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ON SETS CHARACTERIZING ADDITIVE AND MULTIPLICATIVE ARITHMETICAL FUNCTIONS

BY

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1. Introduction

A function f: $\mathbf{N} \to \mathbf{C}$ is called *additive* if

$$(1) \qquad f(mn) = f(m) + f(n)$$

for all coprime $m, n \in \mathbf{N}$. If (1) holds for all pairs of integers $m, n \in \mathbf{N}$, we say that f is *completely additive*. A function g: $\mathbf{N} \to \mathbf{C}$ is called *multiplicative* (resp. *completely multiplicative*) if

$$(1') \qquad g(mn) = g(m)g(n)$$

for all coprime $m, n \in \mathbf{N}$ (resp. for all $m, n \in \mathbf{N}$).

Because of the canonical representation

$$(2) \qquad n = \prod_{p \text{ prime}} p^{\alpha_p} \quad \text{with} \quad p^{\alpha_p} \| n$$

of the integers $n \in \mathbf{N}$ we have $f(n) = \sum_{p \text{ prime}} f(p^{\alpha_p})$ (resp. $g(n) = \prod_{p \text{ prime}} g(p^{\alpha_p})$). An additive f can be extended uniquely to an "additive" function f^*: $\mathbf{Q}^+ \to \mathbf{C}$, where $\mathbf{Q}^+ = \{a/b\colon (a, b) = 1;\ a, b \in \mathbf{N}\}$, by $f^*(a/b) = f(a) - f(b)$. In a similar manner we get an extension g^* of a multiplicative function g by $g^*(a/b) = g(a)/g(b)$ in case $g(b) \neq 0$ for all $b \in \mathbf{N}$. In the following we denote by $\mathfrak{A}$ the set of all additive f: $\mathbf{Q}^+ \to \mathbf{C}$ and by $\mathfrak{M}$ the set of all multiplicative g: $\mathbf{Q}^+ \to \mathbf{C}$ with $g(b) \neq 0$ for all $b \in \mathbf{N}$. We write $\mathfrak{A}_c$ (resp. $\mathfrak{M}_c$) for the subsets of completely additive (resp. completely multiplicative) functions in $\mathfrak{A}$ (resp. $\mathfrak{M}$).

DEFINITIONS. Let $\mathscr{A} = \{a_n\} \subset \mathbf{Q}^+$. We say that $\mathscr{A}$ is a

(a) *U-set* for $\mathfrak{A}$ in case $f \in \mathfrak{A}$, $f(\mathscr{A}) = \{0\}$ implies $f = 0$,
(b) *U-set* for $\mathfrak{M}$ in case $g \in \mathfrak{M}$, $g(\mathscr{A}) = \{1\}$ implies $g = 1$,
(c) *C-set* for $\mathfrak{A}$ in case $f \in \mathfrak{A}$, $\lim_{n\to\infty} f(a_n) = 0$ implies $f = 0$,
(d) *C-set* for $\mathfrak{M}$ in case $g \in \mathfrak{M}$, $\lim_{n\to\infty} g(a_n) = 1$ implies $g = 1$.

In an obvious manner U-sets and C-sets are defined for $\mathfrak{A}_c$ (resp. $\mathfrak{M}_c$).

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Examples. G. Freud [5] gave examples of U-sets and C-sets $\mathscr{A} \subset \mathbf{N}$ for $\mathfrak{A}$ (resp. $\mathfrak{A}_c$) characterized by density properties. P. D. T. A. Elliott [3] showed that the set of $\{p+1: p \text{ prime}\}$ is a U-set for $\mathfrak{A}$. K.-H. Indlekofer [6] investigated the family of sets $\mathscr{A} = \{a_n\} \subset \mathbf{N}$ defined by the following conditions:

(i) $a_n \ll n$, $n \in \mathbf{N}$;

(ii) $\sum_{n, a_n = k} 1 = O(1)$ for all $k \in \mathbf{N}$.

(iii) $\sum_{n \le x, a_n \equiv 0(d)} 1 = x\rho(d)/d + o(x)$ for all $d \in \mathbf{N}$, where $\rho \ge 0$ is multiplicative and $o(\cdot)$ depends only on d and ρ.

