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ALEXANDER MODULES OF SUBLINKS AND AN INVARIANT OF CLASSICAL LINK CONCORDANCE

BY

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Introduction

A link in S^3 is an ordered collection $L = \{K_1, \ldots, K_m\}$ of smooth, oriented, pair-wise disjoint knotted circles in the 3-dimensional sphere. To any link, one may associate the *complement* $X = S^3 - \bigcup_{i=1}^{m} N(K_i)$, where $N(K_i)$ denotes an open tubular neighborhood of K_i in S^3. The *Alexander modules* of L are the homology groups of the universal abelian cover $\tilde{X}$ of X, viewed as modules over the integral group ring Λ of the group of covering transformations. By Alexander duality, this group is the free abelian group on m generators, so we may identify Λ with the Laurent polynomial ring

$$\mathbf{Z}[x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}].$$

We begin by studying the relation between the Alexander modules of a link

$$L = \{K_1, \ldots, K_m\}$$

in S^3 and those of a sublink $L' = \{K_2, \ldots, K_m\}$. The first result is the discovery of certain short exact sequences which express this relationship (see Theorem 1.1). An interesting corollary of this result is a new proof of a well-known formula of Torres [12] which related the Alexander polynomial of L to that of L', one which does not use the free differential calculus. The proof of the formula turns out to carry other important information as well. Investigation of a certain map (which is always zero in the cases of interest for the proof of the formula) leads to the discovery of a new link invariant $I_1(L)$ which detects non-boundary-linking. In many instances this invariant is quite easy to compute; in fact, in the examples in Section 3, it is far easier to compute than the Alexander polynomial. The proof of the fact that $I_1(L)$ vanishes for boundary links (Theorem 3.1) indicates that $I_1(L)$ is related to a certain rank invariant $r(L)$ which we define in Section 4. This invariant turns out to be an invariant of link concordance (Theorem 4.4). An application of $r(L)$ to the Whitehead link shows that it is not concordant to a boundary link, and therefore not a slice link. Finally, we show how $I_1(L)$ and $r(L)$ are related, and note that this implies that the examples of Section 3 are not concordant to boundary links.

The results of the first two sections are revisions of a chapter of my Ph.D. thesis, Brandeis University 1978. I wish to thank my advisor, Jerome Levine,

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for his guidance. The rank invariant and Theorem 4.4 were discovered independently and earlier by Kawauchi [4]. I would also like to thank Jonathan Hillman, who pointed out a correction to the original manuscript, and the referee, who made numerous helpful suggestions.

1. The Alexander modules of a sublink

Let $L = \{K_1, \ldots, K_m\}$ be a link in S^3, and let $L' = \{K_2, \ldots, K_m\}$ be the sublink of $m-1$ components missing K_1. We would like to relate the Alexander modules of L' to those of L. The Alexander modules of L are modules over

$$\Lambda = \mathbf{Z}[x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}],$$

while those of L' are modules over the ring

$$R = \mathbf{Z}[x_2, \ldots, x_m, x_2^{-1}, \ldots, x_m^{-1}],$$

and this fact must be taken into account. If Y is the complement of L' and $\tilde{Y}$ is its universal abelian cover, we would be looking for a map $\tilde{X} \to \tilde{Y}$ which reflects this algebraic information. The easiest way is to interpose a certain space $\hat{X}$, to obtain maps $\tilde{X} \to \hat{X} \to \tilde{Y}$.

The space $\hat{X}$ we want turns out to be a certain covering space of X. Specifically, it is the covering space associated to the composition

$$\pi_1 X \to H_1 X \to H_1 Y$$

where the first map is abelianization and the second is induced by inclusion. It is not difficult to see that $\tilde{X} \to \hat{X}$ is an infinite cyclic cover, and that $\hat{X}$ is a subspace of $\tilde{Y}$ (in fact, $\hat{X}$ is obtained from $\tilde{Y}$ by removing all the pre-images of $N(K_1)$ in $\tilde{Y}$). The homology groups of $\tilde{X}$ are naturally modules over R.

We regard R as a Λ-module via the isomorphism $R \cong \Lambda/(x_1 - 1)$.

