots, \mathfrak{z}_i) \in lG^i\colon \mathfrak{z}_j \cdot \mathfrak{z}_k = \delta_{jk}\}.$$

*Note 4.1.* Let $x \in G$; every element $u \in T_x G$ can be extended to an element of $lG$ by $\mathfrak{z}(y) = \tau_*^{yx^{-1}}(u)$. Therefore any orthonormal $i$-tuple $\beta$ in $T_x G$ can be extended to an element of $\Omega_i$ because of (*).

LEMMA 4.1. *Let $A$ belong to $\mathscr{O}(T_e G)$ and let $x \in G$. Then*

(i) $q(\tau_*^x(A); \mathfrak{z}(x)) = q(A, \mathfrak{z}(e))$, *for all* $\mathfrak{z} \in \Omega_1$, *and*
(ii) $V_i(\tau_*^x(A); \omega(x)) = \tau_*^x(V_i(A; \omega(e)))$, *for all* $\omega \in \Omega_i$, $i = 1, \ldots, n$.

*Proof.*

(i) $$q(\tau_*^x(A); \mathfrak{z}(x)) = \sup_{a \in A} \tau_*^x(a) \cdot \mathfrak{z}(x) = \sup_{a \in A} a \cdot \mathfrak{z}(e) = q(A; \mathfrak{z}(e)).$$

(ii) $$\begin{aligned} V_1(\tau_*^x(A); \mathfrak{z}(x)) &= \{a_* \in \tau_*^x(A)\colon a_* \cdot \mathfrak{z}(x) = q(\tau_*^x(A), \mathfrak{z}(x))\} \\ &= \{\tau_*^x(a)\colon a \in A,\ \tau_*^x(a) \cdot \mathfrak{z}(x) = q(A; \mathfrak{z}(e))\} \\ &= \tau_*^x(V_1(A; \mathfrak{z}(e))). \end{aligned}$$

The proof is completed by induction. ∎

The following proposition allows us to give a natural definition of differentiability for m.v.f. on a Lie group.

PROPOSITION 4.1. *For each $i = 0, 1, \ldots, n$, the map*

$$\tilde{\mathscr{T}}_i\colon G \times \tilde{\mathscr{O}}_i(T_e G) \to \tilde{\mathscr{O}}_i(TG) = \bigcup_{x \in G} \tilde{\mathscr{O}}_i(T_x G),$$

*defined by*

$$\tilde{\mathscr{T}}_i(x, (A, B)) = (\tau_*^x(A), \tau_*^x(B)) \text{ for } x \in G,\ (A, B) \in \tilde{\mathscr{O}}_i(T_e G),$$

*defines a unique structure of trivial vector bundle* $\tilde{\pi}_i\colon \tilde{\mathcal{O}}_i(TG) \to G$. *Moreover*

$$\tilde{\mathcal{T}}_{i,x}\colon \tilde{\mathcal{O}}_i(T_e G) \to \tilde{\mathcal{O}}_i(T_x G)$$

*is an isometry.*

*Proof.* The proof of the first part follows easily from [4 p. 43]. We prove that $\mathcal{T}_{0,x}$ is an isometry. In fact, if $(A, B)$ belongs to $\tilde{\mathcal{O}}_0(T_e G)$ we have

$$\begin{aligned}\|(\tau_*^x(A), \tau_*^x(B))\|_0 &= \sup_{v \in \Omega_1} |q(\tau_*^x(A)); v(x) - q(\tau_*^x(B); v(x))| \\ &= \sup_{v \in \Omega_1} |q(A; v(e)) - q(B; v(e))| \\ &= \|(A, B)\|_0.\end{aligned}$$

In an analogous way we obtain the statement for $i = 1, \ldots, n$. ∎

From Proposition 4.1, if $A \in \mathcal{O}(T_x G)$ is identified with $(A, 0) \in \tilde{\mathcal{O}}(T_x G)$, then m.v.f. $F$ is a section of the $C^\infty$ vector bundle $\tilde{\pi}_i\colon \tilde{\mathcal{O}}_i(TG) \to G$, $i = 0, \ldots, n$. If such a section is of class $C^l$ (locally Lipschitz) we say that $F$ belongs to $C_i^l$ ($i$-Lipschitz). Since $h^0 \leq h^1 \leq \cdots \leq h^n$, it is obvious that $F \in C_i^l$ implies $F \in C_0^l$. In particular, if $F \in C_i^l$, $l \geq 1$, $F$ is a locally Lipschitz m.v.f. as in Definition 2.2.