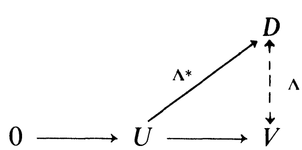
A special example of these results is the following: *If $\mathscr{A} = \{a_n\}$ fulfills the conditions* (i)–(iii) *and if the set $\{d: \rho(d) = 0\}$ is empty, then $\mathscr{A}$ is a C-set for $\mathfrak{A}_c$.*

2. Results

The aim of this paper is to handle U-sets and C-sets from a different point of view. For this purpose we remind the reader of two well-known facts of linear algebra and group theory.

(I) *Let U be a subspace of the $\mathbf{Q}$-vector space V. Then $U \ne V$ if and only if there exists a linear functional $\Lambda: V \to \mathbf{Q}$, $\Lambda \ne 0$, with $\Lambda U = \{0\}$.*

(II) (See [7, p. 183]) *Let U be a subgroup of the abelian group V and let $\Lambda^*: U \to D$ be a homomorphism, where D is divisible (i.e., for each $x \in D$ and for every $n \in \mathbf{N}$ there exists a $y \in D$ with $ny = x$). Then Λ^* can be extended to a homomorphism $\Lambda: V \to D$, i.e., a Λ exists making the following diagram commute*:



An easy (and well-known) consequence of (II) is the following.

(II′) *Let U be a subgroup of the abelian group V and let*

$$\mathbf{C}^* := \{z \in \mathbf{C}: |z| = 1\}$$

denote the (multiplicative) circle group. Then $U \ne V$ if and only if there exists a homomorphism $\Lambda: V \to \mathbf{C}^*$, $\Lambda \ne 1$, with $\Lambda U = \{1\}$.

Proof. Let $U \ne V$ and let $\pi: V \to V/U$ be the natural homomorphism. Then $V/U \ne \{0\}$ and by (II) there exists a homomorphism $\Lambda^*: V/U \to \mathbf{C}^*$, $\Lambda^* \ne 1$. Thus the homomorphism $\Lambda = \Lambda^* \circ \pi: V \to \mathbf{C}^*$ has the desired properties. The proof for the other direction is obvious.

For each $q \in \mathbf{Q}^+$ we have the "canonical" representation $q = \prod_{i=1, p_i \text{ prime}}^{l} p_i^{\alpha_i}$ with $\alpha_i \in \mathbf{Z}$. The mapping $q \mapsto (\alpha_1, \ldots, \alpha_l, 0, \ldots)$ provides an isomorphism between the multiplicative group $\mathbf{Q}^+$ and the free (additive) abelian group $V = \sum_{i=1}^{\infty \oplus} \mathbf{Z}_i$ with $\mathbf{Z}_i = \mathbf{Z}$. Then, to the subset $\mathscr{A} \subset \mathbf{Q}^+$ there corresponds a subgroup $U \lhd V$. On the other hand the set V generates the **Q**-vector space $V^* = \sum^{\oplus} \mathbf{Q}$ and the set U generates the subspace U^*. Now, from these facts, (I) and (II′) we deduce the following.[2]

THEOREM 1. *Let* $\mathscr{A} = \{a_n\} \subset \mathbf{Q}^+$. *Then the following two assertions are equivalent:*

(1) $\mathscr{A}$ *is a U-set for* $\mathfrak{A}_c$.

(2) *For each* $n \in \mathbf{N}$ *there exist* $\alpha_1, \ldots, \alpha_k \in \mathbf{Q}$ *and* $n_1, \ldots, n_k \in \mathbf{N}$ *such that*

$$n = \prod_{i=1}^{k} a_{n_i}^{\alpha_i}. \tag{3}$$

THEOREM 2. *Let* $\mathscr{A} = \{a_n\} \subset \mathbf{Q}^+$. *Then the following two assertions are equivalent:*

(1) $\mathscr{A}$ *is a U-set for* $\mathfrak{M}_c$.