THEOREM 1.1. *There are exact sequences of R-modules*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & C \arrow[d] & & \\
0 \arrow[r] & R \otimes_\Lambda H_1\tilde{X} \arrow[r] & H_1\hat{X} \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] & 0 \\
 & & H_1\tilde{Y} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where $C \cong \operatorname{cok} j_*: H_2\tilde{Y} \to H_2(\tilde{Y}, \hat{X})$ in the exact sequence of the pair $(\tilde{Y}, \tilde{X})$, which is isomorphic to $R/(x_2^{l_2}x_3^{l_3} \cdots x_m^{l_m} - 1)$ if $H_2\tilde{Y} \cong 0$. Here, l_1 denotes the linking number of K_1 with K_i, and $\mathbf{Z}$ is regarded as an R-module with trivial action, isomorphic to

$$R/(x_2 - 1, \ldots, x_m - 1).$$

Remarks. (1) $H_2 \tilde{Y} \not\cong 0$ exactly when $\Delta(x_2, \ldots, x_m) = 0$; see Proposition 2.5.

(2) The theorem generalizes, with the same method of proof, to give the relation between the Alexander modules of a link of n-spheres in S^{n+2} and those of a sublink.

Proof of 1.1. Since $\tilde{X} \to \hat{X}$ is an infinite cyclic cover, as in [6] we have an exact sequence of chain complexes over Λ and R:

$$0 \longrightarrow C_*\tilde{X} \xrightarrow[x_1 - 1]{} C_*\tilde{X} \longrightarrow C_*\hat{X} \longrightarrow 0.$$

This yields a long exact homology sequence in the standard fashion. The tail end of the homology sequence is as follows:

$$\cdots \longrightarrow H_1\tilde{X} \xrightarrow[x_1 - 1]{} H_1\tilde{X} \longrightarrow H_1\hat{X} \longrightarrow$$
$$H_0\tilde{X} \xrightarrow[x_1 - 1]{} H_0\tilde{X} \longrightarrow \cdots$$

We identify $H_1\tilde{X}/(x_1 - 1)H_1\tilde{X}$ with $R \otimes_\Lambda H_1\tilde{X}$, and we note that x_1 acts as the identity on $H_0\tilde{X} \cong \mathbf{Z}$, so that we obtain the horizontal exact sequence in the statement of the theorem. The vertical exact sequence is a consequence of the long exact homology sequence of the pair $(\tilde{Y}, \hat{X})$. It remains only to calculate $H_*(\tilde{Y}, \hat{X})$.

If $l_2 = l_3 = \cdots = l_m = 0$, then by excision,

$$H_*(\tilde{Y}, \hat{X}) \cong R \otimes_{\mathbf{Z}} H_*(S^1 \times D^2, S^1 \times S^1).$$

Thus, by the Künneth formula, $H_*(\tilde{Y}, \hat{X}) \cong R$ if $* = 2$ or 3, and $H_*(\tilde{Y}, \hat{X}) \cong 0$ otherwise.

If some $l_i \neq 0$, we have $H_*(\tilde{Y}, \hat{X}) \cong \oplus H_*(D^2, S^1)$, since the pre-images of $\overline{N(K_1)}$ will be copies of its simply-connected cover $D^2 \times \mathbf{R} \simeq D^2$. The image of $\pi_1[N(K_1)]$ in $H_1 Y$ is the infinite cyclic group generated by $x_2^{l_2} \cdots x_m^{l_m}$, so

$$H_2(\tilde{Y}, \hat{X}) \cong R/(x_2^{l_2} \cdots x_m^{l_m} - 1)$$

and $H_*(\tilde{Y}, \hat{X}) \cong 0$ for $* \neq 2$.

This computation completes the proof of Theorem 1.1.

2. A formula of Torres

In 1953, Torres [12] proved, among other things, that the Alexander polynomial of a link $L = \{K_1, \ldots, K_m\}$ in S^3 is related via a formula to the Alexander polynomial of $L' = \{K_2, \ldots, K_m\}$. His proof involved a careful study of the matrix produced by the free differential calculus of Fox from the Wirtinger presentation of $\pi_1 X$. We will show how this formula can be derived from Theorem 1.1.

