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LOCAL FIXED POINT INDEX THEORY FOR NON SIMPLY CONNECTED MANIFOLDS[1]

BY

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1. Introduction

This paper is a sequel to [1]. There we associated to a globally defined map $f: M \to M$ on a compact manifold an obstruction class $o(f) \in H^m(M; \mathscr{B}(f))$, $m = \dim M$, where $\mathscr{B}(f)$ is an appropriate bundle of groups on M, with local group isomorphic to $\mathbf{Z}[\pi]$, $\pi = \pi_1(M)$. We also identified $o(f)$ with an element $\mathscr{L}_\pi(f) \in ZR[\pi, \varphi]$, where $R[\pi, \varphi]$ is the set of Reidemeister classes of π induced by the homomorphism $\varphi = f_\#: \pi \to \pi$. $\mathscr{L}_\pi(f)$ had the form

$$\mathscr{L}_\pi(f) = \pm \sum_{\rho \in R} I(\rho)\rho$$

where $R = R[\pi, \varphi]$ and $I(\rho)$ is the index of the Nielsen class of f corresponding to ρ. This gave us a specific relationship between the obstruction $o(f)$ and the Nielsen number $n(f)$ of f, or, more precisely, between $o(f)$ and a *generalized Lefschetz number* $\mathscr{L}_\pi(f)$ which played the role of a global index and which, in turn, was expressible in terms the Nielsen classes of f. As a consequence, for example, $\mathscr{L}_\pi(f) = 0$ forces $o(f) = 0$ and one obtains the appropriate converse of the Lefschetz Fixed Point Theorem for non-simply connected manifolds.

Our objective here is to carry out this program locally and thereby give a generalized local index theory.

Section 2 is devoted to the local obstruction index. Starting with a smooth or PL manifold M, $\dim M \geq 3$, the inclusion map $M \times M - \Delta \hookrightarrow M \times M$ is replaced by a fiber map $p: E \to M \times M$ and the bundle $\mathscr{B}$ of coefficients is the local system $\pi_{m-1}(F)$ on $M \times M$, where F is the fiber of p. The group $\pi_{m-1}(F)$ is identified in [1] as $\mathbf{Z}[\pi]$, where $\pi = \pi_1(M)$ and the action of $\pi \times \pi$ on $\mathbf{Z}[\pi]$ is given by the right action

$$\alpha \circ (\sigma, \tau) = (\operatorname{sgn} \sigma)\sigma^{-1}\alpha\tau.$$

Now, we suppose that we are given a map $f: U \to M$, which is *compactly fixed* on U (i.e. Fix f is compact), U an open set in M. Let $\mathscr{B}(f)$ denote the bundle of groups on U induced from $\mathscr{B}$ by $i \times f: U \to M \times M$. The *local obstruction index*

$$o(f) = o(f, U) \in H_c^m(U; \mathscr{B}(f))$$

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is defined by first taking a compact m-manifold K with boundary ∂K such that $K \subset U$ and $\operatorname{Fix} f \subset \operatorname{int} K$. Then, if $E(f)$ is the induced fiber space $(i \times f)^*(E)$, there is a natural partial section $s_0(f)\colon \partial K \to E(f)$ and, consequently, a primary obstruction

$$o(f, K) \in H^m(K, \partial K; \mathscr{B}(f, K))$$

with the property that f is deformable (rel ∂K) to a fixed point free map (into M) if, and only if, $o(f, K) = 0$. By letting C denote a slightly smaller copy of K, $o(f, K)$ determines an element of $H^m(U, U - C)$ and consequently the element

$$o(f) \in H_c^m(U; \mathscr{B}(f))$$

called the *local obstruction index of f on U*. Among others, it has the property that f can be deformed by a compactly fixed homotopy to a fixed point free map g if, and only if, $o(f) = 0$.

In Section 3 we study local Nielsen numbers in a more general situation. Here $f\colon U \to X$ is a compactly fixed map and X is a Euclidean neighborhood retract (ENR [2]). Given two points x_1 and x_2 in $\operatorname{Fix} f$ we say that x_1 and x_2 are Nielsen equivalent if there is a path C in U from x_1 to x_2 such that C and Cf are homotopic in X, modulo endpoints. The resulting classes (finite in number) are called Nielsen classes of f in U. Such a Nielsen class $N(f, U)$ is essential if the local (numerical) index [2] is not zero on $N(f, U)$. The local Nielsen number $n(f, U)$ on U is just the number of such essential classes. We also express the local Nielsen classes in terms of the universal covers $\eta_U\colon \tilde{U} \to U$, $\eta\colon \tilde{X} \to X$. One takes lifts $\tilde{\imath}\colon \tilde{U} \to \tilde{X}$, $\tilde{f}\colon \tilde{U} \to \tilde{X}$ of the inclusion i and the map f and identifies π and $\pi(U)$ with the covering groups of η and η_U, respectively. Then, a typical Nielsen class has the form

$$\eta_U(\operatorname{Coin}[\tilde{f}\alpha, \tilde{\imath}]), \quad \alpha \in \pi.$$

where $\operatorname{Coin}[\cdot, \cdot]$ is the coincidence set of two maps. Next, we employ the notion of Reidemeister classes in the situation of *two* homomorphisms,

$$\psi\colon \pi' \to \pi, \qquad \varphi\colon \pi' \to \pi,$$

which induces the right π'-action on π by $\alpha * \sigma = \varphi(\sigma^{-1})\alpha\psi(\sigma)$. The resulting set of orbits (Reidemeister classes) is denoted by $R[\psi, \varphi]$. The relationship between local Nielsen classes and Reidemeister classes is as follows: Let

$$i_U\colon \pi(U) \to \pi, \quad \varphi_U\colon \pi(U) \to \pi$$

denote the homomorphisms induced by the inclusion and the map f. The correspondence $\Gamma\colon [\alpha] \mapsto \eta_U(\operatorname{Coin}[\tilde{f}\alpha, \tilde{\imath}])$ takes $R[i_U, \varphi_U]$ bijectively to the set of Nielsen classes of f on U, if we ignore those Reidemeister classes for which $\eta_U(\operatorname{Coin}[\tilde{f}\alpha, \tilde{\imath}]) = \phi$. Using the correspondence Γ the index $I(\rho)$ of a Reidemeister class $\rho \in R[i_U, \varphi_U]$ is defined to be the index of the corresponding Nielsen class $\Gamma(\rho)$.

In order to calculate the local obstruction index $o(f)$ when U is connected, (Sections 4 and 5) we make use of a bilinear pairing of local systems

$$P\colon \mathscr{B}(f) \otimes \mathscr{T}(U) \to \mathscr{R}(f)$$

where $\mathscr{T}(U)$ is the orientation sheaf on U and $\mathscr{R}(f)$ is the local system on U with local group $\mathbf{Z}[\pi]$ and action

$$\alpha^*\sigma = \varphi_U(\sigma^{-1})\alpha i_U(\sigma).$$

Then, if $\mu(U) \in H_m^c(U; \mathscr{T}(U))$ is the twisted fundamental class on U we have a cap product based on the above pairing and a Kronecker product

$$\langle \cdot, \mu(U) \rangle\colon H_c^m(U; \mathscr{B}(f)) \to H_o(U; \mathscr{R}(f)) \equiv \mathbf{Z}R[i_U, \varphi_U].$$

We are now in a position to state the main theorem which expresses the local obstruction index $o(f)$ in terms of Reidemeister (Nielsen) class of f on U.

THEOREM. *Let $R = R[i_U, \varphi_U]$. Then*

$$\langle o(f), \underline{\mu}(U) \rangle = (-1)^m \sum_{\rho \in R} I(\rho)\rho \in \mathbf{Z}R[i_U, \varphi_U].$$

COROLLARY. *$f\colon U \to M$ is deformable via a compactly fixed homotopy to a fixed point free map $g\colon U \to M$ if, and only if, the local Nielsen number $n(f, U) = 0$.*

2. The local obstruction

Let M denote a connected (not necessarily compact) manifold of dimension $m \geq 3$, and $\Delta_M = \Delta \subset M \times M$ the diagonal. Then, if we replace the inclusion map $i\colon M \times M - \Delta \subset M \times M$ by a fiber map $p\colon E \to M \times M$, we recall [1] that

$$E = \{(\alpha, \beta) \in M^I \times M^I\colon \alpha(0) \neq \beta(0)\}$$

where I is the interval $[0, 1]$ and $p(\alpha, \beta) = (\alpha(1), \beta(1))$. Furthermore, if $b = (x, y) \in M \times M$, the fiber

$$F_b = p^{-1}(b) = \{(\alpha, \beta) \in E\colon \alpha(1) = x, \beta(1) = y\}$$

is 1-connected, so that F_b is k-simple for every k and $\pi_{m-1}(F_b)$ is a bundle (local system) of groups on $M \times M$. We denote this bundle by $\mathscr{B} = \mathscr{B}(M \times M)$. In [1], we obtained a description of the structure of $\mathscr{B}$ as follows: We fix a base point $b = (x, y) \in M \times M - \Delta$ and let $\bar{b}$ denote the constant path at b. Then we identify π with $\pi_1(M, x)$ and $\pi \times \pi$ with $\pi_1(M, x) \times \pi_1(M, y)$, with x near, but distinct from, y. Then, there is an isomorphism of local systems (on $M \times M - \Delta$)

$$\psi\colon \pi_m(M \times M, M \times M - \Delta, b) \to \pi_{m-1}(F_b, \bar{b})$$

given by the exponential map and ψ was employed to establish the following theorem.