PROPOSITION 4.2. *The following maps are* $C^\infty$ *vector bundle morphisms:*

(i) *For all* $\mathfrak{z} \in \Omega_1$, $q_{\mathfrak{z}}\colon \tilde{\mathcal{O}}_0(TG) \to G \times R$ *given by*

$$(A, B) \mapsto (x; q(A; \mathfrak{z}(x)) - q(B; \mathfrak{z}(x))) \quad \textit{for } (A, B) \in \tilde{\mathcal{O}}_0(T_x G).$$

(ii) *For all* $\omega \in \Omega_i$, $i = 1, \ldots, n-1$, $V_\omega\colon \tilde{\mathcal{O}}_i(TG) \to \tilde{\mathcal{O}}_0(TG)$ *given by*

$$(A, B) \mapsto (V_i(A; \omega(x)), V_i(B; \omega(x))) \quad \textit{for } (A, B) \in \tilde{\mathcal{O}}_i(T_x G).$$

(iii) *For all* $\omega \in \Omega_n$, $v_\omega\colon \tilde{\mathcal{O}}_n(TG) \to TG$ *given by*

$$(A, B) \mapsto V_n(A; \omega(x)) - V_n(B; \omega(x)) \quad \textit{for } (A, B) \in \tilde{\mathcal{O}}_n(T_x G).$$

*Proof.* (i) It is sufficient to show that $q_{\mathfrak{z}(x)}\colon \tilde{\mathcal{O}}_0(T_x G) \to R$ is a continuous linear map, and that the map from $G$ to Hom $(\tilde{\mathcal{O}}_0(T_e G), R)$ defined by $x \mapsto q_{\mathfrak{z}(x)} \circ \tilde{\mathcal{T}}_{0,x}$ is $C^\infty$ [4 p. 44]. For linearity and continuity of $q_{\mathfrak{z}(x)}$ see [9]. Moreover from Lemma 4.1 we have

$$q_{\mathfrak{z}(x)} \circ \tilde{\mathcal{T}}_{0,x}(A, B) = q_{\mathfrak{z}(x)}(\tau_*^x(A), \tau_*^x(B)( = q_{\mathfrak{z}(e)}(A, B),$$

for each $x \in G$ and $(A, B) \in \tilde{\mathcal{O}}_0(T_e G)$. Then the map $x \to q_{\mathfrak{z}(x)} \circ \tilde{\mathcal{T}}_{0,x}$ is the constant map $x \to q_{\mathfrak{z}(e)}$.

(ii) For linearity and continuity of $V_{\omega(x)}\colon \mathcal{O}_i(T_x G) \to \mathcal{O}_0(T_x G)$ see [9]. Moreover, from Lemma 4.1 we have $\tilde{\mathcal{T}}_{0,x}^{-1} \circ V_{\omega(x)} \circ \tilde{\mathcal{T}}_{i,x}(A, B) = V_{\omega(e)}(A, B)$, for each $x \in G$ and $(A, B) \in \tilde{\mathcal{O}}_i(T_e G)$.

(iii) This can be shown by an analogous procedure. ∎

PROPOSITION 4.3. *If $F$ belongs to $C_n^l$ (n-Lipschitz), then $F$ has the $C^l$ (Lipschitz) selection property.*

*Proof.* Let Ext $F(x)$ be the set of extremal points of $F(x)$ [12 p. 87]. $F(x) =$ co Ext $F(x)$. So, for $u \in F(x)$, $u = \sum_{i=1}^{k} \lambda_i b_i$, $b_i \in$ Ext $F(x)$. For each $i$, there exists a orthonormal base $\beta^i$ of $T_x G$, such that $b_i = V_n(F(x); \beta^i)$ [8], [9]. For any such a base $\beta^i$, let $\omega^i \in \Omega_n$ be such that $\beta^i = \omega^i(x)$. The field $f = \sum_{i=1}^{k} \lambda_i v_{\omega^i} \circ F\colon G \to TG$ is a $C^l$ (Lipschitz) selection such that $f(x) = u$. ∎