The Alexander polynomial is an element of Λ, defined only up to multiplication by units of Λ, and is an invariant of the Λ-module $H_1 \tilde{X}$ called an elementary divisor. These are defined for any finitely generated Λ-module M as follows. Choose a presentation

$$\Lambda^r \xrightarrow[d]{} \Lambda^s \to M \to 0$$

with $r \geq s$, and represent d by a matrix D with entries in Λ. For $i = 0, 1, \ldots, s$ let $U_i(d)$ be the ideal of $i \times i$ minor determinants of D, and let $\Delta_j(M)$ be a generator of the smallest principal ideal of Λ containing $U_{s-j}(d)$. The $\Delta_j(M)$ depend only on M and are the elementary divisors. The Alexander polynomial of L is $\Delta_0(H_1 \tilde{X})$.

Let l_i, $i = 2, 3, \ldots, m$ be the linking number of K_1 and K_i. If $\Delta(x_1, \ldots, x_m)$ is the Alexander polynomial of L and $\Delta(x_2, \ldots, x_m)$ is the Alexander polynomial of L', then the result of Torres is the following.

THEOREM 2.1 (Torres, [12, Theorem 3]). (i) *If* $m = 2$,

$$\Delta(1, x_2) = \frac{x_2^{l_2} - 1}{x_2 - 1} \Delta(x_2).$$

(ii) *If* $m > 2$,

$$\Delta(1, x_2, \ldots, x_m) = (x_2^{l_2} x_3^{l_3} \cdots x_m^{l_m} - 1) \Delta(x_2, \ldots, x_m).$$

We will show that this result follows from Theorem 1.1. The main tool is the following technical result of Levine.

LEMMA 2.2 (Levine. [5, Lemma 5]). *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of finitely generated Λ*-modules. Then*

$$\Delta_0(B) = \Delta_0(A) \cdot \Delta_0(C).$$

Applying this in turn to the horizontal and vertical exact sequences in Theorem 1.1 we obtain

$$\Delta_0(R \otimes_\Lambda H_1 \tilde{X})\, \Delta_0[R/(x_2 - 1, \ldots, x_m - 1)] = \Delta_0(C)\, \Delta_0(H_1 \tilde{Y}).$$

Now, it is easy to see that

$$\Delta_0(H_1 \tilde{Y}) = \Delta(x_2, \ldots, x_m)$$

and that $\Delta_0[R/(x_2 - 1, \ldots, x_m - 1)] = 1$ unless $m = 2$, in which case $\Delta_0[R/(x_2 - 1)] = x_2 - 1$. Therefore, we have

$$\Delta_0(R \otimes_\Lambda H_1 \tilde{X}) = \frac{\Delta_0(C)}{x_2 - 1} \Delta(x_2, \ldots, x_m) \quad \text{if } m = 2,$$

and

$$\Delta_0(R \otimes_\Lambda H_1 \tilde{X}) = \Delta_0(C)\,\Delta(x_2, \ldots, x_m) \quad \text{if } m > 2.$$

The theorem will follow from the next two lemmas.

LEMMA 2.3. $\Delta_0(R \otimes_\Lambda H_1 \tilde{X}) = \Delta(1, x_2, \ldots, x_m)$.

Proof of Lemma 2.3. We begin by fixing a resolution for $H_1 \tilde{X}$ as a Λ-module:

$$\Lambda^s \xrightarrow[d]{} \Lambda^r \to H_1 \tilde{X} \to 0.$$

Since a resolution for $R \otimes_\Lambda H_1 \tilde{X}$ is obtained by tensoring the above with R, we see that $U_r(1 \otimes R)$ is the image of $U_r(d)$ under the map $\Lambda \to R$ which sends x_1 to 1. But a result of Crowell and Strauss [2] states that

$$U_r(d) = \Delta(x_1, \ldots, x_m) \cdot (x_1 - 1, \ldots, x_m - 1)^p$$

where p depends only on m. Hence, $\Delta_0(R \otimes_\Lambda H_1 \tilde{X}) = \Delta(1, x_2, \ldots, x_m)$.

