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SYMMETRIC PRODUCTS AND THE STABLE HUREWICZ HOMOMORPHISM

BY

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0. Introduction

Let $\mathbf{S}$ be the sphere spectrum. Let $SP^n\mathbf{X} = \mathbf{X}^n/\Sigma_n$ be the pth symmetric product of the spectrum $\mathbf{X}$. Let $\mathbf{Z}_{(p)}$ be the localization of the spectrum $\mathbf{Z}$ at the prime p, i.e. we invert all other primes. We observe that the Dold-Thom theorem [13] lets us identify the inclusion $\mathbf{S} \to SP^\infty\mathbf{S} = \bigcup_n SP^n\mathbf{S}$ with the Hurewicz map $h: \mathbf{S} \to \mathbf{KZ}$. Let $i: \mathbf{S} \to SP^p\mathbf{S}$ be the inclusion. Our main result is then:

THEOREM 0.1.

$$\ker\left(k: \pi_*\mathbf{S}_{(p)} \to H_*\mathbf{S}_{(p)} = \pi_* SP^\infty\mathbf{S}_{(p)}\right) = \ker\left(i: \pi_*\mathbf{S}_{(p)} \to \pi_* SP^p\mathbf{S}_{(p)}\right).$$

This is also true for some spectra other than $\mathbf{S}$; see Section 6, where we also discuss a counter-example showing that 0.1 does not generalize well (cf. the list of unsolved problems in [21]). It appears that S. D. Liao first suggested that symmetrization has the effect of killing homotopy.

We now list and briefly discuss the main results of each section. We will also indicate how they may be assembled to prove 0.1 above.

Section 1 contains general definitions and notation.

Section 2 deals with a topological version of the notion of *convergent functor* of [19]. The basic result that we need from this section is that the functors we deal with may be studied in a somewhat simpler fashion than what one might first think.

THEOREM 2.1. *Let SP^n, $\overline{SP^n}$, Z^n, $\bar{Z}^n$ be the various symmetric (resp. cyclic) product functors of Section 1. Let T be any of these. Then:*

(i) *T extends in a certain sense to a functor of spectra.*

(ii) *$\pi_* T(\cdot)$ is a homology theory of spectra, since for any space X, $T(\mathbf{S} \wedge X) \simeq_w X \wedge T(\mathbf{S})$.*

Thus we are really studying the much simpler functors which smash a space or spectrum with $T\mathbf{S}$.

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Section 3 recalls results from various papers and studies questions of connectivity (at a given prime p). The techniques are classical in nature.

PROPOSITION 3.1. *Let* $\mathbf{X}$ *be a strongly convergent spectrum which is homologically* $(n-1)$*-connected. Then*

$$SP^{p^r}(\mathbf{X})_{(p)} \to SP^{\infty}\mathbf{X}_{(p)} \simeq \mathbf{KZ} \wedge \mathbf{X}_{(p)}$$

is at least $(2p^{r+1}+n-4)$*-connected.*

This improves the classical estimate by roughly a power of p. We also show:

PROPOSITION 3.2. *Let* $\mathbf{X}$ *be a spectrum. Then:*
(a) $(SP^p\mathbf{X}/\mathbf{X})_{(p)} \simeq_w (\overline{SP^p\mathbf{X}})_{(p)}$;
(b) $(SP^{p^r}\mathbf{X}/SP^{p^{r-1}}\mathbf{X})_{(p)} \simeq_w (\overline{SP^{p^r}}\mathbf{X})_{(p)}$;
(c) $(Z^p\mathbf{X}/\mathbf{X})_{(p)} \simeq_w \bar{Z}^p\mathbf{X}$.
Note however that in general
(d) $(Z^{p^r}\mathbf{X}/Z^{p^{r-1}}\mathbf{X})_{(p)} \not\simeq_w (\bar{Z}^{p^r}\mathbf{X})_{(p)}$.

PROPOSITION 3.3. *Let* X *be a* $(k-1)$*-connected space,* $k \geq 2$.
(a) *The spectrum structure map* f_n: $S^1 \wedge SP^nX \to SP^n(S^1 \wedge X)$ *is at least* $(2k+1)$*-connected, i.e. is an isomorphism on* π_i *for* $i \leq 2k$.
(b) $S^1 \wedge Z^pX \to Z^p(S^1 \wedge X)$ *is at least* $2k$*-connected.*

The significance of 3.3 is that it indicates that our symmetric/cyclic product functors of spectra "extend" space-level functors, in giving homotopy information in a stable range. We have traded the property that $T(\mathbf{S} \wedge X) \simeq \mathbf{S} \wedge T(X)$, perhaps a more natural requirement of a functor which "extends" a space-level functor, for the properties of 2.1, which still give us stable information about the symmetric/cyclic products of spaces, by 3.3. This stable information about spaces was our original goal, although we will seldom spell out the space-level implications of statements about spectra (for brevity).

PROPOSITION 4.1. $\overline{SP^2}\mathbf{S} = \bar{Z}^2\mathbf{S} \simeq \mathbf{S} \wedge S^1 \wedge RP^{\infty}$ [17]. *In general*

$$\bar{Z}^p\mathbf{S} \simeq \mathbf{S} \wedge S^1 \wedge BZ_p, \quad p \text{ prime.}$$

PROPOSITION 4.2. $\overline{SP^p}\mathbf{S}_{(p)} \simeq (\mathbf{S} \wedge S^1 \wedge B\Sigma_p)_{(p)}$.

Putting these latter results together with 3.2, we then see that the cofibre of the inclusion $\mathbf{X}_{(p)} \to SP^p\mathbf{X}_{(p)}$ is $\mathbf{S} \wedge S^1 \wedge B\Sigma_p \wedge \mathbf{X}_{(p)}$, up to homotopy. Thus the Puppe map from the cofibration sequence is a map

$$\delta\colon (\mathbf{S} \wedge S^1 \wedge B\Sigma_p)_{(p)} \to \mathbf{S} \wedge S^1_{(p)}.$$

The purpose of Section 5 is to identify this map.

We recall from [4, p. 49]:

THEOREM (Kahn-Priddy). *Let* $\mathbf{L} = (\mathbf{S} \wedge S^1 \wedge B\Sigma_p)_{(p)}$.
(i) *There is a map of spectra* ϕ: $\mathbf{L} \to \mathbf{S}^1_{(p)}$ *which induces an isomorphism onto* $\pi_{2p-3}\mathbf{S}_{(p)} = \pi_{2p-2}S^1 \wedge \mathbf{S}_{(p)}$.

(ii) *Such a map is unique up to an equivalence* $\mathbf{L} \to \mathbf{L}$.