THEOREM 2.1. *There is an equivariant isomorphism*

$$\xi\colon \mathbf{Z}[\pi] \to \pi_{m-1}(F_b, \bar{b})$$

where the action of $\pi \times \pi$ *on* $\pi_{m-1}(F_b, \bar{b})$ *is given by* $\mathscr{B}$ *and the action of* $\pi \times \pi$ *on* $\mathbf{Z}[\pi]$ *is given by the right action*

$$\alpha \circ (\sigma, \tau) = (\operatorname{sgn} \sigma)\sigma^{-1}\alpha\tau.$$

σ *and* τ *belong to* π *and* sgn σ *is* ± 1 *according as* σ *preserves or reverses a local orientation at* $x \in M$.

Remark 2.2. If π is identified with covering transformations of $\eta\colon \tilde{M} \to M$, the universal cover of M, then $\sigma^{-1}\alpha\tau$ is to be read as composition of functions from left to right. In fact, we will, in general, write compositions of functions from left to right. However, we will still write $\alpha(x)$ for the value of the function α at x and thus we will also write, for example,

$$(\alpha\beta\gamma)(x) = \gamma(\beta(\alpha(x))).$$

In general group actions will be from the right and if π acts on X, $x\alpha$ may be used for the action of $\alpha \in \pi$ on $x \in X$ as well as $\alpha(x)$. In [1], we used the corresponding left action

$$(\sigma, \tau) \circ \alpha = (\operatorname{sgn} \sigma)\tau\alpha\sigma^{-1}$$

reading composition of functions from right to left.

We review briefly this isomorphism ξ in Theorem 2.1. ξ is obtained by establishing an isomorphism

$$\nu\colon \mathbf{Z}[\pi] \to \pi_m(M \times M, M \times M - \Delta, b)$$

and setting $\xi = \nu\psi$. The structure of ν is a bit involved and takes the following form.

Again, let $\eta\colon \tilde{M} \to M$ denote the universal cover of M. Choose a base point $\tilde{x}_1 \in \tilde{M}$ over x. We identify π with the covering group of η and if we set $\tilde{x}_\alpha = \tilde{x}_1\alpha$, $\alpha \in \pi$, then $\eta^{-1}(x) = \{\tilde{x}_\alpha, \alpha \in \pi\}$. The diagram

$$(1)\qquad \begin{array}{ccc} \tilde{M} & \xleftarrow{\text{proj}_1} & (\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \zeta} \\ M & \xleftarrow{\text{proj}_1} & (M \times M, M \times M - \Delta) \end{array}$$

where $\zeta = \eta \times \eta$ and the horizontal maps of (1) are fibered pair projections on the first coordinate, gives rise to isomorphisms for each σ, τ.

$$(2)\qquad \begin{array}{ccc} \pi_m(\tilde{M}, \tilde{M} - y^{-1}(x), \tilde{y}_\tau) & \xrightarrow{\approx} & \pi_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta), (\tilde{x}_\sigma, \tilde{y}_\tau)) \\ \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} \\ \pi_m(M, M - x, y) & \xrightarrow{\approx} & \pi_m(M \times M, M \times M - \Delta, (x, y)) \end{array}$$

where $(M, M - x)$ and $\tilde{M}, \tilde{M} - \eta^{-1}(x))$ are the fiber pairs of the horizontal maps in (1). In (2), $\tilde{y}_\tau = \tau\tilde{y}_1$, where $\tilde{y}_1$ lies over y and $\tilde{y}_1$ is chosen near $\tilde{x}_1$. Also, the top horizontal isomorphism in (2) is induced by the fiber inclusion

$$\theta_\sigma\colon (\tilde{M}, \tilde{M} - \eta^{-1}(x)) \subset (\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta))$$

given by $\theta_\sigma(u) = (\tilde{x}_\sigma, u)$. Applying the Hurewicz Isomorphism Theorem, we have

$$\begin{array}{ccc}
\pi_m(\tilde{M}, \tilde{M} - \eta^{-1}(x), \tilde{y}_\tau) & \xrightarrow{\theta_{\sigma\#}} & \pi_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta), (\tilde{x}_\sigma, \tilde{x}_\tau)) \\
\downarrow \approx & & \downarrow \approx \\
H_m(\tilde{M}, \tilde{M} - \eta^{-1}(x)) & \xrightarrow{\theta_{\sigma *}} & H_m(\tilde{M} \times \tilde{M}, \tilde{M} - \zeta^{-1}(\Delta)).
\end{array}$$

Now, choose a cell neighborhood V of x and corresponding neighborhoods $\tilde{V}_\alpha$ of $\tilde{x}_\alpha$, evenly covering V so that $\tilde{V}_1\alpha = \tilde{V}_\alpha$. Choose a local orientation at x, thereby determining a generator

$$\gamma_1 \in H_m(\tilde{V}_1, \tilde{V}_1 - \tilde{x}_1)$$

and since

$$H_m(\tilde{M}, \tilde{M} - \eta^{-1}(x)) \approx \sum_{\alpha \in \pi} H_m(\tilde{V}_\alpha, \tilde{V}_\alpha - x_\alpha),$$

the correspondences $\alpha \mapsto \gamma_1 \alpha \mapsto \theta_{1*}(\gamma_1 \alpha)$ give rise to the isomorphism ν as the following composition

$$\begin{array}{ccccc}
\mathbf{Z}[\pi] & \xrightarrow{\approx} & \sum_{\alpha \in \pi} H_m(\tilde{V}_\alpha, \tilde{V}_\alpha - \tilde{x}_\alpha) & \xrightarrow{\approx} & H_m(\tilde{M}, \tilde{M} - \eta^{-1}(x)) \\
& & & & \downarrow \theta_{1*} \\
& & \pi_m(M \times M, M \times M - \Delta, b) & \xleftarrow{\approx} & H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta).
\end{array}$$

This completes the sketch of the structure of ξ. While ξ does not depend on the choice for $\tilde{x}_1$ over x_1, ξ does depend on the orientation chosen at x and the choice of the base point $b = (x, y)$.

There is also an alternative description of ξ. Define a correspondence

$$\mu\colon \mathbf{Z}[\pi \times \pi] \to H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta))$$

by setting

$$\mu(\alpha, \beta) = \theta_{1*}\gamma_1(\alpha \times \beta).$$

We factor out the subgroup D of $\mathbf{Z}[\pi \times \pi]$ generated by elements of the form

$$\operatorname{sgn} \sigma(\alpha\sigma, \beta\sigma) - (\alpha, \beta),\ \sigma, \alpha, \beta \in \pi.$$

Since [1], for every $\sigma \in \pi$,

$$\theta_{1*}\gamma_1(\sigma \times \sigma) = (\operatorname{sgn} \sigma)\theta_{1*}\gamma_1.$$

μ induces

$$\bar{\mu}\colon \mathbf{Z}[\pi \times \pi]/D \to H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)).$$

Now, let $\omega\colon \mathbf{Z}[\pi \times \pi] \to \mathbf{Z}[\pi]$ be defined by

$$\omega(\alpha, \beta) = (\operatorname{sgn} \alpha)\alpha^{-1}\beta.$$

Then, $\omega(D) = 0$, and we have an induced isomorphism

$$\bar{\omega}\colon \mathbf{Z}[\pi \times \pi]/D \to \mathbf{Z}[\pi].$$

Thus, ξ is also given by the following composition

$$\begin{array}{ccccc} \mathbf{Z}[\pi] & \xleftarrow[\approx]{\bar{\omega}} & \mathbf{Z}[\pi \times \pi]/D & \xrightarrow[\approx]{\bar{\mu}} & H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)) \\ & \searrow & & \swarrow \approx & \\ & & \pi_m(M \times M, M \times M - \Delta, b) & & \end{array}$$

and $\bar{\omega}$ and $\bar{\mu}$ are equivalent with respect to the right actions of $\pi \times \pi$ given respectively, when $(\sigma, \tau) \in \pi \times \pi$, by

$$\alpha(\sigma, \tau) = \operatorname{sgn} \sigma\sigma^{-1}\alpha\tau, \quad \alpha \in \pi,$$

$$[(\alpha, \beta)](\sigma, \tau) = [(\alpha\sigma, \beta\tau)], \quad (\alpha, \beta) \in \pi \times \pi,$$

$$u(\sigma, \tau) = (\sigma \times \tau)_*(u), \quad u \in H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)).$$

We now consider the following data.

2.3. *The data (M, f, U).*

(i) M is a smooth or PL manifold of dimension $m \geq 3$.
(ii) U is an open subset of M.
(iii) $f\colon U \to M$ is a map with compact fixed point set $\operatorname{Fix} f \subset U$; i.e. f is *compactly fixed.*

This data is accompanied by the following ingredients with notation as follows:

(iv) $i\colon U \hookrightarrow M$, inclusion map,
(v) $\mathscr{B}(f)$ the bundle of coefficients (local system) on U induced by $i \times f\colon U \to M \times M$ from $\mathscr{B} = \mathscr{B}(M \times M)$, i.e. $\mathscr{B}(f) = (i \times f)^*(\mathscr{B}(M \times M))$,
(vi) $p_U\colon E(f) \to U$, the fiber space over U induced from $p\colon E \to M \times M$ by $i \times f$, i.e., $E(f) = (i \times f)^*(E)$.