LEMMA 2.4. *If* $\Delta(x_2, \ldots, x_m) \neq 0$, *then* $\Delta_0(C) = x_2^{l_2} \cdots x_m^{l_m} - 1$.

Lemma 2.4 will follow from the description of C given in Theorem 1.1 and the next proposition. If M is any R-module, the rank of M is the dimension of the vector space $R_0 \otimes_R M$, where R_0 is the quotient field of R.

PROPOSITION 2.5. *Let $\tilde{Y}$ be the universal abelian cover of the complement Y of a link of circles in S^3. Then $H_1 \tilde{Y}$ has the same rank as $H_2 \tilde{Y}$. In particular, since $H_2 \tilde{Y}$ is R-torsion-free, if $H_1 \tilde{Y}$ is R-torsion* (i.e. if $\Delta(x_2, \ldots, x_m) \neq 0$) *then* $H_2 \tilde{Y} \cong 0$.

Remark. I am indebted to the referee for the information that, for unsplittable links, part of this proposition is contained in the work of Cochran and Crowell [1].

Proof. As in [7], Y collapses to a finite 2-complex K. Since $H_2 \tilde{Y} \cong H_2 \tilde{K}$ which is a submodule of the free R-module $C_2 \tilde{K}$, $H_2 \tilde{Y}$ must be R-torsion-free. The Euler characteristic of Y is zero. Hence, the Euler characteristic over R of $\tilde{Y}$ is zero. Since $H_3 \tilde{Y} \cong 0$ and $H_0 \tilde{Y} \cong \mathbf{Z}$ so that the ranks of both of these modules are zero, we have $\operatorname{rk} H_1 \tilde{Y} = \operatorname{rk} H_2 \tilde{Y}$.

3. A linking invariant

There is a question which arises naturally from consideration of Lemma 2.4 and Proposition 2.5. Namely, what happens when $H_2 \tilde{Y} \neq 0$? The question breaks up into two cases: when all the l_i are zero and when some $l_i \neq 0$. The case when all the l_i are zero turns out to be more interesting, as it will be seen that analysis of this case relates to the question of which links are concordant to boundary links.

A boundary link is one in which the components bound disjoint Seifert surfaces. It is easy to see how this implies that all the linking numbers must be zero. Since linking numbers are preserved under concordance (for definitions, see Section 4) any link concordant to a boundary link will have all zero linking numbers; in particular, this will be true of slice links.

When $H_2\tilde{Y} \neq 0$ and $l_2 = l_3 = \cdots = l_m = 0$, we have $H_2(\tilde{Y}, \hat{X}) \cong R$, generated by a transverse disk to the first component K_1. Thus, the image of $H_2\tilde{Y} \to H_2(\tilde{Y}, \hat{X})$ will be an ideal in R, and it is easily seen that this is an invariant of the link. We will first show that this ideal must be zero for any boundary link. We will then interpret the inclusion map $H_2\tilde{Y} \to H_2(\tilde{Y}, \hat{X})$ geometrically to give us a method of computing the image from a diagram of the link. Finally, we will use this invariant to obtain results about specific links.

Let $I_1(L)$ be the ideal in R defined above. Note that $I_1(L)$ is always zero if $m = 2$, since $H_2\tilde{Y} \cong 0$ in that case (Y is a *knot* complement). Thus, it is an invariant of links of at least three components. Clearly, if enough linking numbers are zero, we can analogously define invariants $I_j(L)$, for $2 \leq j \leq m$. This will be the case, for example, when the link is boundary.