(iii) $\Omega\Omega^\infty\phi\colon \Omega\Omega^\infty\mathbf{L} \to \Omega\Omega^\infty(\mathbf{S} \wedge S^1)_{(p)} = \Omega^\infty\mathbf{S}_{(p)}$ *is split on the component of the basepoint by a map of spaces*

$$k'\colon (\Omega^\infty\mathbf{S}_{(p)})_0 \to \Omega^\infty(\mathbf{S} \wedge B\Sigma_p)_{(p)},$$

so that $\pi_i\mathbf{S}_{(p)}$ *is a direct summand of* $\pi_i(\mathbf{S} \wedge B\Sigma_p)_{(p)}$ *for* $i > 0$. *Here* Ω^∞ *is a functor from spectra to* CW *complexes defined in Section* 1.

(We have changed the statement somewhat to be closer to our notation and point of view. In particular, the statements in [4] and [18] apply to p-primary factors, rather than p-localizations.)

We show in Section 5 that our map δ satisfies condition (i) and thus by (ii) we may apply (iii). Then 0.1 becomes equivalent to the Kahn-Priddy theorem. For note that the preceding implies that there is a commutative diagram in which the rows are cofibrations:

$$\begin{array}{ccccc}
S^{-1} \wedge \overline{SP}^p\mathbf{S}_{(p)} & \xrightarrow{\ \delta\ } & \mathbf{S}_{(p)} & \longrightarrow & SP^p\mathbf{S}_{(p)} \\
\downarrow & & \| & & \downarrow \\
(\mathbf{S} \wedge B\Sigma_p)_{(p)} & & & & \\
{\scriptstyle a}\downarrow & & & & \\
\mathbf{A}_{(p)} & \longrightarrow & \mathbf{S}_{(p)} & \xrightarrow{\ h\ } & \mathbf{KZ}_{(p)}.
\end{array}$$

We note in passing that $T(\mathbf{X}_{(p)}) \simeq (T\mathbf{X})_{(p)}$ for any convergent functor T. Taking π_* of this diagram we get

$$\begin{array}{ccccc}
\sigma_*(B\Sigma_p)_{(p)} & \xrightarrow{\ h\ } & \sigma_* & \xrightarrow{\ i\ } & \pi_*SP^p\mathbf{S}_{(p)} \\
{\scriptstyle a_*}\downarrow\uparrow{\scriptstyle k'} & & \| & & \downarrow \\
\pi_*\mathbf{A}_{(p)} & \longrightarrow & \sigma_* & \xrightarrow{\ h\ } & H_*\mathbf{S}_{(p)}.
\end{array}$$

Here the rows are exact and k' is the splitting of Kahn-Priddy. The kernel of the Hurewicz homomorphism is the image of $\pi_*\mathbf{A}$ in σ_*. So $\ker h \subseteq \ker i$. The reverse inclusion is clear, as the map h factors through i.

COROLLARY 0.2. *The statement* 0.1 *is still true if* SP^p *is replaced with* Z^p. ∎

We note that the map ϕ of (i) above may be given by a transfer construction, t say, as in [18]. Also, 0.1 extends to the statement that either of the groups there is equal to the image of

$$t_*\colon \pi_*B\Sigma_p \wedge \mathbf{S}_{(p)} \to \pi_*\mathbf{S}_{(p)}.$$

1. Notation and basic definitions

1.1. We will usually work in the category $\mathscr{C}_*$ of based (compactly generated) CW-complexes and continuous maps. We let $*$ be the non-degenerate basepoint of such a complex. Also, $\simeq_w$ means a weak homotopy equivalence, and $\sigma_* X$ is the stable homotopy of the space X.

1.2. As far as spectra, we use those of [1]; we also use his smash products of spectra. Not all spectra will be useful to us. We will occasionally have to restrict ourselves with the following.

DEFINITION 1.3 (cf. also p. 42 of [8]). A spectrum $\mathbf{X}$ is *strongly convergent* if and only if there is an N with each $\mathbf{X}_n$ $(n + N)$-connected for $n \gg 0$, *and* for each q, the structure maps

$$f_k\colon S^1 \wedge \mathbf{X}_k \to \mathbf{X}_{k+1}$$

are $(q + k)$-connected for almost all k.

1.4. *The functor* Ω^∞. Let $\mathbf{Y}$ be a strongly convergent spectrum with cellular structure maps. If X is a finite CW complex, $H(X) = [S \wedge X, \mathbf{Y}]$ satisfies the axioms for Adams' version of Brown's Representability Theorem [2], and therefore is isomorphic to $[X, \Omega^\infty \mathbf{Y}]$ for some CW complex $\Omega^\infty \mathbf{Y}$. A map of spectra $\mathbf{Y} \to \mathbf{Z}$ induces $[X, \Omega^\infty \mathbf{Y}] \to [X, \Omega^\infty \mathbf{Z}]$, a natural transformation. So there is a homotopy class of maps $f\colon \Omega^\infty \mathbf{Y} \to \Omega^\infty \mathbf{Z}$ inducing this natural transformation.

If $\mathbf{Y} = \mathbf{S} \wedge Y'$ for some CW complex Y', $\Omega^\infty \mathbf{Y} \simeq \Omega^\infty \mathbf{S} \wedge Y'$ may be taken to be given by the construction of Barratt-Eccles, [7].

Our only use of Ω^∞ is in discussing the Kahn-Priddy theorem.

1.5. *Symmetric products.* Let Σ_n be the symmetric group on n letters. Let X^n be the n-fold CW product of the CW complex X with itself (retopologized as per 1.1). We let Σ_n act on X^n by permuting the coordinates. The *nth symmetric product of* X is $SP^nX = X^n/\Sigma_n$, the orbit space of the action of Σ_n. Note that there is an *inclusion* of X into SP^nX, given by $x \to x \times * \times * \times \cdots \times *$. We will *not* be concerned with the *diagonal inclusion*, $x \to x \times x \times \cdots \times x$, in this paper. Note that there is an analogous inclusion of SP^mX into SP^nX, $m \leq n$. At the end of this section we discuss the cell structure of SP^nX (and the other functors we are about to introduce: $\overline{SP}^n$, Z^n, and $\overline{Z}^n$). For now it suffices to know that SP^nX can be given a CW structure. Furthermore, we will make $SP^{n-1}X$ a subcomplex of SP^nX. We define the *nth reduced symmetric product* to be

$$\overline{SP}^nX = SP^nX/SP^{n-1}X = X^{(n)}/\Sigma_n = X \wedge \cdots \wedge X/\Sigma_n.$$

There is a map $S^1 \wedge SP^nX \to SP^n(S^1 \wedge X)$ given by

$$t \wedge (x_1 \times \cdots \times x_n) \to (t \wedge x_1) \times \cdots \times (t \wedge x_n).$$

This means we may extend to a definition of SP^n on our category of spectra.

