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POINTS OF SUPPORT FOR CLOSED CONVEX SETS

BY
A. J. LAZAR

A point x_0 of a closed convex subset K of a real Banach space X is called a point of support for K if there is a functional $x^* \in X^*$ such that $x^*(x_0) \leq x^*(x)$ for every $x \in K$ and $x^*(x_0) < x^*(x')$ for some $x' \in K$. S. Rolewicz proved in [4] that every separable K contains a point which is not a point of support for K and asked if every non-separable Banach space must contain a closed convex subset consisting only of points of support. He further asked what is the situation for $L^\infty[0, 1]$. We shall give below a partial answer to the first question and show that $L^\infty[0, 1]$ does indeed contain a subset with the required property.

All the Branch spaces considered are over the real field. The notation and the terminology are those of [1].

THEOREM 1. *A Banach space X whose dual is not weak* separable contains a closed convex subset consisting only of support points.*

Proof. For each countable ordinal α we shall construct, by transfinite induction, elements $x_\alpha \in X$ and functionals $x_\alpha^* \in X^*$ so that $x_\alpha^*(x_\alpha) = 1$ and $x_\alpha^*(x_\beta) = 0$ if $\beta \neq \alpha$. Choose $x_1 \in X$ and $x_1^* \in X^*$ such that $x_1^*(x_1) = 1$. Suppose that α is a countable ordinal and for each ordinal $\beta < \alpha$ we have chosen $x_\beta \in X$, $x_\beta^* \in X^*$ which satisfy $x_\beta^*(x_\beta) = 1$ and $x_\beta^*(x_\gamma) = 0$ if $1 \leq \gamma < \beta$ or $\beta < \gamma < \alpha$. Let

$$X_\alpha = \{x \in X : x_\beta^*(x) = 0 \text{ for every } \beta < \alpha\}.$$

We claim that X_α is a closed non-separable linear subspace of X. Indeed, if X_α were separable, there would be a sequence $\{y_n^*\}_{n=1}^\infty \subset X^*$ which is total over X_α. But then the linear span of $\{x_\beta^*\}_{\beta<\alpha} \cup \{y_n^*\}_{n=1}^\infty$ would be weak* dense in X^*, contrary to the hypothesis. Thus X_α is non-separable and we can choose $x_\alpha \in X_\alpha$ which is not in the closed linear span of $\{x_\beta: \beta < \alpha\}$. By the separation theorem there is $x_\alpha^* \in X^*$ such that $x_\alpha^*(x_\alpha) = 1$ and $x_\alpha^*(x_\beta) = 0$ for $\beta < \alpha$. Hence the existence of the families $\{x_\alpha\} \subset X$, $\{x_\alpha^*\} \subset X^*$ with the desired properties is proved.

Now, let

$$K = \overline{\text{conv}}\, \{x_\alpha: \alpha \text{ a countable ordinal}\}.$$

Clearly $K = \bigcup_\alpha \overline{\text{conv}}\, \{x_\beta: \beta < \alpha\}$ and for each $x \in \overline{\text{conv}}\, \{x_\beta: \beta < \alpha\}$ we can find

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a countable ordinal $\gamma > \alpha$. But then

$$x_\gamma^*(x) = 0 \quad x_\gamma^*(x_\gamma) = 1 \quad \text{and} \quad x_\gamma^*(y) \geq 0 \text{ for each } y \in K.$$

Thus every point of K is a point of support for it.

A Banach space is called weakly compactly generated (WCG) if it is the closed linear span of a weakly compact subset [3].

COROLLARY. *Every non-separable* WCG *Banach space contains a closed convex subset whose every point is a point of support.*

Proof. By [3, Proposition 2.2.], if X is a non-separable WCG then X^* is not weak* separable and the previous result can be applied.

We are going to discuss now the above problem for the spaces of continuous functions and the spaces of integrable functions.

THEOREM 2. *Let S be a compact Hausdorff space satisfying one of the following conditions:*
(i) *S contains a non-G_δ closed subset;*
(ii) *S is non-separable.*
Then $C(S)$ contains a closed convex set consisting only of support points.

Proof. (i) Let Ω be the first uncountable ordinal. For each ordinal $\alpha < \Omega$ we shall produce a point $s_\alpha \in S$ and a function $f_\alpha \in C(S)$ such that $f_\alpha \geq 0, f_\alpha(s_\alpha) = 1$ and $f_\alpha(s_\beta) = 0$ for each β with $\alpha < \beta < \Omega$. Let $F \subset S$ be a non-G_δ closed subset. Choose $s_1 \in S\backslash F$ and $f_1 \in C(S)$ so that $f_1 \geq 0$, $f_1(s_1) = 1$ and $f_1|_F \equiv 0$. Let $1 < \alpha < \Omega$ and suppose that for each β with $1 \leq \beta < \alpha$ points $s_\beta \in S$ and functions $f_\beta \in C(S)$ have been chosen so that, if

$$F_\beta = \{s \in S : f_\beta(s) = 0\},$$

we have $f_\beta \geq 0, f_\beta(s_\beta) = 1$, $F_\beta \supset F$ and $s_\beta \in \bigcap_{1 \leq \gamma < \beta} F_\gamma$ for $\beta > 1$. Then $\bigcap_{\beta < \alpha} F_\beta$ is a closed G_δ containing F. Hence $(\bigcap_{\beta < \alpha} F_\beta)\backslash F \neq \emptyset$. Choose