THEOREM 3.1. *Let $L = \{K_1, K_2, \ldots, K_m\}$, $m \geq 3$, be a boundary link. Then the invariants $I_j(L)$, $1 \leq j \leq m$, are all defined and equal to zero.*

Proof. There are two ingredients in the argument. First, there is the fact that, for a boundary link, the rank of $H_1\tilde{X}$ over Λ is $m - 1$. This is well known; it follows for example from the vanishing of the elementary ideals, or from a description of $H_1\tilde{X}$ as the direct sum of a Λ-torsion module and the abelianized commutator subgroup of the free group on m letters, which has rank $m - 1$ over Λ (see [10, Lemma 2.5]). The second is a general fact about Betti numbers. Let C_* be a free chain complex of finite type over an integral domain R, and let $f\colon R \to S$ be a ring map where S is also an integral domain. The Betti numbers of C_* are the ranks (over R) of the homology groups of C_* and the Betti numbers of $C_* \otimes_R S$ are the ranks, over S, of the homology of that complex. It is an easy observation of Milnor (see [7, page 145]) that the Betti numbers of C_* are bounded above by those of $C_* \otimes_R S$.

We put these two ingredients together as follows. Since $C_*\hat{X} = C_*\tilde{X} \otimes_\Lambda R$, the ranks of the R-modules $H_*\hat{X}$ must at least equal the ranks of the Λ-modules $H_*\tilde{X}$. Since $H_1\tilde{X}$ has rank $m - 1$, by Proposition 2.5, so does $H_2\tilde{X}$. Thus $H_2\hat{X}$ has rank at least $m - 1$. Since the removal of the first component leaves an $(m - 1)$-component boundary link, $H_2\tilde{Y}$ has rank $m - 2$. We know from an earlier calculation that $H_3(\tilde{Y}, \hat{X}) \cong H_2(\tilde{Y}, \hat{X}) \cong R$. The long exact sequence of the pair $(\tilde{Y}, \hat{X})$ reads

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_3(\tilde{Y}, \hat{X}) & \longrightarrow & H_2\hat{X} & \xrightarrow{i_*} & H_2\tilde{Y} & \xrightarrow{j_*} & H_2(\tilde{Y}, \hat{X}) & \longrightarrow & \cdots \\ & & \| & & \uparrow & & \uparrow & & \| & & \\ & & R & & \text{rank} \geq m-1 & & \text{rank} = m-2 & & R & & \end{array}$$

By exactness and rank counting, $H_2\hat{X}$ must have rank $m-1$ and cok i_* must be an R-torsion module. But cok $i_* = \operatorname{im} j_* \subset R$, and so j_* must be the *zero map*.

In order to make use of $I_1(L)$, we must be able to compute this. To do this, we first interpret j_* geometrically, Since K_1 has intersection number ± 1 with its transverse disk, a lift of which generates $H_2(\tilde{Y}, \hat{X})$, j_* is given by equivariant intersection with a lift of K_1, as follows:

$$j_*(c) = \sum_{g \in G} \langle c, g\tilde{K}_1 \rangle g \in R$$

where $c \in H_2(\tilde{Y})$, $G \cong \mathbf{Z}^{m-1}$ is the group of covering transformations of $\tilde{Y} \to Y$, $\langle \ , \ \rangle$ is the ordinary integral intersection number, and $\tilde{K}_1$ is a pre-chosen lift of K_1 to $\tilde{Y}$. This interpretation, which allows us to work in the base space rather than the covering space, allows us to work from a diagram of the link.

The method is best illustrated by means of an example. Consider a diagram of the Borromean rings (Figure 1). Removal of any components of L as pictured in Figure 1 results in the trivial link of two components. The generator of $H_2\tilde{Y}$ is (a lift of) the 2-sphere which separates K_2 and K_3, pictured as a plane between K_2 and K_3. The sphere is given an orientation, so that passing through the plane from left to right is the negative direction. We also orient K_1.