1.6. *Cyclic products.* Let $Z_n = Z/nZ$. We may analogously define the *nth cyclic product of* X to be $Z^nX = X^n/Z_n$. The cyclic group acts by cyclic permutation of coordinates. We have $X \to Z^nX$ as before. If $km = n$, $k \neq 0$, then

$$x_1 \times \cdots \times x_m \to x_1 \times \underbrace{* \times \cdots \times *}_{k-1} \times x_2 \times \underbrace{* \times \cdots \times *}_{k-1} \times \cdots \times *$$

defines a map $Z^mX \to Z^nX$. If $m \nmid n$, then it is not clear how to proceed. All this extends to spectra as with the symmetric products.

1.7. Now $SP^\infty X = \bigcup_{n \geq 0} SP^nX$, where we set the conventions $SP^1X = X$, $SP^0X = *$. We note Deleanu's [10] version of the Dold-Thom [13] theorem:

THEOREM 1.7. *Let f: $X \to Y$ be a map of connected* CW-*spaces. Then the homology sequence of f is naturally isomorphic with the homotopy sequence of $SP^\infty f$: $SP^\infty X \to SP^\infty Y$.*

Note. We have obvious natural maps which make the following diagram commute:

$$\begin{array}{ccccc} Z^nX & \longrightarrow & SP^nX & \longrightarrow & SP^\infty X \\ \downarrow & & \downarrow & & \\ \bar{Z}^nX & \longrightarrow & \overline{SP}^nX & & \end{array}$$

There is also a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & SP^nX & \longrightarrow & SP^nX \bigcup_i CX \\ \downarrow & & \| & & \downarrow \\ SP^{n-1}X & \longrightarrow & SP^nX & \longrightarrow & \overline{SP}^nX \end{array}$$

in which the rows are cofibrations. We will show that (localized at a prime p) the two rows are homotopy equivalent in the stable range.

1.8. *Localization.* We use $\mathbf{Z}_{(p)}$ for the localization of the spectrum $\mathbf{Z}$ at the prime p (cf. [1, p. 201]).

If G is an abelian group, we use ${}_pG$ for the p-primary part of G.

Appendix. Symmetric and cyclic products as CW complexes

We describe how to obtain a CW structure on SP^nX, given a CW complex X. This is done in such a way that it is natural in X, $SP^{n-1}X$ is a subcomplex of SP^nX, and so that if f: $X \to Y$ is cellular then so is SP^nf. In particular $\overline{SP}^nX$ is given cellular structure as $SP^nX/SP^{n-1}X$. The structure for cyclic products will have analogous properties. For all of this, note that it suffices to give a natural cell structure to X^n such that each $\sigma \in \Sigma_n$ is either the identity on a cell e_α or is a homeomorphism of e_α onto some other cell e_β. This is the cellular analog of the simplicial decomposition of [24]. We will not attempt to do more than this, as the notation would get rather ghastly, to no real purpose.

The whole point of what we do is that the sets of fixed points of $\sigma \in \Sigma_n$ are all subcomplexes. We arrange this by "subdividing" certain cells of X^n. Let A be an indexing set for the cells of X. Recall that the cells of X^n are of the form $e_{\alpha_1} \times \cdots \times e_{\alpha_n}$. We wish to alter the structure we give to the images of those cells $e_{\alpha_1} \times \cdots \times e_{\alpha_n}$ with $\alpha_i = \alpha_j$ for some $i \neq j$. Up to permutation of coordinates then, we have to describe what to do with a cell of the form

$$\underbrace{e_{\alpha_1} \times \cdots \times e_{\alpha_1}}_{r_1} \times \underbrace{e_{\alpha_2} \times \cdots \times e_{\alpha_2}}_{r_2} \times \cdots \times \underbrace{e_{\alpha_j} \times \cdots \times e_{\alpha_j}}_{r_j},$$

where $\Sigma r_i = n$. Such a cell will be given the product structure arising from the structures we place on each

$$\underbrace{e_{\alpha_i} \times \cdots \times e_{\alpha_i}}_{r_i} = (e_{\alpha_i})^{r_i}.$$

The structure on $(e_{\alpha_i})^{r_i}$ will match the structure on $(e_{\alpha_i})^{r_i+1}$ under all possible inclusions.

Specifically, what we do is describe a cellular decomposition of $(I^d)^{r_i}$, where d is the dimension of the cell e_{α_i}. The cells in X^n are the images of these cells under the maps of the $e_{\alpha_1} \times \cdots \times e_{\alpha_n}$ into X^n. Let $r = r_i$. Triangulate $(I^d)^{r_i}$ as a simplicial (hence CW) complex by taking as rd-simplices the sets of points

$$i_1 \times \cdots \times i_r \quad \text{with } i_{\sigma(1)} \leq i_{\sigma(2)} \leq \cdots \leq i_{\sigma(r)},$$

using the lexicographic ordering. We do this for each $\sigma \in \Sigma_r$. The other cells are to be the faces of these simplices. Note that the lowest dimension in which we get something new is d, corresponding to the diagonal $i_1 = \cdots = i_r$. Cells in lower dimensions arise from products of cells of lower dimensions. We then use a homeomorphism

$$(I^d, \dot{I}^d)^r \xrightarrow{h} (e_{\alpha_i}, \dot{e}_{\alpha_i})^r$$

to tell us how to attach these new cells: map them by inclusion to I^d, follow by h, and then use the map including $(e_{\alpha_i}, \dot{e}_{\alpha_i})^r$ into (X^r, X^r_{rd-1}).

2. Convergent functors

In their paper [19], Kan and Whitehead present a notion of "convergent functor", defined on the category of semisimplicial spectra. Examples include the set functors SP^n, $\overline{SP}^n$, Z^n, and $\bar{Z}^n$, as extended to that category. The other standard example is Milnor's F, the "loops on suspension" functor. In this section, we will discuss topological analogues of the semi-simplicial version.

A comment on all this fuss is perhaps appropriate. One might well ask why the semi-simplicial version isn't good enough. The problem is that the topological functors so not seem to be well-related to their semi-simplicial analogues under the realization and semi-simplification functors. Also the axioms in [19] are such as to give little hint of what sort of ingredients are essential to the proofs, the price of being so elegantly simple.

The first requirement on a convergent functor is that it be a functor taking spaces to spaces. It should preserve homotopy; if one wishes to have this as an axiom then one should work in a category where the morphisms are homotopy classes of maps. We give our axioms for a convergent functor below, after stating what is to be proved about such a functor. The concerns of this section have been dealt with in a number of other ways, and the proofs will be deleted in the interest of brevity. See also [5], [6], [7], [22], and of course [19].

Properties of convergent functors. Let T be a convergent functor. Then we wish to have the properties of Theorem 2.1 above: the functor T extends in a certain sense to spectra, and the homotopy of T is a homology theory of spectra. This latter follows from the property that applying T to a cofibration of spectra gives a cofibration of spectra.