COROLLARY 4.1. *Let $F$ be n-Lipschitz. If $D = \{v_\omega \circ F\colon \omega \in \Omega_n\}$, then*

$$\text{cl } A(x_0, D) = \text{cl } \mathscr{A}(x_0, F) \quad \text{and} \quad \text{cl } A(x_0, T, D) = \text{cl } \mathscr{A}(x_0, T, F)$$

*for each $x_0 \in G$ and $T \in R^+$.*

The proof follows from Theorem 3.1 and Corollary 3.1, taking Proposition 4.3 into account. ∎

COROLLARY 4.2. *Let $F \in C_n^l$, $l \geq 1$, and let $F(x)$ be symmetric for each $x \in G$. If $D$ is defined as in Corollary 4.1 then $A(x_0, D) = \mathscr{A}(X_0, F) = M_{x_0}$ for each $x_0 \in G$, where* id: $M_{x_0} \to G$ *is a $C^l$ immersion.*

*Proof.* By Corollary 3.2 it is sufficient to prove that $D$ is symmetric. In fact for each $A \in \mathcal{O}(T_x G)$, and for each $v_1, \ldots, v_n \in T_x G$, we have

$$-V_n(A; v_1, \ldots, v_n) = V_n(-A; -v_1, \ldots, -v_n).$$

So, since $F(x) = -F(x)$, we have $-v_\omega \circ F = v_{-\omega} \circ F$ if $\omega = (\mathfrak{z}_1, \ldots, \mathfrak{z}_n)$ and $-\omega = (-\mathfrak{z}_1, \ldots, -\mathfrak{z}_n)$. ∎

The following propositions give some examples of classes of $n$-Lipschitz ($C_n^l$) m.v.f.

PROPOSITION 4.4. *Let $F(x) = \text{cl co } \{\mathfrak{z}_u(x)\colon \mathfrak{z}_u \in lG\}_{u \in A}$, where $A$ is an index set. If $F(x)$ belongs to $\mathcal{O}(T_x G)$ for each $x \in G$, then $F$ is a $C_n^\infty$ m.v.f.*

*Proof.* We have $\mathscr{T}_n^{-1} \circ F(x) = (x, (\tau_*^{x^{-1}} \circ F(x), 0))$. Moreover,

$$\tau_*^{x^{-1}}(\text{cl co } \{\mathfrak{z}_u(x)\}_{u \in A}) = \text{cl co } \{\mathfrak{z}_u(e)\}_{u \in A},$$

so $\mathscr{T}_n^{-1} \circ F$ is a constant map. ∎

PROPOSITION 4.5. *Let $A \in \mathcal{O}(R^m)$ be an r-convex set (that is a finite intersection of r-balls). Let $f\colon G \to$ Hom $(G \times R^m, TG)$ be a locally Lipschitz section such that* rank $f(x) = n$ *for each $x \in G$. Then $F\colon G \to (TG)$, defined by $F(x) = (f(x) \times (A), 0)$, is an n-Lipschitz m.v.f.*

*Proof.* As rank $f(x) = n$, then ker $(f(x))^* = 0$ for each $x \in G$. So since $A$ is a strictly convex set for each $\omega = (\mathfrak{z}, \ldots)$ we have

$$V_n(A; (f(x))^*(\omega(x))) = V_1(A; (f(x))^*(\mathfrak{z}(x))).$$

Then, taking Lemma 4.1 into account we get

$$\begin{aligned} &h^n(\tau_{*,x}^{x^{-1}} \circ F(x), \tau_{*,y}^{y^{-1}} \circ F(y)) \\ &\quad = \sup_{\mathfrak{z} \in \Omega} (\|\tau_{*,x}^{x^{-1}} \circ f(x)[V_1(A; (f(x))^*(\mathfrak{z}(x))) - V_1(A; (f(y))^*(\mathfrak{z}(y)))]\| \\ &\qquad + \|(\tau_{*,y}^{y^{-1}} \circ f(y) - \tau_{*,x}^{x^{-1}} \circ f(x))[V_1(A; (f(y))^*(\mathfrak{z}(y)))]\|). \end{aligned}$$

Moreover, as $A$ is $r$-convex, $V_1(A; \cdot): \Omega_1(R^m) \to R^m$ is a Lipschitz map with Lipschitz constant $r$ [7]. So $V_1(A; \cdot): R^m \backslash \{0\} \to R^m$ is locally Lipshitz. The proof follows taking into account that $f$ is Lipschitz and $\mathfrak{z}$, $\tau$ are $C^\infty$ maps. ∎

## 5. Applications

In the following, $M$ will be as in Section 2 and $\acute{G}$ will be as in Section 4.