$$s_\alpha \in \left(\bigcap_{\beta < \alpha} F_\beta\right)\backslash F \quad \text{and} \quad f_\alpha \in C(S)$$

satisfying $f_\alpha \geq 0, f_\alpha(s_\alpha) = 1, f_\alpha|_F \equiv 0$. This establishes the claim made in the first sentence of the proof. Let

$$K = \overline{\text{conv}}\, \{f_\alpha : 1 \leq \alpha < \Omega\}.$$

Again, since $K = \bigcup_{2 \leq \alpha < \Omega} \overline{\text{conv}}\, \{f_\beta : 1 \leq \beta < \alpha\}$, it is easily seen that each point of K is a point of support. The support functionals are this time the Dirac measures corresponding to the points s_α.

(ii) By the first part of the proof we may suppose that each closed subset of S is a G_δ. We claim that for each $\alpha < \Omega$ there are $s_\alpha, t_\alpha \in S$ and $f_\alpha \in C(S)$ such that $0 \leq f_\alpha \leq 1$, $f_\alpha(s_\alpha) < f_\alpha(t_\alpha)$, $f_\beta(s_\alpha) = f_\beta(t_\alpha)$ for $\beta < \alpha$ and $f_\beta(s_\alpha) = f_\beta(t_\alpha) = 0$ if

$\alpha < \beta < \Omega$. Let s_1, t_1 be any two distinct points of S and choose $f_1 \in C(S)$ which satisfies

$$0 \leq f_1 \leq 1, f_1(s_1) < f_1(t_1).$$

Suppose that $\alpha > 1$ is a countable ordinal and for each ordinal $\beta < \alpha$ points s_β, $t_\beta \in S$ and functions $f_\beta \in C(S)$ have been distinguished subject to the following conditions: $0 \leq f_\beta \leq 1$, $f_\beta(s_\beta) < f_\beta(t_\beta)$, $f_\gamma(s_\beta) = f_\gamma(t_\beta)$ if $\gamma < \beta$ and $f_\gamma(s_\beta) = f_\gamma(t_\beta) = 0$ if $\beta < \gamma < \alpha$. Let $S_\alpha = \overline{\bigcup_{\beta<\alpha} \{s_\beta, t_\beta\}}$. We claim that there are s_α, $t_\alpha \in S \backslash S_\alpha$ such that $s_\alpha \neq t_\alpha$ but $f_\beta(s_\alpha) = f_\beta(t_\alpha)$ for every $\beta < \alpha$. Assume, to the contrary, that the map of $S \backslash S_\alpha$ into $[0, 1]^{\aleph_0}$ given by $s \to (f_\beta(s))_{\beta<\alpha}$ is one-to-one (if α is a finite ordinal then $\aleph_0$ should be changed with a suitable finite cardinal). Since $S \backslash S_\alpha$ is open, it is a countable union of compact subsets. On each of these the above map is a homeomorphism hence $S \backslash S_\alpha$ is separable, a contradiction. Thus, there are s_α, t_α as needed. Choose now $f_\alpha \in C(S)$ such that $0 \leq f_\alpha \leq 1$, $f_\alpha(s_\alpha) < f_\alpha(t_\alpha)$, $f_\alpha | S_\alpha \equiv 0$. The transfinite induction argument is now complete.

Let $K = \overline{\text{conv}} \{f_\alpha: 1 \leq \alpha < \Omega\} = \bigcup_{1<\alpha<\Omega} \overline{\text{conv}} \{f_\beta: 1 \leq \beta < \alpha\}$ and suppose $f \in \overline{\text{conv}} \{f_\beta: 1 \leq \beta < \alpha\}$. The functional $\delta \in C(S)^*$ defined by $\delta(g) = g(t_\alpha) - g(s_\alpha)$, $g \in C(S)$ satisfies $\delta(f) = 0$, $\delta(f_\alpha) > 0$ and $\delta(h) \geq 0$ for each $h \in K$. This shows that f is a point of support for K.

The maximal ideal space of $L^\infty[0, 1]$ has no G_δ points by [5, Corollary 4.13, Corollary 4.10] therefore satisfies the condition (i) of Theorem 2. This solves Rolewicz's question for $L^\infty[0, 1]$. Also, the maximal ideal space of l^∞ has non-G_δ points [2, 9.6] thus l^∞ too contains a closed convex subset consisting only of support points.

The proof of the following lemma is immediate so we omit it.

LEMMA. *Let Γ be an uncountable set. Every point of*

$$K = \{x \in l^1(\Gamma): x(\gamma) \geq 0, \gamma \in \Gamma\}$$

is a point of support for K in $l^1(\Gamma)$.

THEOREM 3. *Let (S, Σ, μ) be a measure space such that $L^1(S, \Sigma, \mu)$ is non-separable. Then $L^1(S, \Sigma, \mu)$ contains a closed convex subset consisting only of support points.*

Proof. If μ is a σ-finite measure then $L^1(S, \Sigma, \mu)$ is WCG [3, p. 240]. Otherwise, it is easily seen that $L^1(S, \Sigma, \mu)$ contains a subspace isometric to $l^1(\Gamma)$ for some uncountable set Γ and the previous lemma can be used.

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Montana State University
Bozeman, Montana