Here extending to spectra means simply that there is a natural map

$$S^1 \wedge TX \to T(S^1 \wedge X)$$

for all spaces X. The alternative approach is to view this as giving an extension of T, say T', which maps spaces to infinite loop spaces, as in [22], by taking the adjoint of the above natural map, and forming a limit construction. In all cases the existence of such a map seems to either be one of the axioms or to follow fairly easily from them.

All of the other properties follow by classical techniques from one hard result, which appears as one of the axioms in [22]:

PROPOSITION. *Let T be a convergent functor. Then T preserves cofibrations in a stable range. In other words let $A \subseteq B$ be n-connected spaces and let $C = B/A$; then (for $n > 1$) $\pi_k TC = \pi_k TB/TA$ for $k < 2n - 1$.*

To prove this, one needs some connectivity axioms. In particular, if $f: X \to Y$ is n-connected, Tf must also be n-connected. Then [19] or [7] show (in a semi-simplicial context) how to reduce the above to the following, which seems to be best approached by building it into the axioms:

Property. Let X be $(m - 1)$-connected and let Y be $(n - 1)$-connected. Then the natural map $TX \vee TY \to T(X \vee Y)$ is $(m + n - 1)$-connected.

Here is our version of a convergent functor: Let $\mathscr{C}_*$ be the category of CW complexes with basepoint, with all continuous maps as morphisms, as in Section 1. Let $\mathscr{T}_*$ be the category of based compactly generated Hausdorff spaces. Throughout this section *T is a functor from $\mathscr{C}_*$ to $\mathscr{C}_*$* induced by a functor of $\mathscr{T}_*$. Thus given a nice space X, T gives us another space, TX; when we add in a specific cell structure on X, we get back a specific cell structure on TX. *We assume that if A is a subcomplex of X, TA is a subcomplex of TX.*

DEFINITION. Let $T: \mathscr{C}_* \to \mathscr{C}_*$ be as above. T is a *chain functor* provided there is a functor T' of graded free abelian groups with $C_* T(\cdot) = T'C_*(\cdot)$, where C_* is the cellular chains. T is *diagonally continuous* provided a certain map θ is continuous for all spaces X and Y in $\mathscr{C}_*$. We define θ as follows. Let $x \in X$. Define i_x: $Y \to X \wedge Y$ by $i_x(y) = x \wedge y$. Then θ: $X \wedge TY \to T(X \wedge Y)$ is given by $\theta(x \wedge z) = T(i_x)(z)$. Note that this is natural in X and Y. We say T is *convergent* provided:

(1) $T(*) = *$

(2) T preserves direct limits of directed sets.

(3) $(T(X \vee Y))_n \subseteq \bigcup_{p+q=n} T(X_p \vee Y_q)$. (*n-skeleta condition*).

(4) T is a chain functor.

(5) T is diagonally continuous.

Please note that the comments of Section 1 make (3)–(5) above fairly clear for the functors we deal with in this paper.

Note. It appears that the work of Brown-Douglas-Filmore could to some extent be dealt with in the framework of convergent functors. On the other hand, cellular structure for their functors would be a nuisance to define, and an approach such as [22] might be better.

3. Basic technical connectivity results

This section contains assorted connectivity results, most of which are corollaries to results which may be found in several papers. These papers are referred to at the appropriate places.

Our proofs deal with symmetric products first, followed by discussion of properties of the cyclic products.

Proof of 3.3(a). The pertinent reference is [28]. There is a filtration

$$SP^n(S^1 \wedge X) = A_n \supseteq A_{n-1} \supseteq \cdots \supseteq A_1 = S^1 \wedge SP^n X.$$

Here $A_i = \{(t_1 \wedge x_1) \times \cdots \times (t_n \wedge x_n) \in A_n \mid$ there are at most i different numbers among the t_j's$\}$. Let

$$\pi = [i_1 : i_2 : \cdots : i_q]$$

be a partition of n. Let $A_{q\pi} \subseteq A_q$ be the points with representatives having i_1 of the t_j equal to s_1, i_2 equal to s_2, and so on, for some $s_1 \geq s_2 \geq \cdots \geq s_q$. Then

$$A_q/A_{q-1} = \bigvee_{\pi} (A_{q\pi}/A_{q-1}).$$

Snaith and Ucci show

$$\frac{A_{q\pi}}{A_{q-1}} \simeq S^q \wedge (SP^{i_1}X \wedge SP^{i_q}X) \vee S^q \left(SP^{i_1}X \wedge \prod_{j=2}^{q-1} SP^{i_j}X \wedge SP^{i_q}X \right),$$

Now SP^iX is $(k-1)$-connected if X is, for $1 \leq i \leq n$. So $A_{q\pi}/A_{q-1}$ is at least $(q+2k-1)$-connected. Hence A_q/A_{q-1} is $(q+2k-1)$-connected $(q \geq 2)$. Thus

$$A_n/A_1 = SP^n(S^1 \wedge X)/S^1 \wedge SP^nX$$

is at least $(2k+1)$-connected. This proves 3.3(a).

We will need more detailed information about the SP^n. One of our first concerns is the connectivity of $\overline{SP}^nS^k$:

LEMMA 3.5. *Let $k \geq 1$. Let $n = \sum_{i=0}^{r(p,n)} a_i p^i$ $(0 \leq a_i < p)$ be the p-adic decomposition of n. Set*

$$l = \sum_{i=1}^{r(p,n)} a_i\left(k + 2(p-1)\sum_{j=0}^{i-1} p^j\right) + a_0 k - 2.$$

Then $(SP^nS^k)_{(p)}$ is at least l-connected.

Note. We haven't defined what we mean by localization of a space. One may use a constructive method on the (homotopy) category of CW complexes, as in [15]. Alternatively, one may regard the statements involving $X_{(p)}$ as the obvious equivalent statement concerning the homotopy of X modulo the Serre class $\mathcal{G}$ of torsion groups with no element having order equal to a non-zero power of p. Either interpretation leads to the given statements concerning localized spectra.

Proof of 3.5. This follows fairly easily from [23, Corollary 2, p. 69]. The point is that $l+2$ is the least degree in which an element of $H^*(\overline{SP}^nS^k; Z_p)$ can occur. So $l+1$ is the similar degree for H_*. But as $k \geq 1$, $\overline{SP}^nS^k$ is $(k+1)$-connected [24]; in particular it is 1-connected. So the Hurewicz theorem mod p applies; note that $H_*(\overline{SP}^nS^k; Z)$ is either torsion prime to p or else is detected by the mod-p homology.

Remark 3.6. Note that $2(p-1)\sum_{j=0}^{i-1} p^j = 2(p^i - 1)$, so l can also be written as

$$l = 2n - 2\left(\sum_{i=1}^{r(p,n)} a_i\right) + \sum_{i=0}^{r(p,n)} a_i k - 2.$$

As $\sum_1^r a_i \leq n$, l is definitely larger than $n + a_0 k - 2$. Thus $a_0 \neq 0, 1$ already puts us in the stable range.