PROPOSITION 5.1. *Let $f$: $M \times R^m \to TM$ be such that $f(x, \cdot)$: $R^m \to T_x M$ is an affine map for each $x \in M$, and that $f(\cdot, u)$: $M \to TM$ is a locally Lipschitz vector field for each $u \in R^m$. If we consider the control system $\dot{x}(t) = f(x(t), u(t))$, then, for each $x_0 \in M$ and $A \in \mathcal{O}(R^m)$, we have*

$$\text{cl } \mathscr{A}(x_0, M(A)) = \text{cl } \mathscr{A}(x_0, C(\text{Ext } A))$$

*and*

$$\text{cl } \mathscr{A}(x_0, T, M(A)) = \text{cl } \mathscr{A}(x_0, T, C(\text{Ext } A)),$$

*where $\mathscr{A}(x_0, M(A))$ is the set attainable from $x_0$ by means of the family $M(A)$ of measurable controls with values in $A$, and $C(\text{Ext } A)$ is the set of piecewise constant controls with values in* Ext $A$.

*Moreover, if $f(\cdot, u)$ is a $C^1$ vector field for each $u \in R^m$, we get the following results:*

(a) *If, for each $x \in M$, the null vector of $T_x M$ belongs to the interior of $f(x, A)$, relative to the minimal linear variety of $T_x M$ containing $f(x, A)$, then*

$$\mathscr{A}(x_0, M(A)) = \mathscr{A}(x_0, C(A))$$

(b) *If, for each $x \in M$, $f(x, 0) = 0 \in T_x M$ and $A$ is a symmetric set, then*

$$\mathscr{A}(x_0, M(A)) = \mathscr{A}(x_0, C(\text{Ext } A)).$$

*Proof.* For the first part it is sufficient to apply Proposition 3.1, Theorem 3.1 and Corollary 3.1, noting that, from the above assumption, that $f(x, u) = f_1(x) + f_2(x)(u)$ with $f_2(x) \in \text{Hom }(R^m, T_x M)$. For (a) and (b) it is sufficient to apply Corollaries 3.2 and 3.3.

We note that an explicit form of control system in Proposition 5.1 is

$$\dot{x}(t) = X^0(x(t)) + \sum_{i=1}^m u_i(t) X^i(x(t)),$$

where $u_i(t) \in R$ and $X^j$ $(j = 0, 1, \ldots, m)$ are Lipschitz vector fields on $M$. Moreover, if $X^0 = 0$ and $X^i$ $(i = 1, \ldots, m)$ are $C^1$ vector fields on $M$, the previous system satisfies condition $(b)$.

PROPOSITION 5.2. *Let $f$ be as in Proposition 5.1. Denoting the unit cube of $R^m$ by $B$ we get*

$$\mathrm{cl}\ \mathscr{A}(x_0, M(R^m)) = \mathrm{cl}\ \mathscr{A}(x_0, C(\mathrm{Ext}\ B))$$

*and*

$$\mathrm{cl}\ \mathscr{A}(x_0, T, M(R^m)) = \mathrm{cl}\ \mathscr{A}\left(x_0, T, \bigcup_n C(\mathrm{Ext}\ nB)\right),$$

*where $M(R^m)$ is the set of measurable, essentially bounded control functions with values in $R^m$. Moreover, if $f(x, 0) = 0$ for all $x \in M$, and if $f(\cdot, u)$ is a $C^1$ vector field for all $u \in R^n$, we get*

$$\mathscr{A}(x_0, M(R^m)) = \mathscr{A}(x_0, C(\mathrm{Ext}\ B))$$

*Proof.* Let $B_n = nB$ and $F_n(x) = f(x, B_n)$. By Proposition 5.1,

$$\mathrm{cl}\ \mathscr{A}(x_0, M(B_n)) = \mathrm{cl}\ \mathscr{A}(x_0, C(\mathrm{Ext}\ B_n)).$$

Since $\mathscr{A}(x_0, C(\mathrm{Ext}\ B_n) = \mathscr{A}(x_0, C(\mathrm{Ext}\ B))$, we obtain

$$\mathscr{A}(x_0, M(R^m)) = \bigcup_n \mathscr{A}(x_0, M(B_n)) \subset \mathrm{cl}\ \mathscr{A}(x_0, C(\mathrm{Ext}\ B)).$$