Our objective is to define a local obstruction index $o(f) \in H_c^m(U, \mathscr{B}(f))$. To this end let K denote a triangulable compact m-manifold in U with boundary ∂K such that $(\operatorname{Fix} f) \cap \partial K = \phi$. Define a partial section $s_o(f)\colon \partial K \to E(f)$ by

$$s_o(f)(x) = (\bar{x}, \overline{f(x)})$$

where $\bar{u}$ denotes the constant path at u. Furthermore, let $\mathscr{B}(f, K)$ denote the restriction of $\mathscr{B}(f)$ to K.

LEMMA 2.4. *Let K be as above. Then, $f \mid K$ is deformable, relative to ∂K, to a map $g\colon K \to M$ which is fixed point free on K, iff, $s_0(f)$ admits an extension to a section over K.*

Proof. The "only if" part is obvious. The "if" part requires a simple covering homotopy argument to adjust the section to have a constant path in the first coordinate [1].

DEFINITION 2.5. *Let* $o(f, K) \in H^m(K, \partial K; \mathscr{B}(f, K))$ denote the primary obstruction to extending $s_0(f)$ to a section $s(f)$ over K. $o(f, K)$ will be called the *local obstruction index of f on $K \subset U$.*

General obstruction theory ([3]) implies the following proposition.

PROPOSITION 2.6. *$f \mid K$ is deformable to be fixed point free, relative to ∂K, iff the local obstruction index of f on K, $o(f, K)$,* is 0.

Now let $\Gamma(U)$ denote the compact subsets C of U directed by inclusion and consider

$$H_c^m(U; \mathscr{B}(f)) = \varinjlim H^m(U, U - C; \mathscr{B}(f))$$

where the direct limit is over $\Gamma(U)$. Also suppose that $\operatorname{Fix} f \subset \operatorname{int} K$. The "excision" isomorphism,

$$H^m(U, U - K_o, \mathscr{B}(f)) \approx H^m(K, \partial K; \mathscr{B}(f, K)),$$

where K_o is K minus a "collar" of ∂K, tells us that $o(f, K)$ determines an element $o(f) \in H_c^m(U; \mathscr{B}(f))$.

DEFINITION–PROPOSITION 2.7. *$o(f)$ is independent of K and is called the local obstruction index of f.*

Proof (of independence on K). Given K and K', choose K'' such that $K \cup K' \subset K''$. The diagram

$$\begin{array}{ccc}
H^m(U, U - K''_o; \mathscr{B}(f)) & \longleftarrow & H^m(U, U - K_o; \mathscr{B}(f)) \\
\approx\downarrow & & \downarrow\approx \\
H^m(K'', \partial K''; \mathscr{B}(f, K'')) & & H^m(K, \partial K; \mathscr{B}(f, K)) \\
\nwarrow & & \nearrow\approx \\
& H^m(K'', L; \mathscr{B}(f, K'')) &
\end{array}$$

where $L = K'' - K$, and the corresponding diagram where K' replaces K, tells that $o(f, K)$ and $o(f, K')$ coalesce in $H^m(U, U - K''_o; \mathscr{B}(f))$ and hence determine the same element in $H_c^m(U; \mathscr{B}(f))$.

PROPOSITION 2.8 (HOMOTOPY INVARIANCE). *Suppose* $\Gamma\colon U \times I \to M$ *denotes a homotopy such that* $\bigcup_t \operatorname{Fix} \Gamma_t$ *is compact; i.e. the homotopy is compactly fixed. Set* $\Gamma_0 = f$ *and* $\Gamma_1 = g$. *The induced homotopy*

$$i \times f \sim i \times g\colon U \to M \times M$$

induces a bundle equivalence

$$\begin{array}{ccc} \mathscr{B}(f, U) & \xrightarrow{\ \Gamma\ } & \mathscr{B}(g, U) \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

which, in turn, establishes a (coefficient) isomorphism

$$\Gamma^*\colon H_c^m(U, \mathscr{B}(f, U)) \to H_c^m(U; \mathscr{B}(g, U)).$$

Then

$$\Gamma^*(o(f)) = o(g).$$

Proof. Let K denote a compact m-manifold with boundary ∂K such that $\bigcup_t \operatorname{Fix} \Gamma_t \subset \operatorname{int} K$, so that K may be used to determine both $o(f)$ and $o(g)$. The remainder of the proof is standard.

THEOREM 2.9. *Given* $f\colon U \to M$. *Then there is a compactly fixed homotopy* $\Gamma\colon U \times I \to M$ *such that* $H_0 = f$ *and* $H_1 = g$ *is fixed point free iff the local obstruction index*

$$o(f) = 0 \in H_c^m(U, \mathscr{B}(f, U)).$$

Proof. An immediate consequence of 2.7 and 2.8.

Remark 2.10. Sometimes we will display U in the notation for $o(f)$, i.e., $o(f) = o(f, U)$. Also, if $f\colon M \to M$ is globally $o(f, U)$ will denote $o(f \mid U)$.

In order to state the "addivity" property of the local index, we recall some facts. Suppose $V_1, V_2, \ldots, V_k$ are mutually disjoint open subsets of the open set U and $C_l \subset V_l$ are compact subsets. Suppose furthermore, that $\mathscr{G}$ is a local system on U and $\mathscr{G}_l = \mathscr{G} \mid V_l$. Then, for each l we have

$$H^m(V_l, V_l - C_l; \mathscr{G}_l) \underset{\approx}{\xleftarrow{\ j_l^*\ }} H^m(U, U - C_l; \mathscr{G}) \xrightarrow{\ i_l^*\ } H^m(U, U - C; \mathscr{G})$$

where i_l, j_l are inclusions and $C = \bigcup_l C_l$. The homomorphism $i_l^{*-1} j_l^*$ induces a homomorphism

$$\alpha_l\colon H_c^m(V_l; \mathscr{G}_l) \to H_c^m(U; \mathscr{G})$$

and consequently a homomorphism

$$\alpha = \sum \alpha_l\colon \sum_l H_c^m(V_l; \mathscr{G}_l) \to H_c^m(U; \mathscr{G}).$$

The proof of the following proposition is now a simple exercise.

PROPOSITION 2.10 (ADDITIVITY). *Given $f\colon U \to M$ (compactly fixed as in 2.3). Suppose $V_1, \ldots, V_k$ are finitely many mutually disjoint open sets such that $\operatorname{Fix} f \subset \bigcup_l V_l$. Let $f_l = f \mid V_l\colon V_l \to M$. Then under the homomorphism*

$$\alpha\colon \sum H_c^m(V; \mathscr{B}(f_l)) \to H_c^m(U; \mathscr{B}(f))$$

we have

$$\alpha(\sum o(f_l, V_l)) = o(f, U).$$

3. Local Nielsen numbers

In this section we consider compactly fixed maps $f\colon U \to X$, where U is an open set in a Euclidean neighborhood retract (ENR [2]). In particular, then, X may be manifold (possibly with boundary) or a locally finite polyhedron. Notice that we do not require X to be compact, nor do we require the map f to be compact. The fact that $\operatorname{Fix} f$ is compact is what is essential. We recall also that for ENR's we have a local index theory with the usual properties [2] for maps $f\colon U \to X$ with compact fixed point set. $I(f, U)$ will denote the index of f on U.

Our objective here is to take a compactly fixed $f\colon U \to X$ and classify the points of $\operatorname{Fix} f$ into *local* Nielsen classes and develop the necessary elementary properties. Since there is a distinct parallel between the local theory and the well-known global theory [4] we will often omit details.

DEFINITION 3.1. Let x_0 and x_1 denote fixed points of $f\colon U \to X$. x_0 and x_1 are Nielsen equivalent in U proved there is a path C *in* U from x_0 to x_1 such that C and Cf are homotopic with endpoints fixed *in* X. (Recall that composition of functions is read from left to right.) The resulting equivalence classes are called the local Nielsen classes of f in U. $\mathscr{N}(f, U)$ will denote the set of such classes.

PROPOSITION 3.2. *The local Nielsen classes of $f\colon U \to X$ are finite in number.*

Proof. Since X is an ANR, it is ULC [5] and this forces each Nielsen class to be open in Fix (f). Since $\operatorname{Fix} f$ is compact the result follows.

Notation 3.3. We designate the local Nielsen classes of $f\colon U \to X$ by

$$\mathscr{N}(f, U) = \{N_1(f, U), N_2(f, U), \ldots\}.$$

Furthermore, if $f\colon X \to X$ is globally defined, we set $N(f, U) = N(f \mid U, U)$; i.e. a local Nielsen class of $f\colon X \to X$ on U is taken to be a local Nielsen class of $f \mid U\colon U \to X$.

DEFINITION 3.4. The *index* $I(N_j(f, U))$ of a Nielsen class $N_j(f, U)$ is defined to be $I(f, V_j)$ where V_j is an open set in U such that $V_j \cap (\operatorname{Fix} f) = N_j(f, U)$. If the index $I(N_j(f, U)) \neq 0$, we recall $N_j(f, U)$ an essential class. Finally, the *Nielsen number* $n(f, U)$ of $f: U \to X$ is defined to be the number (finite) of *essential* Nielsen classes.