We note in passing that if X is $(k-1)$-connected, SP^nX/X is $(k+1)$-connected (at least) (cf. [24]).

We make a slight digression at this point in order to look directly at what $\overline{SP}^n(\cdot)$ does to cofibrations at the space level. Results sharper than those of Section 2 are possible. There is a natural map p_n: $\overline{SP}^nX/\overline{SP}^nA \to \overline{SP}^n(X/A)$, where $n \geq 1$ and $A \subseteq X$.

LEMMA 3.7. *Let A, X, and X/A be connected spaces. Then if $X_{(p)}$ and $A_{(p)}$ are $(k-1)$-connected ($k \geq 1$), there are recursively definable functions $l(n, p, k)$ and $l'(n, p, k)$ such that $\overline{SP}^n X_{(p)}$ is at least l-connected ($n \geq 2$) and the map $(p_n)_{(p)}$ is at least l'-connected.*

Let the connectivity of $\overline{SP}^n S^k_{(p)}$ be $m(n, p, k)$ (see above). We have

$$l(n, p, k) = \min\,(l''(n, p, k), m(n, p, k))$$

and

$$l'(n, p, k) = l''(n, p, k) + 1,$$

where we define

$$l''(n, p, k) = \min\,\{k + l(n-1, p, k), l(2, p, k) + l(n-2, p, k), \\ l(3, p, k) + l(n-3, p, k), \ldots, l([n/2], p, k) + l([(n+1)/2], p, k)\}.$$

Proof. We begin with $n = 2$. The proof will then proceed by induction on n, with a subsidiary induction on the number of cells on the CW-complexes in question.

There is a cofibration (up to homotopy):

$$X/A \wedge A \to \overline{SP}^2 X/\overline{SP}^2 A \xrightarrow{p_2} \overline{SP}^2(X/A) \to \cdots.$$

If $A_{(p)}$ and $X_{(p)}$ are $(k-1)$-connected, $(X/A \wedge A)_{(p)}$ is $(2k-1)$-connected. Thus p_2 is an isomorphism on H_i for $i \leq 2k-1$. Using a result of Nakaoka mentioned above, $SP^n X/X$ is $(k+1)$-connected [24]. Thus if X, A, and X/A are connected, the $\overline{SP}^2$ terms above are 2-connected. In particular, the Whitehead theorem applies, so p_2 is $2k$-connected. $\overline{SP}^2 X_{(2)}$ is $(k+1)$-connected, as may be seen now by induction over cells and attaching maps. For $p \neq 2$, $\overline{SP}^2 X_{(p)}$ is $(2k-1)$-connected, since $\overline{SP}^2 S^k$ is. For $n > 2$ we proceed inductively, supposing both results known for $2 \leq m < n$. Let F be the points collapsed to the basepoint under p_n, so that we have the cofibration (up to homotopy type)

$$F_{(p)} \to (\overline{SP}^n X/\overline{SP}^n A)_{(p)} \to (\overline{SP}^n(X/A))_{(p)}.$$

Filter $F_{(p)}$ by letting F_k be the points of $F_{(p)}$ with at least k coordinates in $A_{(p)}$. Then we have

$$* = F_n \subset \cdots \subset F_1 = F_{(p)}.$$

Now

$$F_1/F_2 \simeq \overline{SP}^{n-1}(X/A) \wedge A_{(p)},$$

$$F_{n-1} \simeq (X/A \wedge \overline{SP}^{n-1} A)_{(p)},$$

$$F_{n-2}/F_{n-1} \simeq (\overline{SP}^2(X/A) \wedge \overline{SP}^{n-2} A)_{(p)},$$

and in general,

$$F_{n-1}/F_{n-j+1} \simeq (\overline{SP}^j(X/A) \wedge \overline{SP}^{n-j} A)_{(p)} \quad \text{for } 1 \leq j < n-1.$$

Thus $F_{(p)}$ is at least l-connected, where l is the minimum of the connectivities of the F_{n-j}/F_{n-j+1}. This gives rise to the ugly recursive relations appearing in the statement of the lemma, since l then determines the connectivity of $(p_n)_{(p)}$, as indicated. ∎

Of course, the problem remains: we need to solve the number-theoretical questions implicit in the definitions of the recursive functions l and l'. Consider a fixed prime p, and suppose $n < p^2$. Then

$$m(n, p, k) = \begin{cases} 2n + nk - 2 & \text{if } n < p, \\ 2p + k - 4 & \text{if } n = p, \\ 2n + (k-2)(n-p+1) + (n - [n/p]p)k - 2 & \text{if } p < n < p^2. \end{cases}$$

Suppose $n \leq p$ now. Then

$$l''(n, p, k) = \min \{nk - 1\} = nk - 1,$$

so $l'(n, p, k) = nk$. Now

$$l(n, p, k) = \min (nk - 1, 2n + nk - 1) = nk - 1 \quad \text{if } n < p,$$

$$l(p, p, k) = \min (pk - 1, 2p + k - 4).$$

If $k \geq 2$ this is $2p + k - 4$; if $k = 1$ it is $p - 1$. This proves:

LEMMA 3.8. *If $n \leq p$ then $l'(n, p, k) = nk$. If $k \geq 2$*

$$l(n, p, k) = \begin{cases} nk - 1 & \text{if } n < p, \\ 2p + k - 4 & \text{if } n = p. \end{cases}$$

The same holds for $n < p$ if $k = 1$.

We can carry this a bit further. Suppose $n \geq p$. Then we may inductively show that $\overline{SP}^n X_{(p)}$ is $(2n + k - 4)$-connected, using the definition of l''. In particular, we have:

COROLLARY 3.9. *$\overline{SP}^n X$ is at least $(2n + k - 4)$-connected, if X is $(k-1)$-connected, $k \geq 1$.*

This agrees with results obtained by Nakaoka. We restate 3.8 more explicitly:

COROLLARY 3.10. *Let $2 \leq n \leq p$, p prime. Let $k \geq 1$. Suppose $A_{(p)}$ and $X_{(p)}$ are $(k-1)$-connected, where A and X are connected* CW-*complexes. Then $\overline{SP}^n X_{(p)}$ is $(nk - 1)$-connected if $n < p$ and is $(2p + k - 4)$-connected if $n = p$, $k \geq 2$. Also*

$$(p_n)_{(p)}\colon (\overline{SP}^n X / \overline{SP}^n A)_{(p)} \to \overline{SP}^n (X/A)_{(p)}$$

is nk-connected.