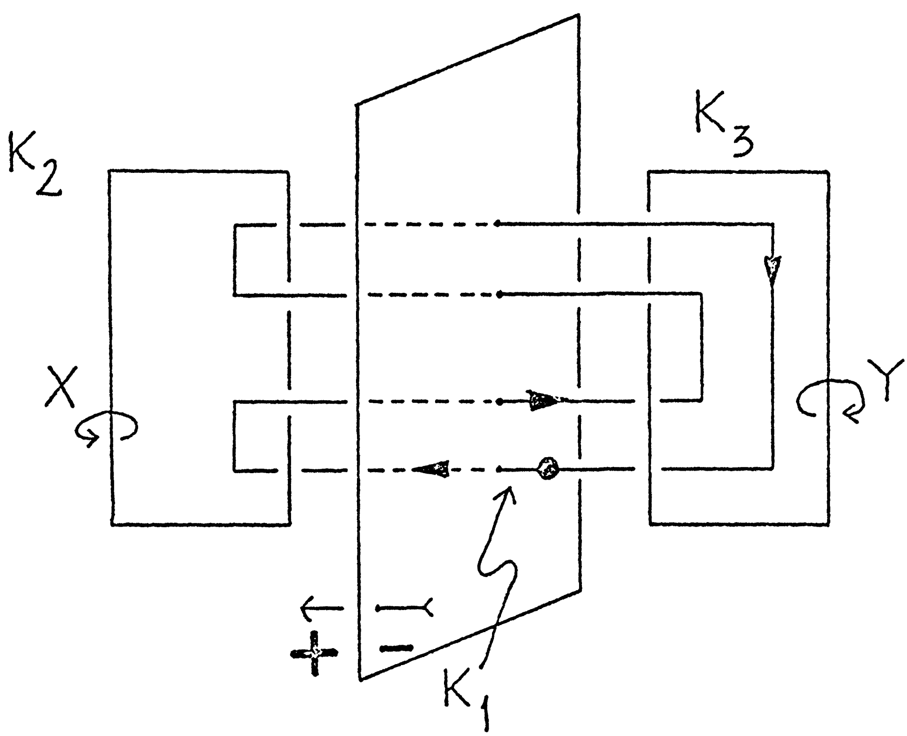


FIG. 1

If x represents the element of $H_1 x$ which comes from the meridian drawn around K_2 and y represents the element which comes from the meridian drawn around K_3, then $R = \mathbf{Z}[x, y, x^{-1}, y^{-1}]$ and $I_1(L)$ will be a principal ideal generated by a polynomial in x and y.

We begin at the base point of K_1 (marked with a heavy black dot). We pass through the sphere in the positive direction, which gives $+1$, around K_2 and back through in the negative direction which gives $-x$, around K_3 and back through in the positive direction getting $+xy$, around K_2 in the inverse direction, and back through the sphere in the negative direction picking up a $-xyz^{-1} = -y$ and back around K_3 and to the base point again. Adding, we get $I_1(L) = (1 - x + xy - y)$. The reader may verify that this computation yields the equivariant intersection of the sphere and K_1 up to multiplication by units of R and replacing a variable by its inverse.

A consequence of this calculation is that the Borromean rings are not a boundary link: in particular, the link is non-trivial.

Of course, this is not surprising, because the Alexander polynomial of L is

$$(1 - x)(1 - y)(1 - z)$$

according to Fox [3]; the advantage here is ease of computation. The next example is a link of four components whose Alexander polynomial is zero, according to O'Neill [8], who studied this link and showed it was non-trivial using a higher order Mas y product. We will show that it is in fact not a boundary link using our method. The link is pictured in Figure 2; note that the removal of any component leaves a trivial link of three components. Both pictures represent the same link, which the reader may verify. The plane in the right-hand picture is part of the sphere which separates K_2 from K_3 and K_4, and is one of the generators of $H_2 \tilde{Y}$. Following the same method as before, we see that the equivariant intersection of this sphere and K_1 is given by $(1 - x) \times (1 - y)(1 - z)$, and therefore $I_1(L) \neq 0$. In fact, it turns out that $I_1(L)$ is just the principal ideal generated by $(1 - x)(1 - y)(1 - z)$ in

$$R = \mathbf{Z}[x, y, z, x^{-1}, y^{-1}, z^{-1}].$$

If L_1 is a boundary link, then the role of the spheres in the computation process is played by the boundaries of regular neighborhoods of Seifert surfaces for L_1.