THEOREM 3.5. (HOMOTOPY INVARIANCE). *Suppose* $H: U \times I \to X$ *is a compactly fixed homotopy, i.e. there is a compact set* $K \subset U$ *such that* $K \supset \bigcup_t \operatorname{Fix} H_t$, $0 \leq t \leq 1$. *Then*, $n(H_0, U) = n(H_1, U)$.

Proof. The proof proceeds in a manner parallel to the proof for compact ANR's in [4]. First, set $f = H_0$ and $g = H_1$ and if C is a path *in* U set

$$\langle H, C\rangle(t) = H(C(t), t) = H_t(C(t)), \quad 0 \leq t \leq 1.$$

Thus, $\langle H, C\rangle$ is a path in X. Now, if $x_0 \in \operatorname{Fix} f$ and $x_1 \in \operatorname{Fix} g$, we say that $x_0 H x_1$ (x_0 is H-related to x_1) provided there exists a C *in* U from x_0 to x_1 with $C \sim \langle H, C\rangle$ (endpoint homotopic) *in* X. This relation H induces a one-one correspondence $\hat{H}$ from a subset of $\mathcal{N}(f, U)$ to a subset of $\mathcal{N}(g, U)$ via the relation between Nielsen classes

$$[N(f, U)]H[N(g, U)] \Leftrightarrow x_0 H x_1, \quad x_0 \in N(f, U), \quad x_1 \in N(g, U)$$

(see [4, page 92]). Up to this point the fact that the homotopy is compactly fixed is not used. It is used, however, at this point to show that $\hat{H}$ is bijective from the *essential* Nielsen classes of f to the essential Nielsen classes of g. Because X is locally compact one can assume that the compact set K above contains $\operatorname{Fix} H_t$ in its interior for all t, $0 \leq t \leq 1$. Now, open sets in the interior of K may be used to compute indices of H_t and furthermore $H: K \times I \to X$ may be considered a path in X^K where the compact open topology on X^K coincides with the uniform topology. Now, the proof in [4, pages 93–94] applies to show

(a) $[N(f, U)]H[N(g, U)] \Rightarrow I(N(f, U)) = I(N(g, U))$,

(b) $N(f, U)$ is not H-related to some $N(g, U) \Rightarrow I(f, U) = 0$.

This completes the sketch of the proof.

We will also find it useful to express local Nielsen classes in terms of universal covers after the manner of Jiang [6]. Given $f: U \to X$, where X is an ENR, the components of U are open and since $\operatorname{Fix} f$ is assumed compact, $\operatorname{Fix} f$ lies in a finite number of these components and each of these components produces distinct local Nielsen classes. There is, therefore, no essential loss of generality if we assume U and X are connected.

Let $\eta: \tilde{X} \to X$, $\eta_U: \tilde{U} \to U$ denote the universal covers of X and U, respectively and $i: U \hookrightarrow X$ the inclusion map. Choose

$$u_0 \in U, \tilde{u}_0 \in \eta_U^{-1}(u_0), \tilde{x}_0 \in \eta^{-1}(i(u_0)), \tilde{y}_0 \in \eta^{-1}(f(u_0)).$$

These choices uniquely determine fixed lifts $\tilde{\imath}$ and $\tilde{f}$ such that $\tilde{\imath}(\tilde{u}_0) = \tilde{x}_0$, $\tilde{f}(\tilde{u}_0) = \tilde{y}_0$:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{\imath}} & \tilde{X} \\ {\scriptstyle \eta_U}\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\ U & \xrightarrow{i} & X \end{array} \qquad \begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle \eta_U}\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\ U & \xrightarrow{f} & X. \end{array}$$

Furthermore, if we let $\pi(U)$ and π denote, respectively, the covering groups of η_U and η, i and f induce homomorphisms $i_U: \pi(U) \to \pi$ and $\varphi_U: \pi(U) \to \pi$ with characterizing equations

$$\sigma\tilde{\imath} = \tilde{\imath}i_U(\sigma), \quad \sigma\tilde{f} = \tilde{f}\varphi_U(\sigma), \quad \sigma \in \pi(U).$$

We should also note that all the lifts of f have the form $\tilde{f}\alpha$, $\alpha \in \pi$ and $\tilde{f}\alpha = \tilde{f}\beta$ iff $\alpha = \beta$.

Now, let Coin $[\tilde{f}\alpha, \tilde{\imath}]$ denote the coincidence set of $\tilde{f}\alpha$ and $\tilde{\imath}$; i.e.

$$\text{Coin } [\tilde{f}\alpha, \tilde{\imath}] = \{\tilde{u} \in \tilde{U}: (\tilde{f}\alpha)(\tilde{u}) = \tilde{\imath}(\tilde{u})\}.$$

PROPOSITION 3.6. *Each set $\eta_U(\text{Coin } [\tilde{f}\alpha, \tilde{\imath}])$, $\alpha \in \pi$, is a Nielsen class or empty. Furthermore,*

$$\eta_U(\text{Coin } [\tilde{f}\alpha, \tilde{\imath}]) = \eta_U(\text{Coin } [\tilde{f}\beta, \tilde{\imath}])$$

iff there is a $\sigma \in \pi(U)$ such that

$$\sigma^{-1}\tilde{f}\alpha i_U(\sigma) = \tilde{f}\beta$$

or, equivalently, for some $\sigma \in \pi_U$,

$$\varphi_U(\sigma^{-1})\alpha i_U(\sigma) = \beta.$$

Proof. (a) Suppose $\tilde{u}$ and $\tilde{v}$ belong to Coin $[\tilde{f}\alpha, \tilde{\imath}]$. Then a path $\tilde{C}$ in $\tilde{U}$ from $\tilde{u}$ to $\tilde{v}$ induces a path C from $u = \eta_U(\tilde{u})$ to $v = \eta_U(\tilde{v})$ in U which does the job for showing that u and v are Nielsen equivalent fixed points in U. Thus,

$$\eta_U(\text{Coin } [\tilde{f}\alpha, \tilde{\imath}]) \subset \text{some Nielsen class } N(f, U).$$

(b) Each fixed point $u \in U$ determines an $\alpha \in \pi$ as follows. Choose $\tilde{u} \in \eta_U^{-1}(u)$. α is determined by the condition $(\tilde{f}\alpha)(\tilde{u}) = \tilde{\imath}(\tilde{u})$ so that $\tilde{u} \in \text{Coin } [\tilde{f}\alpha, \tilde{\imath}]$ and hence

$$u \in \eta_U(\text{Coin } [\tilde{f}\alpha, \tilde{\imath}])$$

for some $\alpha \in \pi$. It is a simple matter to show that where u and v are Nielsen equivalent in U, we may choose $\tilde{u}$ and $\tilde{v}$ above to yield exactly the same $\alpha \in \pi$. Thus, each local Nielsen class is contained in some $\eta_U(\text{Coin } [\tilde{f}\alpha, \tilde{\imath}])$. This verifies the first part of Proposition 3.6.

(c) Now, suppose

$$\eta_U(\text{Coin } [f\alpha, i]) = \eta_U(\text{Coin } [f\beta, i]).$$

Then, we have $\tilde{u}$, $\tilde{u}_1$ in $\tilde{U}$ such that

$$(\tilde{f}\alpha)(\tilde{u}) = \tilde{\imath}(\tilde{u}),\ (\tilde{f}\beta)(\tilde{u}_1) = \tilde{\imath}(\tilde{u}_1),\ \sigma(\tilde{u}) = \tilde{u}_1,\ \sigma \in \pi(U).$$

Then

$$\begin{aligned}(\tilde{f}\alpha)(\tilde{u}) = \tilde{\imath}(\tilde{u}) &\Rightarrow (\sigma^{-1}\tilde{f}\alpha)(\tilde{u}_1) = (\tilde{\imath}i_U(\sigma^{-1}))(\tilde{u}_1)\\ &\Rightarrow (\sigma^{-1}\tilde{f}\alpha i_U(\sigma))(u_1) = \tilde{\imath}(\tilde{u}_1)\\ &\Rightarrow \sigma^{-1}(\tilde{f}\alpha)i_U(\sigma) = \tilde{f}\beta, \quad \alpha \in \pi(U).\end{aligned}$$

Since this last equality is equivalent to $\varphi_U(\sigma^{-1})\alpha i_U(\sigma) = \beta$ and also implies (as a simple exercise)

$$\eta_U(\text{Coin}\,[\tilde{f}\alpha, \tilde{\imath}]) = \eta_U(\text{Coin}\,[\tilde{f}\beta, \tilde{\imath}]),$$

the proof is complete.

Definition 3.7. Given homomorphisms (of groups) $\psi\colon \pi' \to \pi$ and $\phi\colon \pi' \to \pi$. We introduce the right action of π' on π by

$$\alpha * \sigma = \varphi(\sigma^{-1})\alpha\psi(\sigma), \quad \sigma \in \pi', \quad \alpha \in \pi.$$

The resulting set of orbits $R[\psi, \varphi]$ is called the set of Reidemeister classes, i.e. each orbit is a *Reidemeister class.*

Definition 3.8. Given a compactly fixed $f\colon U \to X$ and corresponding homomorphisms $i_U\colon \pi(U) \to \pi$, $\varphi_U\colon \pi(U) \to \pi$ (as above), we call $R[i_U, \varphi_U]$ the set of local Reidemeister classes on U generated by f.