Corollary 3.10 leads easily to:

PROPOSITION 3.11. *$(SP^p X/X)_{(p)} \to \overline{SP}^p X_{(p)}$ is $2k$-connected for $(k-1)$-connected X, $k \geq 1$.*

Proof of 3.2(a) *and* (b). The above proves (a) for strongly convergent spectra. Recall our convergent functors result: $SP^p(\cdot)/(\cdot)$ and $\overline{SP}^p(\cdot)$ are both convergent functors. So

$$(SP^p\mathbf{X}/\mathbf{X})_{(p)} \simeq_w SP^p\mathbf{S}/\mathbf{S} \wedge \mathbf{X}_{(p)} \simeq_w \overline{SP}^p\mathbf{S} \wedge \mathbf{X}_{(p)} \simeq_w \overline{SP}^p\mathbf{X}_{(p)}.$$

To prove (b), use the same technique. The result holds for $\mathbf{S}$ because if $n \neq p^k$, $\overline{SP}^n S^l$ is at least $(2l-1)$-connected (3.5), and the argument of 3.11 now works.

LEMMA 3.12. $\pi_l \overline{SP}^{p^{r+1}}\mathbf{S} = 0$ *for* $0 < l \leq 2p^{r+1} - 3$.
This is an easy corollary of 3.5.

Proof of 3.1. Due to our convergent functor results, we can consider the question in the form

$$SP^{p^r}\mathbf{S} \wedge \mathbf{X}_{(p)} \to SP^{\infty}\mathbf{S} \wedge \mathbf{X}_{(p)} \simeq \mathbf{KZ} \wedge \mathbf{X}_{(p)}.$$

The result follows from homological connectivity, due to our hypotheses on $\mathbf{X}$.

We now consider the analogous results for cyclic products [25].

Proof of 3.3(b). It is fairly easy to see this for $X = S^k$ by using the known cohomology structure. We may then finish with a filtration argument as in the proof of 3.3(a) above, using homological information this time to verify connectivity. If $A \subseteq X$, then we are interested in the map

$$Z^pX/Z^pA \xrightarrow{\pi} Z^p(X/A).$$

The point of Section 2 is that this is a homotopy equivalence in a stable range. So we can use this to do induction over the cells of X.

Proof of 3.2(c). It suffices to consider

$$(Z^p\mathbf{S}/\mathbf{S}) \wedge \mathbf{X} \to \bar{Z}^p\mathbf{S} \wedge \mathbf{X}.$$

The result follows from the fact that $Z^pS^k/S^k \to \bar{Z}^pS^k$ is a (co-)homology isomorphism in a stable range. Probably the easiest way to see this fact is to use the identification of $\bar{Z}^pS^k$ in Section 4 below, which doesn't rely on 3.2(c).

Proof of 3.2(d). For this we need the information from Section 4 below concerning $\bar{Z}^{p^r}S^k$. The reader may find a description of the cohomology of $Z^{p^r}S^k$ in Swan [29]. It is then obvious that $\bar{Z}^{p^r}S^k$ has the wrong cohomology mod p to be the cofibre of $Z^{p^{r-1}}S^k \to Z^{p^r}S^k$, even when attention is restricted to the stable range.

4. Identifying various spaces and spectra

This section is devoted to identifying the representing spectra $Z^p\mathbf{S}_{(p)}$ and $SP^p\mathbf{S}_{(p)}$. The reader will note that our results generalize those of [17]. The basic technique is from Snaith-Ucci [28]. Although $SP^p\mathbf{S}$ has the "right" cohomology, it is hard to identify, so our proofs start with 4.1, since the group action is much closer to being free than with Σ_n. We then go on to prove 4.2, which is more central to our interests.

One interesting property here is that the fixed point set roughly corresponds to the suspension coordinates. The situation seems to be such that the action of Σ_p, in a stable context, might as well be free. Using an induction based on [24, p. 131], one can show that the fixed point set from SP^pS^k is $(2k-1)$-connected at p.

We also note an interesting corollary to Section 2. Let

$$Y^{(k)} = Y \wedge \cdots \wedge Y \quad (k \text{ times})$$

be the k-fold smash product. Let $H \subseteq \Sigma_{p^r}$ be the wreath subgroup $\Sigma_p \int^r \Sigma_p$. Then we have:

PROPOSITION 4.3. $\mathbf{X}_{(p)}^{p^r}/H \simeq (SP^p\mathbf{S})^{(r)} \wedge \mathbf{X}_{(p)}$.

This makes it easy to calculate the cohomology of the Liao Γ-product for H. The action of the Steenrod algebra on this cohomology is clear. The appropriate analogs hold for $\overline{SP}^p$, Z^p, and $\bar{Z}^p$, using $Z_p \int^r Z_p$ for H in the latter two cases.

Proof of 4.3. Note that

$$\mathbf{X}_{(p)}^{p^r}/H = \underbrace{SP^p \cdots SP^p}_{r}\mathbf{X}_{(p)}.$$

As $SP^p\mathbf{X}_{(p)} \simeq SP^p\mathbf{S} \wedge \mathbf{X}_{(p)}$ by Section 2, the right hand side is just $(SP^p\mathbf{S})^{(r)} \wedge \mathbf{X}_{(p)}$, as claimed.

Finally, we also obtain the homotopy type of $\bar{Z}^{p^r}S^k$ as a join:

PROPOSITION 4.4. $\bar{Z}^{p^r}\mathbf{S} \simeq S^r \wedge BZ_p \wedge BZ_{p^2} \wedge \cdots \wedge BZ_{p^r} \wedge \mathbf{S}$.

Again, this makes the cohomology easy to deal with.

4.5. *The technique of Smith-Ucci* [28]. Although they deal with $\overline{SP}^nS^k$, note that Σ_n may be replaced by Z_n without changing the arguments.

Let (D, S) be the unit disc and sphere in R^l with usual inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \Sigma x_i y_i$. If we have an orthogonal decomposition $R^l = W_1 \oplus W_2$ such that $W_1^\perp = W_2$, we may set $D_i = D \cap W_i$ and $S_i = S \cap W_i$ (i = 1, 2). Then if we let $*$ indicate the join of two spaces, we have

$$(D_1 * S_2, S_1 * S_2) \cong (D, S),$$

a homeomorphism of pairs. The map is given by

$$f([s\mathbf{x}, \mathbf{y}, t]) = s\sqrt{1-t}\,\mathbf{x} + \sqrt{t}\,\mathbf{y}.$$

We now apply this with $l = nk$ and $W_1 = \{v \in (R^k)^n \mid \mathbf{v}_1 = \cdots = \mathbf{v}_n\}$. The symmetric group Σ_n acts trivially on W_1, hence on D_1, when considered as acting on R^{nk} by permuting blocks of k coordinates.