(2) *For each* $n \in \mathbf{N}$ *there exist* $\alpha_1, \ldots, \alpha_k \in \mathbf{Z}$ *and* $n_1, \ldots, n_k \in \mathbf{N}$ *such that*

$$n = \prod_{i=1}^{k} a_{n_i}^{\alpha_i}. \tag{4}$$

Remark 1. F. Dress and B. Volkmann [2] give a different proof of Theorem 1. Furthermore, they state the following result (corollary in [2]). *Let* $f, g \in \mathfrak{M}_c$ *and* $\mathscr{A} = \{a_n\} \subset \mathbf{N}$. *Then the following two assertions are equivalent:*

(i) *If* $f(a_n) = g(a_n)$ *for all* $n \in \mathbf{N}$ *then* $f = g$.

(ii) *For each prime p there exists a natural number* $\alpha \geq 1$ *such that* $p^\alpha \in \mathscr{A}$ *and p has a representation* (4).

This result is not correct because of the following:

Example. Let p_0, p_1 be two different primes. Let $\mathscr{A} = \{p_0^2, p_0 p_1\} \cup \mathbf{P}\backslash\{p_0\}$ and define two functions $f, g \in \mathfrak{M}_c$ by

$$f(p_0) = -g(p_0) = 1, \qquad f(p) = g(p) = 0 \text{ if } p \neq p_0.$$

Then (ii) holds but (i) is not valid.

By a slight modification of the arguments used in [2] it is possible to give a different proof of Theorem 2.

DEFINITION. Let

$$n = \prod_{i=1}^{k} a_{n_i}^{\alpha_i} = \prod_{i=1}^{k'} a_{n'_i}^{\alpha'_i}$$

be two representations of n in (3) (resp. (4)). We say the two representations are *different* in case $a_{n_i} \neq a_{n'_j}$ for all $i = 1, \ldots, k$, $j = 1, \ldots, k'$.

COROLLARY 1. *Let* $\mathscr{A}$ *be a C-set for* $\mathfrak{A}_c$ *(resp.* $\mathfrak{M}_c$*). Then there exist infinitely many pairwise different representations* (3) *(resp.* (4)*).*

[2] The author proved Theorem 1 in talks given in Ulm, Germany (1976), and in Oberwolfach, Germany (November 1977).

Proof. If $\{a_n\}$ is a C-set for $\mathfrak{A}_c$ (resp. $\mathfrak{M}_c$) then the same holds for $\{a_{n+n_k}\}$ for each $n_k \in \mathbf{N}$. Therefore Corollary 1 is valid.

A result in the other direction is:

COROLLARY 2. *Assume that, for each $n \in \mathbf{N}$, there exist infinitely many pairwise different representations* (4) *having $\sum_{i=1}^{k} |\alpha_i| = O(1)$. Then $\mathscr{A}$ is a C-set for $\mathfrak{A}_c$ and $\mathfrak{M}_c$.*

Proof. Let $f \in \mathfrak{A}_c$. Then $|f(n)| \le \sum_{i=1}^{k} |f(a_{n_i})|\, |\alpha_i|$ and the assertion of Corollary 2 is obvious.

Remark 2. There is sometimes another way of checking that a given U-set is also a C-set. Let p be a prime and $v \in \mathbf{N}$. If $p^v = \prod_{i=1}^{k} a_{n_i}^{\alpha_i}$, $\alpha_i \in \mathbf{Z}$, and $f \in \mathfrak{A}_c$, then $vf(p) = \sum_{i=1}^{k} \alpha_i f(a_{n_i})$. Now, if the right side is $o(v)$ as $v \to \infty$, then of course $f(p) = 0$.

3. Examples and applications

(1) Let $a_n = [\alpha n]$, where $\alpha > 1$ is irrational. Furthermore, let $q \in \mathbf{N}$ and $0 < \varepsilon < q^{-1}$. Then there exists a sequence $\{n_l\}$ of natural numbers n_l such that $[\alpha n_l] < \alpha n_l < [\alpha n_l] + \varepsilon$. Hence

$$q[\alpha n_l] < \alpha q n_l < q[\alpha n_l] + \varepsilon q < q[\alpha n_l] + 1$$

and so $q = [\alpha q n_l]/[\alpha n_l]$. Now (4) holds with $k = 2$, $\alpha_i \in \{-1, 1\}$. Thus $[\alpha n]$ is a C-set for $\mathfrak{M}_c$ (and $\mathfrak{A}_c$).[3]