Proposition 3.9. *The correspondence* $\Gamma\colon [\alpha] \to \eta_U(\text{Coin}\,[\tilde{f}\alpha, \tilde{\imath}])$ *takes Reidemeister classes to Nielsen classes bijectively provided we ignore those Reidemeister classes* $[\alpha]$ *for which* Coin $[\tilde{f}\alpha, \tilde{\imath}] = \phi$.

Proof. Immediate from Proposition 3.6.

Suppose we let $\hat{U}$ denote the component of $\eta^{-1}(U)$ which contains $\tilde{x}_0 \in \eta^{-1}(i(u_0))$. Then, $\eta \mid \hat{U}\colon \hat{U} \to U$ is a covering map. It is easy to see that $\pi_1(\hat{U}, x_0)$ corresponds to the kernel of $i_U\colon \pi(U) \to \pi$ and hence the covering map $\eta \mid \hat{U}$ is regular and furthermore $f\colon U \to X$ has a unique lift $\hat{f}\colon (\hat{U}, \tilde{x}_0) \to (\tilde{X}, \tilde{y}_0)$ and hence a diagram

$$\begin{array}{ccccc} \tilde{U} & \xrightarrow{\ \tilde{\imath}\ } & \hat{U} & \xrightarrow{\ f\ } & \tilde{X} \\ \eta_U \big\downarrow & & \eta|u \big\downarrow & & \eta \big\downarrow \\ U & \xrightarrow{\ \text{id}\ } & U & \xrightarrow{\ f\ } & X \end{array}$$

The following lemma is easy to prove, because $\tilde{\imath}\hat{f} = \tilde{f}$.

Lemma 3.10. *For* $\alpha \in \pi$, $\tilde{\imath}(\text{Coin}\,[\tilde{f}\alpha, \tilde{\imath}])$ *and hence*

$$\eta_U(\text{Coin}\,[\tilde{f}\alpha, \tilde{\imath}] = \eta \mid \hat{U}(\text{Fix}\,\hat{f}\alpha).$$

We also have the following result. Let $\hat\pi(U) = \pi(U)/\ker i_U$ and $j\colon \pi(U) \to \hat\pi(U)$ the natural projection. We also have diagrams

$$\begin{array}{ccc} \pi(U) & \xrightarrow{i_U} & \pi \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle \hat\imath_U} & \\ & \hat\pi(U) & \end{array} \qquad \begin{array}{ccc} \pi(U) & \xrightarrow{i_U} & \pi \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle \hat\varphi_U} & \\ & \hat\pi(U) & \end{array} .$$

LEMMA 3.11. *Since*

$$\varphi_U(\sigma^{-1})\alpha i_U(\sigma) = \hat\varphi_U(j(\sigma)^{-1})\alpha\hat\imath_U(j(\sigma))$$

the identity map id: $\pi \to \pi$ *induces a bijection*

$$R[i_U, \varphi_U] \xrightarrow{\approx} R[\hat\imath_U, \hat\varphi_U].$$

Thus, Proposition 3.9 may be reformulated as follows:

PROPOSITION 3.12. *The correspondence* $R[\hat\imath_U, \hat\varphi_U] \to \mathcal{N}(f, U)$ *which takes*

$$[\alpha] \mapsto \eta \mid \hat U(\mathrm{Fix}\,(\hat f\alpha))$$

takes Reidemeister classes to Nielsen classes bijectively provided we ignore Reidemeister classes $[\alpha]$ *for which* $\mathrm{Fix}\,(\hat f\alpha) = \phi$.

Suppose $U \subset V \subset X$, where U and V are both open, connected subsets of X, $f_V\colon V \to X$ is a given map, and $\tilde U$, $\tilde V$, $\tilde X$ are the corresponding covering spaces. Then, as before we have fixed lifts

$$\tilde U \xrightarrow{\tilde\imath_U} \tilde X,\ {}^{U}\tilde U \longrightarrow \tilde X,\ \tilde V \xrightarrow{\tilde\imath_V} \tilde X,\ {}^{V}\tilde V \longrightarrow \tilde X$$

where $\tilde\imath_U$ and $\tilde\imath_V$ cover inclusions (which are not designated) and $\tilde f_U, \tilde f_V$ cover f_V and $f_U = f_V \mid U$, respectively. Choose the lift $\tilde\imath_U^V\colon \tilde U \to \tilde V$ of the inclusion map $U \hookrightarrow V$ with the property that $\tilde\imath_U^V \tilde\imath_V = \tilde\imath_U$. Then, $i_U^V\colon \pi(U) \to \pi(V)$ is uniquely determined by the condition

$$\sigma\tilde\imath_U^V = \tilde\imath_U^V i_U^V(\sigma), \quad \sigma \in \pi(U).$$

Now a simple argument shows that $i_U = i_U^V i_V$ and $\varphi_U = i_U^V \varphi_V$. Furthermore, the identity map $\pi \to \pi$ is equivalent with respect to the map $i_U^V\colon \pi(U) \to \pi(V)$, thus inducing

$$h_U^V\colon R[i_U, \varphi_U] \to R[i_V, \varphi_V].$$

Convention 3.13. If $K \subset X$ is an set and $U = \mathrm{int}\, K$, it is convenient to set

$$R[i_K, \varphi_K] = R[i_U, \varphi_U], \quad \mathcal{N}(f, K) = \mathcal{N}(f, U).$$

Given a compactly fixed $f\colon U \to X$, it may be impossible to find a compact set K in U such that the fundamental group $\pi(K)$ "captures" all of $\pi(U)$. Thus, the natural map

$$h_K^U\colon R[i_K, \varphi_K] \to R[i_U, \varphi_U]$$

need not be injective. However, the following result indicates that such a K captures the essential information on $\operatorname{Fix} f$ in U.

PROPOSITION 3.14. *Let $f\colon U \to X$ denote a compact fixed map, where X is an ENR. Then, there exists a compact set $K \subset U$ such that* $\operatorname{Fix} f \subset \operatorname{int} K$ *and* $\mathcal{N}(f, U) = \mathcal{N}(f, K)$.

Proof. First, using the fact that X is locally compact, choose a compact set L such that $\operatorname{Fix} f \subset \operatorname{int} L$. Each Nielsen class $N(f, L)$ of $f \mid L$ lies in a unique Nielsen class $N(f, U)$, thus defining a surjective function $\psi\colon \mathcal{N}(f, L) \to \mathcal{N}(f, U)$. If N_i and N_j are Nielsen classes in $\mathcal{N}(f, L)$ such that $\psi(N_i) = \psi(N_j)$, there is a path α_{ij} in U from N_i to N_j such that $\alpha_{ij} \sim f(\alpha_{ij})$. Only finitely many such pairs N_i, N_j occur so that there is a compact set $K \subset U$ such that $\operatorname{Fix} f \subset \operatorname{int} K$ and the paths α_{ij} are all in $\operatorname{int} K$. Now, it is clear that the corresponding map $\psi\colon \mathcal{N}(f, K) \to \mathcal{N}(f, U)$ is the identity.

COROLLARY 3.15. *Let $f\colon U \to M$ denote a compactly fixed map, where U is an open set in the manifold* M. *Then, there exists a manifold (with boundary) $K \subset U$ such that the Nielsen classes in $\mathcal{N}(f, U)$ and $\mathcal{N}(f, K)$ correspond identically. Furthermore, if U is connected we may choose K to be connected.*

COROLLARY 3.16. *If $f\colon U \to X$ and K are as in Proposition 3.14, the correspondence*

$$h_K^U\colon R[i_K, \varphi_K] \to R[i_U, \varphi_U]$$

is bijective provided (using Proposition 3.9) we restrict ourselves to Reidemeister classes which correspond to Nielsen classes.

4. Preliminaries to calculating $o(f, U)$

Let $p\colon E \to M \times M$ denote the fiber map (Section 2) replacing the inclusion map

$$M \times M - \Delta \subset M \times M.$$

$F_{(u,v)}$ will denote the fiber over (u, v). Given a tubular neighborhood T of the diagonal $\Delta \subset M \times M$, let $T_0 = T - \Delta$. Then, given $u \in M$ and a local orientation of M at μ we can assign an element

$$g_u \in \pi_{m-1}(F_{(u,v)}), \quad (u, v) \in T_0$$

as follows: Let $\tilde{\Delta}$ denote the diagonal in $\tilde{M} \times \tilde{M}$, with corresponding tubular neighborhood $\tilde{T}$. If $\tilde{T}_0$ denotes the complement of the 0-section in $\tilde{T}$, we have

$$\begin{array}{ccc}
H_m(\tilde{T}, \tilde{T}_0) & \xrightarrow{\approx} & H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \tilde{\Delta}) \\
& \searrow & \uparrow \\
& & H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)) \\
& & \uparrow \approx \\
& & \pi_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta), \quad (\tilde{u}, \tilde{v})) \\
& & \downarrow \approx \\
\pi_m(M, M - u, v) & \xrightarrow{\approx} & \pi_m(M \times M, M \times M - \Delta, \quad (u, v)) \\
& & \downarrow \approx \\
& & \pi_{m-1}(F_{(u,v)}, \quad (\bar{u}, \bar{v}))
\end{array}$$