PROPOSITION [28]. *There is a homeomorphism of pairs*

$$(\bar{D}_1 * (S_2/\Sigma_n), \bar{S}_1 * (S_2/\Sigma_n)) \cong SP^n(D^k, S^{k-1}).$$

Here $\bar{D}_1$ is the homeomorphic image (under projection to the orbit space) of D_1. Similarly for $\bar{S}_1$. By $SP^n(D^k, S^{k-1})$ we mean the orbit space under the action of Σ_n on

$$((D^k)^n, (D^k)^{n-1} \times S^{k-1})$$

(end of recollection).

We now do the obvious. Collapsing subspaces to a point we get $\bar{D}_1 * (S_2/\Sigma_n)/\bar{S}_1 * (S_2/\Sigma_n) \cong \overline{SP}^n S^k$. We have a space of the form $A * C/B * C \simeq (A/B) * C$. So

$$\overline{SP}^n S^k = S^k * (S_2/\Sigma_n) = S^{k+1} \wedge (S_2/\Sigma_n).$$

We now note that Σ_n plays an illusory role in the above, and that all this works for any subgroup of Σ_n. In particular, the analysis works for Z_p. We therefore get

LEMMA 4.6. $\bar{Z}^p S^k \simeq S^{k+1} \wedge (S_2/Z_p)$.

The advantage of what we've done so far is that we've isolated the diagonal inclusion, which is what is left fixed by Z_p, p prime. This is seen by the above to correspond to the suspension coordinates.

All we have to do now is note that $S_2 = S \cap W_2$ is the set of points $\mathbf{v}$ of $(R^k)^p$ with $\Sigma \mathbf{v}_i = 0$ and $\Sigma \|v_i\|^2 = 1$. Here $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_p)$. In particular, Z_p acts freely on S_2! But S_2 is a $(pk - k - 1)$-sphere. So $S_2 \to S_2/Z_p$ corresponds to $(EZ_p)_{pk-k-1} \to (BZ_p)_{pk-k-1}$. Stably, this gives us a homotopy equivalence and proves 4.1.

Note this technique fails for composite n. We will return to this technique after identifying $\overline{SP}^p\mathbf{S}$. Note that our natural map $Z^p S^k/S^k \to \bar{Z}^p S^k$ may now be seen to induce homology isomorphisms in at least the stable range. Thus we obtain the rest of the proof (see Section 3) that $Z^p\mathbf{S}/\mathbf{S} \simeq_w \bar{Z}^p S$. Also:

COROLLARY 4.7. $\bar{Z}^p S^k$ *is* min $(2k - 1, k + 1)$*-connected*, $k \geq 1$.

Proof of 4.2. The key here is our identification of $\bar{Z}^p\mathbf{S}$ and of the cohomology structures. So all we need is:

LEMMA 4.8. $\bar{Z}^p\mathbf{S} \to \overline{SP}^p\mathbf{S}$ *in cohomology maps the cohomology of* $\overline{SP}^p\mathbf{S}$ *isomorphically onto the summand corresponding to the factor*

$$(\mathbf{S} \wedge S^1 \wedge B\Sigma_p)_{(p)} \quad \textit{of} \quad (\mathbf{S} \wedge S^1 \wedge BZ_p)_{(p)}.$$

Thus the map

$$(\mathbf{S} \wedge S^1 \wedge B\Sigma_p)_{(p)} \to (\mathbf{S} \wedge S^1 \wedge BZ_p)_{(p)} = \bar{Z}^p\mathbf{S}_{(p)} \to \overline{SP}^p\mathbf{S}_{(p)}$$

induces a cohomology, hence homotopy, isomorphism.

(Concerning the word "factor", note the wedge splitting of [16]. One of the factors has the cohomology mod p of $\mathbf{S} \wedge S^1 \wedge B\Sigma_p$.)

Proof of 4.8. We begin by noting that $H^*SP^p\mathbf{S}$ and $H^*Z^p\mathbf{S}$ map onto $H^*\mathbf{S}$, so the long exact sequences in cohomology of $\mathbf{S} \subseteq Z^p\mathbf{S}$ and $\mathbf{S} \subseteq SP^p\mathbf{S}$ break into short exact sequences. So for our purposes it suffices to look at the map $Z^p\mathbf{S} \xrightarrow{f} SP^p\mathbf{S}$ and see what effect f has on integral cohomology. As $H^*(SP^p\mathbf{S}; Z)$ is a direct summand of $H^*(\mathbf{KZ}; Z)$, all elements have the right order, and we may as well look at cohomology with Z_p coefficients. We know the stable zero-dimensional class maps correctly, because the corresponding homotopy classes do: $\iota \in \pi_0\mathbf{S}$ survives in $\pi_0\mathbf{KZ}$. It remains to note that in $H^*(Z^p\mathbf{S}; Z_p)$ the corresponding Steenrod operations are also non-zero on the zero-dimensional cohomology class. This follows from Nakaoka's calculations [25, p. 89, Theorem 13.2].

4.9. *Proof of* 4.4. We first work through the case $n = p^2$. Consider applying 4.5 with W_1 the subspace consisting of all vectors of the form $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \ldots)$. Then Z_{p^2} acts freely on S_2, and $(D_1/S_1)/Z_{p^2}$ is homotopy equivalent to $\bar{Z}^pS^k$. Thus $\bar{Z}^{p^2}S^k \simeq \bar{Z}^pS^k * (BZ_{p^2})_N$, i.e.

$$\bar{Z}^{p^2}\mathbf{S} \simeq S^2 \wedge BZ_p \wedge BZ_{p^2} \wedge \mathbf{S}.$$

An analogous argument works for $\bar{Z}^{p^r}\mathbf{S}$: taking W_1 to be the subspace of $(R^k)^{p^r}$ with

$$\mathbf{v}_i = \mathbf{v}_{p^{r-1}+i} = \mathbf{v}_{2p^{r-1}+i} = \cdots = \mathbf{v}_{(p-1)p^{r-1}+i} \quad \text{for } 1 \le i \le p^{r-1},$$

we get $\bar{Z}^{p^r}\mathbf{S} \simeq \bar{Z}^{p^{r-1}}\mathbf{S} * BZ_{p^r}$. This leads to an obvious inductive argument. Note that the result is unstable, but we duck the question of figuring out what skeleton of BZ_{p^r} we're dealing with.