4. The rank invariant $r(L)$ and link concordance

The proof of Theorem 3.1 indicates that the vanishing of $I_1(L)$ is related to the rank of $H_2\tilde{X}$, which by Proposition 2.5 is equal to the rank of $H_1\tilde{X}$. With this in mind, we define $r(L) = \text{rank } H_1\tilde{X} = \text{rank } H_2\tilde{X}$. Making use of the observation about Betti numbers in the proof of Theorem 3.1 and noting that, by Alexander duality, $H_2 X$ is a free abelian groups of rank $m - 1$, we obtain:

PROPOSITION 4.1. *$0 \leq r(L) \leq m - 1$ for any m-component link L.*

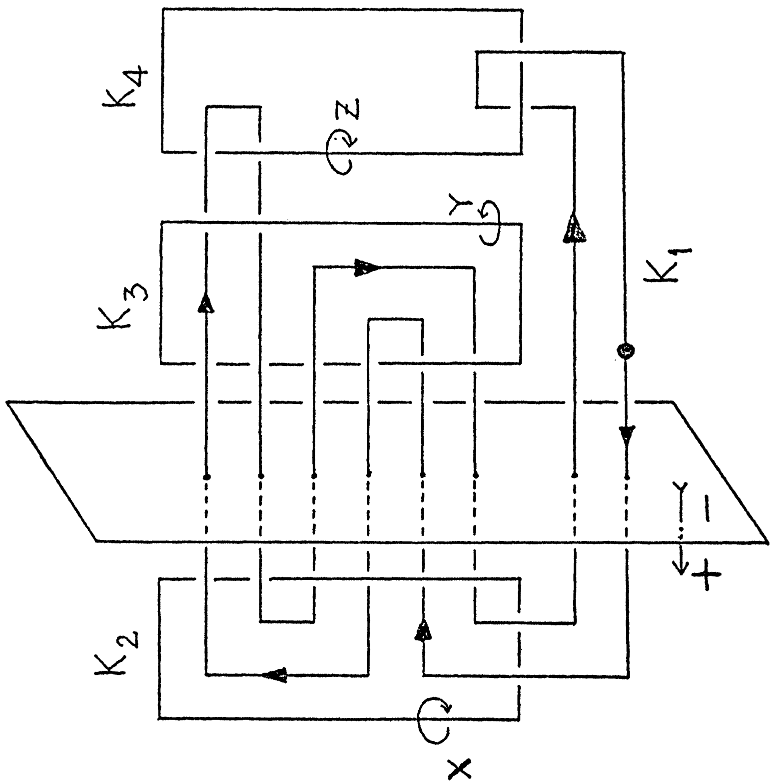
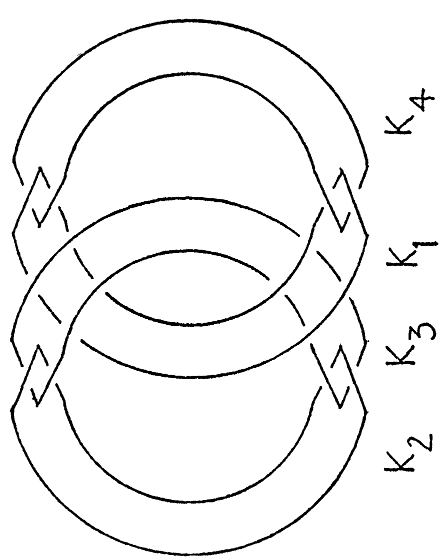


FIG. 2



Furthermore, $r(L)$ carries the following more precise information (compare Cochran and Crowell [1]).:

PROPOSITION 4.2. *If* $r(L) = 1$, *then* $H_2\tilde{X} \cong \Lambda$.

Proof. By the "equivariant" form of Poincaré duality due to Milnor [7, Lemma 1],

$$H_2\tilde{X} \cong \overline{H_e^1(\tilde{H}, \partial\tilde{X}; \Lambda)}.$$

Here, $H_e^*(\tilde{X}, \partial\tilde{X}; \Lambda)$ represents the cohomology of the cochain complex $\mathrm{Hom}_\Lambda\,(C_*(\tilde{X}, \partial\tilde{X}; \Lambda)$, and for a right Λ-module M, $\bar{M}$ represents the left Λ-module with the same underlying albelian group where the Λ-action is twisted as follows. If $\bar{\lambda}$ represents the image of $\lambda \in \Lambda$ under the involution of Λ which carries the variables x_i to their inverses, then for $m \in M$, $\lambda \cdot m = m \cdot \bar{\lambda}$. Furthermore, by the Universal Coefficient Theorem, which in this case takes the form of a spectral sequence (see [10, Theorem 4.2] for a statement),

$$H_e^1(\tilde{X}, \partial\tilde{X}; \Lambda) \cong \mathrm{Hom}_\Lambda\,(H_1(\tilde{X}, \partial\tilde{X}; \Lambda).$$