where $(\tilde{u}, \tilde{v}) \in \tilde{T}$, $(\tilde{u}, \tilde{v}) \mapsto (u, v)$, and $\bar{u}$, $\bar{v}$ are constant paths at u and v, respectively. The isomorphism

$$\pi_m(M, M - u, v) \to \pi_m(M \times M, M \times M - \Delta, \quad (u, v))$$

is induced by the section $M \to M \times M$ given by $y \mapsto (u, y)$. If we choose a Euclidean neighborhood W of u and an orientation of W, an imbedding

$$i_u\colon (D^m, S^{m-1}, a_0) \to (W, W - u, v)$$

(which take 0 to u) determines an element of $\pi_m(M, M - u, v)$ and hence (see the diagram above) an element

$$g_u \in \pi_{m-1}(F_{(u,v)}, \quad (\bar{u}, \bar{v})).$$

g_u may be represented in

$$H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta))$$

as follows: Given $\tilde{u}$ over u, the imbedding i_u lifts to an imbedding

$$\tilde{i}_u\colon (D^m, S^{m-1}, a_0) \to (\tilde{W}, \tilde{W} - \tilde{u}, \tilde{v}),$$

where $\tilde{W}$ covers W. Define $\gamma_u\colon (D^m, S^{m-1}) \to (\tilde{T}, \tilde{T}_0)$ by $\gamma_u(y) = (\tilde{u}, \tilde{i}_u(y))$. $[\gamma_u]$ generates $H_m(T, T_0)$ and determines an element

$$g_{\tilde{u}} \in H_m(\tilde{M} \times \tilde{M}, \tilde{M} \times \tilde{M} - \zeta^{-1}(\Delta)).$$

If $\tilde{u}\sigma = \tilde{u}_1$, then it is easy to see that $g_{\tilde{u}} = (\operatorname{sgn} \sigma) g_{\tilde{u}_1}$. The following lemma is easy to prove.

LEMMA 4.1. *Let U denote a connected open set in M. If U is non-orientable, any choice of local orientations leads to a function $g\colon U \to \mathscr{B}$ with the property* *that for* (x, y) *and* $(u, v) \in T_0 \cap (U \times U)$, *there exists a path* (α, β) *in* $T_0 \cap (U \times U)$ *from* (x, y) *to* (u, v) *such that*

$$(\alpha, \beta)_\# : \pi_{m-1}(F_{(x,y)}) \to \pi_{m-1}(F_{(u,v)})$$

takes g_x *to* g_u. *In the orientable case the result holds provided local orientations are chosen compatibility.*

Now, let (x, y), (u, v), (u', v') belong to $T_0 \cap (\text{int } L \times \text{int } L)$ and consider (x, y) as our base point with $\pi_m(F_{(x,y)})$ identified with $\mathbf{Z}[\pi]$, with g_x corresponding to $1 \in \pi$.

LEMMA 4.2. *Suppose* (α, β) *is any path from* (u, v) *to* (u', v'). *Suppose further that* (α_0, β_0), (α_1, β_1) *are paths in* T_0 *from* (x, y) *to* (u, v) *and from* (x, y) *to* (u', v'), *respectively, as in Lemma 4.1 (see Figure 1). Then, under the isomorphism of local groups*

$$(\alpha, \beta)_\# : \pi_{m-1}(F_{(u,v)}) \to \pi_{m-1}(F_{(u',v')})$$

we have

$$(\alpha, \beta)_\# g_u = (\text{sgn } \sigma)(\alpha_1, \beta_1)_\#(\tau\sigma^{-1})$$

where $(\alpha_0, \beta_0)_\# g_x = g_u$, $(\alpha_1, \beta_1)_\# g_x = g_u$, *and* $\sigma = \alpha_0 \alpha \alpha_1^{-1}$, $\tau = \beta_0 \beta \beta_1^{-1}$.

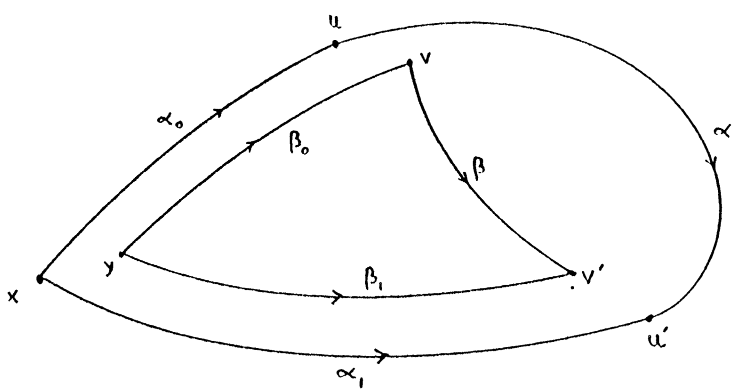

FIG. 1

Proof.

$$\begin{aligned}(\alpha, \beta)_\# g_u &= (\alpha, \beta)_\#(\alpha_0, \beta_0)_\# g_x \\ &= (\alpha_1, \beta_1)_\#(\alpha_1, \beta_1)_\#^{-1}(\alpha, \beta)_\#(\alpha_0, \beta_0)_\# g_x \\ &= (\alpha_1, \beta_1)_\#[g_x \circ (\sigma, \tau)] \\ &= (\text{sgn } \sigma)(\alpha_1, \beta_1)_\#(\tau\sigma^{-1}).\end{aligned}$$

Convention 4.3. If α is a path from u to u' and β is a path from v to v' where u is "close to" v and u' is "close to" v' in the sense that $(u, v) \cup (u', v') \subset T_0$, the statement $\alpha \sim \beta$ (mod endpoints) will mean that there is a homotopy from α to β: $H: I \times I \to M$ such that $H(0, t)$ and $H(1, t)$ trace paths, with $(u, H(0, t))$, $(u', H(1, t))$ in T_0. Alternatively, one may replace β by a path β' from u to u' with β' close to β and then $\alpha \sim \beta$ (mod endpoints) mean $\alpha \sim \beta'$ with endpoints fixed, as usual.

COROLLARY 4.4. *If in Lemma* 4.2, $\alpha \sim \beta$ (*mod endpoints*), *then*

$$(\alpha, \beta)_*(g_u) = (\operatorname{sgn} \sigma) g_u,$$

where $\sigma = \alpha_0 \alpha \alpha_1^{-1}$.

Let f: $U \to M$ denote a compactly fixed map with U connected and choose a base point $x_0 \notin \operatorname{Fix} f$. The local group of $\mathscr{B}(f)$ at x_0 is $\pi_{m-1}(F_b)$, where $b = (x_0, f(x_0))$. $\pi_{m-1}(F_b)$ is identified with $\mathbf{Z}[\pi]$ and the right action of $\pi(U) = \pi_1(U, x_0)$ on $\mathbf{Z}[\pi]$ is given by

$$\alpha \circ \sigma = \operatorname{sgn} \sigma i_U(\sigma^{-1}) \alpha \varphi_U(\sigma), \quad \sigma \in \pi(U), \quad \alpha \in \pi.$$

Define a new right action

$$(*) \qquad \alpha * \sigma = \varphi_U(\sigma^{-1}) \alpha i_U(\sigma), \quad \sigma \in \pi(U), \quad \alpha \in \pi.$$

Now, denote the twisting action of $\pi(U)$ on $\mathbf{Z}$ by

$$n \circ \sigma = (\operatorname{sgn} \sigma) n, \quad \sigma \in \pi(U), \quad n \in \mathbf{Z},$$

and consider the bilinear pairing P_0: $\mathbf{Z}[\pi] \otimes \mathbf{Z} \to \mathbf{Z}[\pi]$ defined by $\alpha \otimes n \mapsto n\alpha^{-1}$.

LEMMA 4.5. *Let* $\sigma \in \pi(U)$. *Then the pairing* P_0 *satisfies the condition*

$$P_0((\alpha \circ \sigma) \otimes (n \circ \alpha)) = P_0(\alpha \circ n) * \sigma$$

i.e. P_0 *is equivariant.*

Proof.

$$\begin{aligned} P_0(\alpha \circ \sigma \otimes n \circ \sigma) &= P_0(\operatorname{sgn} \sigma i_U(\sigma^{-1}) \alpha \varphi_U(\sigma) \otimes (\operatorname{sgn} \sigma) n) \\ &= n \varphi_U(\sigma^{-1}) \alpha^{-1} i_U(\sigma) \\ &= (n\alpha^{-1}) * \sigma \\ &= P_0(\alpha \circ n) * \sigma. \end{aligned}$$

Let $\mathscr{T}(U)$ denote the orientation sheaf of twisted integers over U. Then for $x \in U$, the Hurewicz homomorphism

$$h: \pi_m(M, M - x) \to H_m(M, M - x)$$

induces a coefficient homomorphism $h: \mathscr{B}(U) \to \mathscr{T}(U)$ where $\mathscr{B}(U) = \mathscr{B}(i)$ and $i: U \to M$ is inclusion. In particular, using as base point $x_0 \in U$, we may identify

$$\pi_{m-1}(F_b) \equiv \mathbf{Z}[\pi] \quad \text{with } g_{x_0} \mapsto 1,$$
$$H_m(M, M - x) \equiv \mathbf{Z} \quad \text{with } h(g_{x_0}) \mapsto 1.$$

COROLLARY 4.6. *Let $\mathscr{R}(f)$ denote the local system on U induced by the action* (*). *Then, P_0 induces a bilinear pairing $P: \mathscr{B}(f) \otimes \mathscr{T}(U) \to \mathscr{R}(f)$ so that over every $x \in U$,*

$$P(g_x \otimes h(g_x)) = 1.$$

Remark 4.7. Corollary 4.6 is valid for L a compact connected submanifold with boundary ∂L, $L \subset U$. In particular we have a corresponding pairing

$$P_L: \mathscr{B}(f, L) \otimes \mathscr{T}(L) \to \mathscr{R}(f, L)$$

where the local systems $\mathscr{B}(f, L)$, $\mathscr{T}(L)$, $\mathscr{R}(f, L)$ are restrictions from U to L.