5. Identifying $\overline{SP}^p\mathbf{S} \to S^1 \wedge \mathbf{S}$

In Section 4 we identified the stable homotopy types of $\bar{Z}^p\mathbf{S}$ and $\overline{SP}^p\mathbf{S}$ as $\mathbf{S} \wedge S^1 \wedge BZ_p$ and $\mathbf{S} \wedge S^1 \wedge B\Sigma_p$ (or rather, its p-primary component). As $\overline{SP}^p\mathbf{S}$ is the cofibre of $\mathbf{S} \to SP^p\mathbf{S}$ (when we localize at p), the Puppe sequence gives us a map

$$\delta\colon \overline{SP}^p\mathbf{S} \to \mathbf{S} \wedge S^1.$$

The Kahn-Priddy map $k\colon \mathbf{S} \wedge B\Sigma_p \to \mathbf{S}$ is also such a map (suspend once). See Section 0. We prove that these maps are stably homotopy equivalent by showing that δ is an isomorphism on π_{2p-3+1} (the $+1$ because of the extra suspension in $\overline{SP}^p\mathbf{S}$). We now indicate how this may be done for the prime $p = 2$; the proof for $p \neq 2$ is completely analogous. We're concerned with maps from $\sigma_1 RP^\infty$ to σ_1, the stable 1-stem. As both are Z_2, it suffices to prove that δ_* is onto. Suppose not. Then $\eta \in \sigma_1$ maps to $\eta' \neq 0$ in $\pi_2 SP^2\mathbf{S} \wedge S^1$, where η generates σ_1. As $\pi_2 \overline{SP^n}\mathbf{S} = 0$ for $n \geq 2$, and as these are all torsion groups, η' must map to a non-zero class in

$$\pi_2 SP^\infty \mathbf{S} \wedge S^1 = H_2 S^1 = 0,$$

contradiction. The point is that the $\overline{SP^n}$ represent what could kill η' in passing from SP^2 successively through the SP^n. Alternate proof: we really need show the map is "the" essential map. But the fact that the Steenrod squares are non-zero on $SP^2 S^k$ suffices.

6. Symmetrization is not always fatal

Our original goal was to prove the following generalization of 0.1 which is equivalent to conjecture 84 of [21]:

Conjecture 6.1. Let h and i be as in 0.1. Then

$$\ker\left(h \wedge \mathrm{id}\colon \pi_* \mathbf{X}_{(p)} \to \pi_* SP^\infty \mathbf{X}_{(p)} = H_* \mathbf{X}_{(p)}\right) = \ker\left(i \wedge \mathrm{id}\colon \pi_* \mathbf{X}_{(p)} \to \pi_* SP^p \mathbf{X}_{(p)}\right).$$

Unfortunately, there are some very simple complexes on which 6.1 fails.

Counterexample 6.2. Let ν represent a generator of ${}_2\pi_3 \mathbf{S} \approx Z_8$. Let

$$T_\nu = \mathbf{S} \cup_\nu C(\mathbf{S} \wedge S^3).$$

We claim that 6.1 fails for $\mathbf{X} = T_\nu$. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathbf{S} \wedge RP^\infty & \longrightarrow & T_\nu \wedge RP^\infty & \longrightarrow & \mathbf{S} \wedge RP^\infty \wedge S^4 & \longrightarrow & \mathbf{S} \wedge RP^\infty \wedge S^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbf{S} & \longrightarrow & T_\nu & \longrightarrow & \mathbf{S} \wedge S^4 & \longrightarrow & \mathbf{S} \wedge S^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & SP^2\mathbf{S} & \longrightarrow & SP^2 T_\nu & \longrightarrow & SP^2\mathbf{S} \wedge S^4 & \longrightarrow & SP^2\mathbf{S} \wedge S^1 & \longrightarrow & \cdots
\end{array}$$

Note that our previous work implies that the rows and columns are cofibration sequences (up to homotopy). Passing to homotopy localized at 2 we get an analogous diagram with exact rows and columns. Consider η generating

$$\pi_5 \mathbf{S} \wedge S^4 \approx \pi_1 \mathbf{S} \approx Z_2.$$

Now $\nu_*(\eta) = \nu \circ \eta = 0$, so there is an η' in $\pi_5 T_\nu$ corresponding to η. Also there is $\hat{\eta}$ generating

$$\pi_5 \mathbf{S} \wedge S^4 \wedge RP^\infty \cong \sigma_1 RP^\infty \cong Z_2$$

which maps to η. Furthermore, $(\nu \wedge \mathrm{id})_* \hat{\eta} = \hat{\eta}\nu$ which is non-zero in

$$\pi_5 \mathbf{S} \wedge S^1 \wedge RP^\infty \cong \sigma_4 RP^\infty \cong Z_2.$$

So if there is an $\alpha \in \pi_5 T_\nu \wedge RP^\infty$ which maps to η', it has to map to $\hat{\eta}$ as well, contradiction. So η' survives into $\pi_5 SP^2(T_\nu)_{(p)}$.

6.3. *The prime* 3. Unfortunately, there is no obvious low-dimensional analog of 6.2 (perhaps the size of the prime is related to the complexity needed in a counterexample?). Rather, consider the following approach. Let α_1 generate ${}_3\pi_3\mathbf{S} = Z_3$. Let α_2 generate ${}_3\pi_7\mathbf{S} = Z_3$. The homotopy class with which we can work a trick similar to the above is the Toda bracket $\langle \alpha_1, \alpha_1, \alpha_2 \rangle$. Namely, α_1 lifts in a stable situation to a map from the mapping cone of α_1 to $\mathbf{S}$. As ${}_3\sigma_{14} = 0$, the above Toda bracket is zero. So we have a composition which is zero, factoring through the cone of α_1. There is an analogous Toda bracket in $B\Sigma_3/B\Sigma_2$'s stable homotopy which is non-zero. Thus we're in a situation like that of 6.2 above: looking at Toda brackets as compositions of lifts of maps, we have a composition zero and its analog non-zero. A diagram-chase finishes the argument off.

Remark. As $\ker(\delta_*: \sigma_*(B\Sigma_p)_{(p)} \to \pi_*\mathbf{S}_{(p)}) \neq 0$ for all p, one expects similar counterexamples to exist for other primes.

We note the following proposition (due to G. W. Whitehead).

PROPOSITION 6.4. *Let k' be as in Section* 0. *If* $\deg \beta < \deg \alpha$, *then* $(k'(\alpha))\beta = k'(\alpha\beta)$.

COROLLARY 6.5. 6.1 *holds for T_γ, the cofibre of $\gamma \in \sigma_*$ where γ is one of the following:*

(1) $n\iota$.

(2) $p = 2$: η, η^2.

(3) $p = 3$: α_1, α_2, β_1, α_3', $\alpha_1\beta_1$.

Proof of 6.5. Substitute γ for v in the diagram of 6.2, fixing the dimensions up. One then needs that $\gamma \circ \delta = 0$ implies $k'(\delta)\gamma = 0$ for δ in the stable homotopy of

$$T_\gamma/\text{cells below top dimension} \simeq \mathbf{S} \wedge S^1 \wedge S^{|\gamma|}.$$

This follows from 6.4 and from checking low dimensional cases (the δ with $|\delta| \leq |\gamma|$) by looking at tables (cf. [3], [30], or your favorite compendium of Adams Spectral Sequence calculations).

Proof of 6.4. Work unstably with the $\Omega^n S^n$, using the fact that these approximate $\Omega^\infty\mathbf{S}$.

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