Now $\partial\tilde{X}$ consists of cylinders and planes; the cylinders occur when a component of L has linking number 0 with every other component of L, and the planes occur in the other cases. The important fact is that this implies that the Λ-modules $H_k(\partial\tilde{X})$ are Λ-torsion, and therefore of rank 0. From the long exact sequence in homology of the pair $(\tilde{X}, \partial\tilde{X})$, we see that rank $H_1\tilde{X} =$ rank $H_1(\tilde{X}, \partial\tilde{X})$. Since $r(L) = 1$, $H_1(\tilde{X}, \partial\tilde{X})$ is a Λ-module of rank 1. The proposition now follows from the next lemma, whose proof is left as an exercise for the reader.

LEMMA 4.3. *Let R be a unique factorization domain, and let M be a finitely generated R-module of rank one. Then* $\mathrm{Hom}_R\,(M, R) \cong R$.

Two links L_0 and L_1 of m components are said to be *concordant* if there exists a collection of disjointly and properly embedded smooth annuli $S^1 \times I$ in $S^3 \times I$, such that the restrictions to $S^1 \times 0$ yield $L_0 \subset S^3 \times 0$ and the restrictions to $S^1 \times 1$ yield $L_1 \subset S^3 \times 1$. Often, one requires that the concordance look like a product near 0 and 1, but we will not concern ourselves with that here. Dropping the smoothness condition on the annuli yields the more general notion of I-equivalence (see Stallings [11, p. 176]). Any invariant of I-equivalence is thus an invariant of link concordance.

The complement Z of the I-equivalence in $S^3 \times I$ contains in its boundary both link complements X_0 and X_1. A key fact is that the inclusions of the X_i into Z induce homology equivalences.

Let $\tilde{Z}$ be the universal abelian cover of Z; if $\tilde{X}_i$ is the universal abelian cover of X_i, $i = 0, 1$, then $\tilde{X}_i \subset \tilde{Z}$ for each i.

THEOREM 4.4 *If L_0 is I-equivalent to L_1*, then $r(L_0) = r(L_1)$.

Remark. This result, in the case of smooth concordance, was obtained by Kawauchi [4] independently, with a much different proof.

Proof. Let $i = 0$ or 1. Since $H_* X_i \cong H_* Z$, we have $H_*(Z, X_i) \cong 0$. By the observation about Betti numbers as in the proof of Theorem 3.1, the groups $H_*(\tilde{Z}, \tilde{X}_i)$ have rank zero as Λ-modules. Hence, the rank of $H_q \tilde{X}_i$ equals the rank of $H_q \tilde{Z}$ for each q. This completes the argument.

Combining this with the proof of Theorem 3.1, we observe the following facts:

(1) If $r(L) = m - 1$, then $I_1(L) = 0$

(2) If $r(L) < m - 1$, then L_0 is not I-equivalent (and hence not concordant) to any boundary link.

Putting these together yields the following statement.

COROLLARY 4.5. *If $I_1(L) \neq 0$, then L is not I-equivalent (and hence not concordant) to a boundary link.*

5. Examples

The Whitehead link (see [9, p. 68]) is a two-component link where the two components are unknotted and have linking number 0. However, it has Alexander polynomial $(1 - x)(1 - y)$, according to Rolfsen [9]. Therefore, $r(\text{Whitehead Link}) = 0$ and the Whitehead Link is not I-equivalent to any boundary link; it is *a fortiori* not concordant to the trivial link, i.e., it is not a *slice link.*

The two examples of Section 3 are such that removing any one component leaves a trivial link of one fewer components. However, since $I_1(L) \neq 0$ for both of these, they are not concordant to any boundary links.

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