Now, let L denote a compact, connected triangulated manifold with boundary ∂L such that $L \subset U$. Assume also that L is triangulated so that adjacent m-simplexes are contained in the same Euclidean neighborhood in U. L determines fundamental classes as follows:

If s is an oriented simplex of L and u_s is a point on ∂s, then using Lemma 4.1, the orientation of s determines an orientation around u_s and thereby an element $g_{u_s} \in \pi_{m-1}(F_b)$, $b = (u_s, v_s)$ and v_s is near u_s. Set $g_s = g_{u_s}$.

DEFINITION 4.4. The m-chain $\sum_s g_s s$, where the sum runs over a basis or oriented m-simplexes of $(L, \partial L)$, determines the homology class $\underline{\mu}(L; \pi) \in H_m(L, \partial L; \mathscr{B}(L))$, where $\mathscr{B}(L) = \mathscr{B}(i)$ is induced from $\mathscr{B}$ by $i \times i: L \to \bar{M} \times M$, which we call the *twisted π-fundamental homology class* of $(L, \partial L)$ in M.

Let $\underline{\mu}(L) \in H_m(L, \partial L; \mathscr{T}(L))$ denote the classical twisted integral homology class on $(L, \partial L)$ [7]. Since at the chain level $\underline{\mu}(L)$ has the form $\sum_\sigma h(g_s)s$, one sees that under the induced coefficient homomorphism $h_*: H_m(L, \partial L; \mathscr{B}(L)) \to H_m(L, \partial L; \mathscr{T}(L))$,

$$h_*: \underline{\mu}(L; \pi) \mapsto \underline{\mu}(L).$$

The corresponding dual fundamental cohomology is defined as follows:

DEFINITION 4.5. Let s denote an oriented m-simplex of $(L, \partial L)$. The m-cochain

$$c_s(s') = \begin{cases} g_s & \text{if } s' = s \\ 0 & \text{if } s' \neq s \end{cases}$$

leads to a cohomology class $\bar{\mu}(L; \pi) \in H^m(L, \partial L; \mathscr{B}(L))$ called the twisted π-fundamental cohomology class of $(L, \partial L)$ in M.

Remark 4.6. Using Lemma 4.2 one shows easily that $\bar{\mu}(L; \pi)$ is independent of s, i.e. for $s \neq s'$, c_s and $c_{s'}$ are cohomologous. Also, if we let $\bar{\mu}(L) \in H^m(L, \partial L; \mathscr{T}(L))$ denote the classical twisted (over **Z**) cohomology class [7] $\bar{\mu}(L, \pi)$ maps to $\bar{\mu}(L)$, via

$$h^*: H^m(L, \partial L; \mathscr{B}(L)) \to H^m(L, \partial L; \mathscr{T}(L)).$$

PROPOSITION 4.7. $\langle \bar{\mu}(L, \pi), \underline{\mu}(L) \rangle = [1] \in R[i_U, i_U]$.

Proof. Fix a simplex s and a base point $u_s \in \partial s$. Then,

$$c_s\left(\sum_{s'} h(g_{s'})s' \right) = \Gamma_0(c_s(s) \otimes h(g_s)) = \Gamma_0(g_s \otimes h(g_s)).$$

Therefore,

$$\bar{\mu}(L, \pi) \cap \underline{\mu}(L) = [1 \cdot u_s] \in H_0(L; \mathscr{R}(i))$$

where the cap product is induced by the pairing

$$\mathscr{B}(L) \otimes \mathscr{T}(L) \to \mathscr{B}(L)$$

where $\mathscr{R}(L) = \mathscr{R}(i)$. But, under the isomorphism $H_0(L; \mathscr{R}(L)) \equiv \mathbf{Z}R[i_U, i_U]$, $[1 \cdot u_s]$ corresponds to [1], the Reidemeister class in $R[i_U, i_U]$ containing $1 \in \pi$. Therefore,

$$\langle \bar{\mu}(L, \pi), \underline{\mu}(L) \rangle \equiv \bar{\mu}(L, \pi) \cap \underline{\mu}(L) = [1].$$

These fundamental classes pass to U is the usual fashion as follows. First, if L_0 denotes L minus a small 'collar" around the boundary, then the image of $\bar{\mu}(L; \pi)$ under

$$H^m(L, \partial L; \mathscr{B}(L)) \xrightarrow{\approx} H^m(U, U - L_0, \mathscr{B}(L)) \to H_c^m(U; \mathscr{B}(U))$$

determines $\bar{\mu}(U; \pi) \in H_c^m(U; \mathscr{B}(U))$, the twisted π-fundamental cohomology class of U. Furthermore, if $\mathscr{A}$ is the family of compact, connected manifolds L with boundary ∂L such that $L \subset U$, one can choose a compatible $\mathscr{A}$ family [8]

$$\underline{\mu}(U; \pi) = \{\underline{\mu}(L; \pi) \in H_m(L, \partial L; \mathscr{B}(L)) \equiv H_m(U, U - L_0; \mathscr{B}(U))\}$$

and call $\underline{\mu}(U; \pi)$, the twisted π-fundamental homology class of U. In a similar fashion, a compatible $\mathscr{A}$ family

$$\underline{\mu}(U) = \{\underline{\mu}(L) \in H_m(L, \partial L; \mathscr{T}(L))\}$$

determines the twisted fundamental class (up to sign) of U.

Finally, for any compactly fixed f: $U \to M$, the pairing

$$P: \mathscr{B}(f) \otimes \mathscr{T}(U) \to \mathscr{R}(f)$$

induces a Kronecker product

$$H_c^m(U; \mathscr{B}(f)) \xrightarrow{\langle\ ,\underline{\mu}(U)\rangle} \mathbf{Z}R[i_U, \varphi_U]$$

induced by

$$H^m(L, \partial L; \mathscr{B}(f, L)) \xrightarrow{\langle \cdot, \underline{\mu}(L)\rangle} ZR[i_L, \varphi_L] \xrightarrow{h_L^U} ZR[i_U, \varphi_U]$$

where $\mathscr{B}(f, L)$ is $\mathscr{B}(f)$ restricted to L.

Remark 4.8. A simple direct argument (without invoking duality) shows that

$$\langle \cdot, \underline{\mu}(L)\rangle: H^m(L, \partial L; \mathscr{B}(f, L)) \to ZR[i_L, \varphi_L]$$

is an isomorphism.

5. Calculating the local obstruction index $o(f)$

We assume again the data (M, f, U) of 2.3, with the added assumption that U is connected. We also assume that K is a compact manifold with boundary and $\text{Fix}\, f \subset \text{int}\, K$. Our immediate objective is to compute the local obstruction index $o(f, K) \in H^m(K, \partial K; \mathscr{B}(f, K))$ of f on K (Definition 2.5). We focus our attention first on one of the components L of K and then $o(f, K)$ will be computed in terms of its components $o(f, L) \in H^m(L, \partial L; \mathscr{B}(f, L))$. Thus our immediate objective is to prove, using the notation in Section 4, the following result.

THEOREM 5.1. *Suppose $f: U \to M$ is a compactly fixed map and L a connected compact submanifold with boundary ∂L such that $L \subset U$ and* $(\text{Fix}\, f) \cap \partial L = \phi$. *If $o(f, L)$ is the local obstruction index of f on L in U, then using the pairing* (Section 4) $\mathscr{B}(f, L) \otimes \mathscr{T}(L) \to \mathscr{R}(f, L)$, *under the isomorphism*

$$\langle \cdot, \underline{\mu}(L)\rangle: H^m(L, \partial L; \mathscr{B}(f, L)) \to ZR[i_L, \varphi_L],$$

we have

$$\langle o(f, L), \underline{\mu}(L)\rangle = \sum_{\rho \in R} I(\rho)\rho$$

where $R = R[i_L, \varphi_L]$ is the set of Reidemeister classes and $I(\rho)$ is the index of the Nielsen class corresponding to ρ under the map $\Gamma: R[i_L, \varphi_L] \to \mathscr{N}(f, L)$ of Proposition 3.9.

Before, giving the proof of Theorem 5.1, we prove a succession of lemmas. Some of these closely parallel corresponding ones in the global case [1] so we may omit some details.

We assume now (without loss of generality), in addition to the previous data that Fix $(f) \cap L$ is finite and each fixed point lies in the interior of a maximal simplex of a triangulation of L. Furthermore, each such simplex s is contained in a Euclidean neighborhood V_s and if Fix $(f) \cap s \neq \phi$, then $f(s) \subset V_s$.