(2) Let $a_n = (n+1)/n$. Then, for each $n \in \mathbf{N}$,

$$n = \frac{n!}{(n-1)!} = \frac{n}{n-1} \cdot \frac{n-1}{n-2} \cdot \dots \cdot \frac{2}{1},$$

i.e. $\{a_n\}$ is a U-set for $\mathfrak{M}_c$ and $\mathfrak{A}_c$. Because of $n = n^{l+1}/n^l$ there exist infinitely many different representations (4), and we ask the question whether $\{a_n\}$ is a C-set for $\mathfrak{M}_c$ (resp. $\mathfrak{A}_c$). The answer will be "no".

Indeed, let $f \in \mathfrak{A}_c$ and $f(a_n) = o(1)$ as $n \to \infty$. Then, for a given prime $p \ge 3$ and $v \in \mathbf{N}$, we have the dyadic expansion

$$p^v = 2^{\mu_k} + \cdots + 2^{\mu_1} + 1. \tag{5}$$

Hence

$$\begin{aligned} vf(p) &= f(p^v) = f(p^v) - f(p^v - 1) + \mu_1 f(2) + f(2^{\mu_k - \mu_1} + \cdots + 1) \\ &\ \vdots \\ &= \mu_k f(2) + \sum_{l=0}^{k} \Delta_l f, \end{aligned}$$

[3] $\{[\alpha n]\}$ is also a C-set for $\mathfrak{A}$ (see [6]).

where $k + 1$ is the length of the dyadic expansion (5) and $\Delta_l f$ denotes the difference

$$f(2^{\mu_k - \mu_l} + \cdots + 1) - f(2^{\mu_k - \mu_l} + \cdots + 2^{\mu_{l+1} - \mu_l}) \quad (\mu_0 = 0).$$

Because of $2^{\mu_k} < p^v < 2^{\mu_k + 1}$ we get

$$vf(p) = \mu_k f(2) + o(\mu_k) = v \log p \frac{f(2)}{\log 2} + o(v).$$

Now, dividing by $v \log p$, we obtain

$$f(p) = \frac{f(2)}{\log 2} \cdot \log p \quad \text{for all primes } p. \tag{6}$$

Thus $\{a_n\}$ is not a C-set for $\mathfrak{A}_c$ (similarly for $\mathfrak{M}_c$) provided that $f(2) \neq 0$.[4]

(3) Let $a_n = (an + 1)/n$, where a is an integer > 1. Then we establish the following:

LEMMA. *Let $a_n = (an + 1)/n$. Then, for each $j \in \mathbf{N}$,*

$$\frac{j+1}{j} = \prod_{i=1}^{k} \left(\frac{an_i + 1}{n_i} \right)^{\alpha_i}, \tag{7}$$

where $\alpha_i \in \{-1, 1\}$, $\sum_{i=1}^{k} \alpha_i = 0$, $k = k(j) = O(4^{a-1})$ and $n_i = n_i(j) = O_a(j^{6^{a-1}})$. (The constant in $O_a(\cdot)$ depends only on (a)*).*

Proof. Because of the identity

$$\frac{q^3 + 1}{q^3} \cdot \frac{q}{q+1} \cdot \frac{q(q-1)}{q(q-1)+1} = \frac{q-1}{q}$$

we have, for all $m \in \mathbf{N}$,

$$\frac{am - 1}{am} = \frac{aa^2m^3 + 1}{aa^2m^3} \cdot \frac{am}{am+1} \cdot \frac{am(am-1)}{am(am-1)+1}, \tag{8}$$

i.e. $(am - 1)/am$ is a product of numbers $(al + 1)/al$. On the other hand, putting $n = (a-1)m - 1$,

$$\frac{an+1}{an} = \frac{a(a-1)m - (a-1)}{a((a-1)m - 1)} = \frac{am-1}{am} \cdot \frac{(a-1)m}{(a-1)m - 1}. \tag{9}$$

Thus, by (8) and (9), $((a-1)m - 1)/(a-1)m$ is a product of numbers $(al + 1)/al$.