In an analogous way we get

$$\mathrm{cl}\ \mathscr{A}(x_0, T, M(R^m)) = \mathrm{cl}\ \mathscr{A}(x_0, T, \textstyle\bigcup_n C(\mathrm{Ext}\ nB)).$$

The second statement follows from Proposition 5.1(b). ∎

PROPOSITION 5.3. *Let $A \subset R^m$ be a compact set. Let $f$: $M \times A \to TM$ such that*

$$f(x, \cdot)\colon A \to T_x M$$

*is a continuous map for each $x \in M$, and the set of vector fields*

$$\{\mathfrak{z}_u = f(\cdot, u)\colon u \in A\}$$

*satisfies condition* (ii) *of Proposition* 3.3. *If we consider the control system $\dot{x}(t) = f(x(t), u(t))$ then we get*

$$\mathrm{cl}\ \mathscr{A}(x_0, M(A)) = \mathrm{cl}\ \mathscr{A}(x_0, C(A))$$

*and*

$$\mathrm{cl}\ \mathscr{A}(x_0, T, M(A)) = \mathrm{cl}\ \mathscr{A}(x_0, T, C(A)).$$

*Moreover suppose*

(i) *for every $x \in M$, the null vector of $T_x M$ belongs to the interior of $f(x, A)$, relative to the minimal linear variety of $T_x M$ containing $f(x, A)$, and*

(ii) *for every $u \in A$, $\mathfrak{z}_u$ is a $C^1$ vector field.*

*Then $\mathscr{A}(x_0, M(A)) = \mathscr{A}(x_0, C(A))$.*

*Proof.* For the first part it is sufficient to apply Proposition 3.2, Theorem 3.1 and Corollary 3.1. For the second part it is sufficient to apply Theorem 3.2. ∎

PROPOSITION 5.4. *Let $f_0$ be a $C^l$, $l \geq 1$, vector field on G. Let $A \subset R^m$ be a compact set. Let $f: G \times A \to TG$ be such that $f(x, \cdot): A \to T_xG$ is a continuous map and $f(\cdot, u)$ is a left invariant vector field on G for each $u \in A$. If we consider the control system*

$$\dot{x}(t) = f_0(x(t)) + f(x(t), u(t)),$$

*then we obtain the same results as in Proposition* 5.3.

The proof follows as in Proposition 5.3, taking Proposition 4.4 into account. ∎

REFERENCES

1. R. CONTI and R. M. BIANCHINI, *Processi di controllo bilineari*, Parte I, Ist. di Mat. U. Dini, Univ. di Firenze, Internae Report 1976/77, n. 6.
2. R. M. HIRSCHORN, *Controllability of nonlinear systems*, J. Differential Equations, vol. 19 (1975), pp. 46–61.
3. V. JURDIJEVIC and H. J. SUSSMAN, *Control Systems on Lie group*, J. Differential Equations, vol. 16 (1972), pp. 313–329.
4. S. LANG, *Differential manifolds*, Addison Wesley, Reading, Mass., 1972.
5. C. LOBRY, *Sur l' ensemble des points atteignables par le solutions d' une equation differentielle multivoques*, Publ. Math. Univ. Bordeaux, vol. 1 (1973), pp. 43–60.
6. ———, *Bases mathematiques de la theorie des systèmes asservis non linéaires*, U.E.R. de Mathématiqués e informatique, Bordeaux, 1976.
7. S. LOJASIEWICZ, JR., *The convexity of accessible sets in non linear control system*, unpublished paper.
8. C. OLECH, *A note concerning extremal points of a convex set*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Fis. Nat., vol. 13 (1965), pp. 347–352.
9. G. STEFANI and P. ZECCA, *Properties of convex sets with applications to differential theory of multivalued functions*, Nonlinear Anal. Theory, Methods and Applications, vol. 2 (1978), pp. 583–595.
10. ———, *Multivalued vector fields and multivalued differential equations*, Atti del Convegno "Equadiff 78" Firenze 1978, pp. 458–470.
11. H. J. SUSSMAN, *Orbits of families of vector fields and integrability of a distribution*, Tran. Amer. Math. Soc., vol. 180 (1973), pp. 171–187.
12. F. A. VALENTINE, *Convex sets*, McGraw-Hill, New York, 1964.

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