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[4] P. Erdös [4] proved that (6) holds if $f \in \mathfrak{A}$ and $f(n+1) - f(n) \to 0$; see also A. S. Besicovich [1].

Observing that

$$\frac{b^2m^2 - 1}{b^2m^2} \cdot \frac{bm}{bm - 1} = \frac{bm + 1}{bm} \tag{10}$$

we conclude ($b = a - 1$) that $((a-1)m + 1)/(a-1)m$ is expressible as a product of numbers of $\{a_n\}$ for all $m \in \mathbf{N}$. Repeating these arguments, we obtain assertion (7) of the lemma. The rest of the lemma follows from (8), (9) and (10).

The first consequence of the lemma is that $\{a_n\}$ is a U-set for $\mathfrak{A}_c$ and $\mathfrak{M}_c$. A second consequence is that $\{a_n\}$ is also a C-set for $\mathfrak{A}_c$ and $\mathfrak{M}_c$. To prove this let $f \in \mathfrak{A}_c$. Then, by Example (2), $f(n) = c \log n$, but c is zero because $c \log (an + 1) - c \log n \sim c \log a = 0$. Similarly the assertion for $\mathfrak{M}_c$ is proved.

Remark 3. If $a_n = (an + b)/n$ with $a \in \mathbf{N}$, $b \in \mathbf{Z}$, then, putting $n = |b|m$, we obtain

$$\frac{an + b}{n} = \frac{|b|(am + b/|b|)}{|b|m} = \frac{am + sgn(b)}{m}$$

and because of (10) we conclude that the subsequence $\{a_{|b|m}\}$ (and therefore the whole sequence $\{a_n\}$) is a C-set for $\mathfrak{A}_c$ and $\mathfrak{M}_c$ if $a > 1$.

Remark 4. Let $f \in \mathfrak{A}_c$ and let $f(an + 1) - f(n) = o(\log n)$ as $n \to \infty$. Then, by the Lemma, $f(j + 1) - f(j) = o(\log j)$ and, using a deep new result by E. Wirsing [8], $f(n) = c \log n$.

Remark 5. Let us generalize the concept of C-sets for $\mathfrak{A}$ in the following definition: $\mathscr{A} = \{a_n\}$ is called a Σ-*set for* $\mathfrak{A}$ in case $f \in \mathfrak{A}$, $\sum_{n \le x} |f(a_n)| = o(x)$ as $x \to \infty$ implies $f = 0$.

Now we prove the following:

THEOREM 3. *Let* $\mathscr{A} = \{a_n\}$ *fulfill* (i), (ii) *and* (iii) *of Section* 1 *with* $\rho = 1$. *Then* $\mathscr{A}$ *is a* Σ-*set for* $\mathfrak{A}$.

Proof. We prove a little bit more than the assertion of Theorem 3. Let us assume that $f \in \mathfrak{A}$ and that $\sum_{n \le x} |f(a_n) - c| = o(x)$ holds with a certain constant $c \in \mathbf{C}$. We choose a sequence $x_1 < x_2 < \cdots \infty$, such that $\sum_{n \le x} |f(a_n) - c| \le 4^{-m} x$ for $x > x_m$. If we define a function h: $\mathbf{N} \to \mathbf{R}^+$ by

$$h(n) = \begin{cases} 1 & \text{for } n \in [1, x_1), \\ 2^{-m} & \text{for } n \in [x_m, x_{m+1}) \end{cases},$$

we get

$$\sum_{\substack{n \le x, \\ |f(a_n) - c| > h(n)}} 1 < \sum_{n \le x} |f(a_n) - c|/h(n) \le 2^{-m} x$$

for $x \in [x_m, x_{m+1})$. Now, we omit from $\mathscr{A}$ those a_n for which $|f(a_n) - c| > h(n)$ and obtain a new sequence $\{a'_n\}$. It is easily verified that $\{a'_n\}$ fulfills (i), (i) and (iii). By the fact that $\lim_{n\to\infty} f(a'_n) = c$ we conclude (see K.-H. Indlekofer [6]) that $f = 0$ (and thus c must be zero too).

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