Consider the section $u = u_L: L - \text{Fix}\, f \to E(f)$ given by

$$u(y) = (\bar{y}, \overline{f(y)}), \quad y \in L - \text{Fix}\, f.$$

Thus, the cochain $c(f, L) \in C^m(L, \partial L; \mathscr{B}(f, L))$, representing the obstruction $o(f, L)$ is given by the following: If s is an oriented m-cell, then

$$c(f, L)(s) = \begin{cases} 0 & \text{if } s \cap \operatorname{Fix} f = \phi \\ [\varphi_s] \in \pi_{m-1}(q^{-1}(u_s)) & \text{otherwise} \end{cases}$$

where $u_s \in \partial s$, and when (D^m, S^{m-1}, a_0) and $(s, \partial s, u_s)$ are identified, preserving orientations,

$$\varphi_s(u) = (\overrightarrow{uu_s}, (\overrightarrow{f(u)f(v_s)}))$$

where uv is the directed line segment from u to v (see Figure 2). As noted in [1],

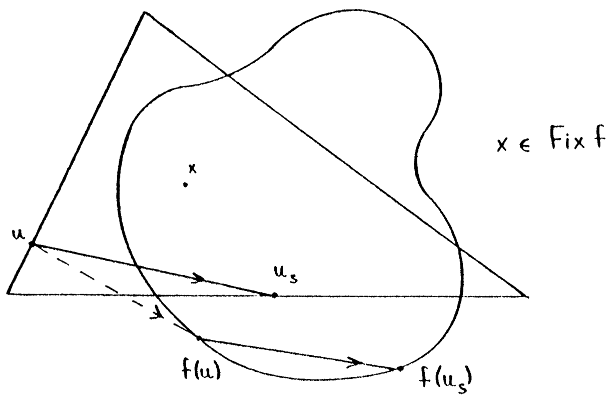

FIG. 2

a simple homotopy argument shows that if we let $\delta_s \colon \partial s \to (M, M - x)$ be given by

$$\delta_s(u) = f(u) - u$$

where $V_s \equiv R^m$ and $x \equiv 0$, $x \in \operatorname{Fix} f \cap (\operatorname{int} s)$, then if we let $\gamma_s = \delta_s + u_s$ (translation by u_s), we have

$$c(f, L)(s) = \begin{cases} 0 & \text{if } s \cap \operatorname{Fix} f = \phi \\ [\gamma_s] & \text{otherwise,} \end{cases}$$

where

$$[\gamma_s] \in \pi_m(M, M - u_s, f(u_s)) \approx \pi_m(M \times M, M \times M - \Delta, \ (u_s, f(u_s)))$$
$$\approx \pi_{m-1}(F_{(u_s, f(u_s))}).$$

Thus, since $u - f(u)$ determines the (numerical) local index $I(f, x)$ at x, we have

$$c(f, L)(s) = \begin{cases} 0 & \text{if } s \cap \operatorname{Fix} f = \phi \\ (-1)^m \operatorname{Ind}(f, x) g_\sigma & \text{otherwise.} \end{cases}$$

Thus, we have the following proposition.

LEMMA 5.2. *The local obstruction index $o(f, L)$ has the cochain representation*

$$c(f, L) = (-1)^m \sum_s [I(f, s)g_s]s$$

where $I(f, s)$ is the local index of f on s.

Remark 5.3. The unhappy sign $(-1)^m$ is the result of using $i \times f\colon U \to M \times M$, rather than $f \times i$; thus encountering $f -$ id, rather than id $- f$.

Let $\mathcal{N}(f, L)$, denote the local Nielsen classes of $f \mid L$, designated individually by $N_1(f, L), \ldots, N_j(f, L), \ldots$. For each j pick a simplex s_j containing a fixed point representing $N_j(f, L)$. If s is another simplex containing a fixed point of $N_j(f, L)$, then there is a path α from s to s_j such that $\alpha \sim f(\alpha)$. Thus, since $g_s s$ to cohomologus to

$$[\operatorname{sgn}(\alpha, s, s_j)(\alpha, f(\alpha))_\#(g_s)]s_j$$

and since $(\alpha, f(\alpha))_\#(g_s) = \operatorname{sgn}(\alpha, s, s_j)g_{s_j}$, we have:

PROPOSITION 5.4. *The local obstruction index $o(f, L)$ has the cochain representation*

$$c'(f, L) = (-1)^m \sum_j [I(N_j(f, L))g_{s_j}]s_j$$

where the sum is over the local Nielsen classes $\mathcal{N}(f, L)$ and $I(N_j(f, L))$ is the (numerical) index of $N_j(f, L)$.

COROLLARY 5.5. (Local Wecken Theorem). *A necessary and sufficient condition that $f \mid L$ be deformable in M (relative to ∂L) to a fixed point free map is that the local Nielsen number $n(f, L) = 0$, i.e. $n(f, L) = 0 \Leftrightarrow o(f, L) = 0$.*

Now, choose a simplex s_1 in L and assume that our base point is $u_1 \in \partial s_1$ and we identify $\pi_{m-1}(F_{(u_1, f(u_1))})$ with $\mathbf{Z}[\pi]$, g_{s_1} corresponding to 1. See Figure 3.

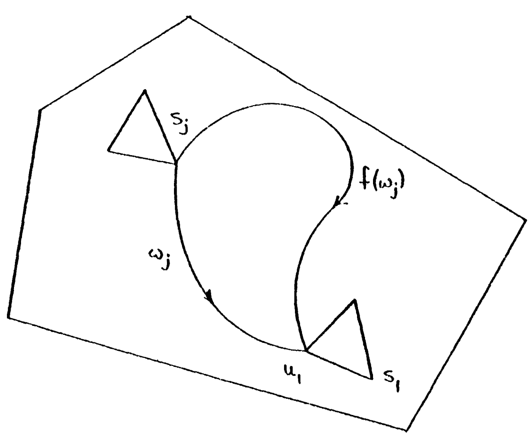

FIG. 3

Choose for each j, a path ω_j in L such that

$$(\omega_j, \omega_j)_\#(g_{s_j}) = g_{s_1}.$$

Then, $g_{s_j}s_j$ is cohomologus to $[\operatorname{sgn}(\omega_j, s_j, s)(\omega_j, f(\omega_j))_\#(g_{s_j})]s_1$ where, by Lemma 4.2,

$$(\omega_j, f(\omega_j))_\#(g_{s_j}) = \operatorname{sgn} \sigma_j(\tau_j \sigma_j^{-1})$$

with $\sigma_j = [\omega_j^{-1}\omega_j]$, $\tau_j = [\omega_j^{-1}f(\omega_j)]$. Since $\sigma = 1$ and $\operatorname{sgn}(\omega_j, s_j, s) = 1$, we have $g_{s_j}s_j$ cohomologous to $\tau_j s_1$ where $\tau_j = [\omega_j^{-1}f(\omega_j)]$. See Figure 3.

LEMMA 5.6. *The local obstruction index $o(f, L)$ has the following cochain representation concentrated at s_1 where the local group at s_1 is identified with* $\mathbf{Z}[\pi]$:

$$c''(f, L) = (-1)^m\left(\sum_j I(N_j(f, L))\tau_j\right)s_1$$

where $\tau_j \in \pi$ is given by $\tau_j = [\omega_j^{-1}f(\omega_j)]$ for an appropriate path ω_j from the Nielsen class $N_j(f, L)$ to the Nielsen class $N_1(f, L)$.

LEMMA 5.7. *If x_s and x_t are fixed points of $f \mid L$ in simplexes s and t, respectively and if ω_s, and ω_t are paths from s to s_1 and t to s_1 such that*

$$(\omega_s, \omega_s)_\# g_s = g_{s_1}, \ (\omega_t, \omega_t)_\# g_t = g_{s_1}$$

then x_s and x_t are Nielsen equivalent in L if, and only if,

$$\tau_s^{-1} = [f(\omega_s^{-1})\omega_s] \quad and \quad \tau_t^{-1} = [f(\omega_t^{-1})\omega_t]$$

are Reidemeister equivalent on L, i.e.

$$\tau_s = \varphi_L(\sigma^{-1})\tau_t i_L(\sigma), \quad \sigma \in \pi(L).$$

Proof. By the argument preceding Lemma 5.6, we have $(\omega_s, f(\omega_s)) \times {}_\#(g_s s) = \tau_s = [\omega_s^{-1}f(\omega_s)]$. Suppose γ is a path in L from s to t with $\gamma \sim f(\gamma)$. Then,

$$\tau_s^{-1} = [f(\omega_s^{-1})\omega_s] = [f(\omega_s^{-1})f(\gamma)f(\omega_t)f(\omega_t^{-1})\omega_t\omega_t^{-1}\gamma^{-1}\omega_s] = \varphi_L(\sigma^{-1})\tau_t i_L(\sigma),$$

where $\sigma = [\omega_t^{-1}\gamma^{-1}\omega_s] \in \pi(L)$.

LEMMA 5.8. *Let Γ denote the correspondence of Proposition 3.9 from the Reidemeister classes $R[i_L, \varphi_L]$ to the Nielsen classes $\mathcal{N}(f, L)$. Then, if $\tau_j = [\omega_j^{-1}f(\omega_j)]$, as in Proposition 5.6, we have $\Gamma([\tau_j^{-1}]) = N_j(f, L)$*

Proof. Let x_j denote the fixed point in s_j, and x_1 the fixed point in x_1. Use x_1 as base point and then apply part (b) of the proof of Proposition 3.6.

If Γ: $R[i_L, \varphi_L] \to \mathcal{N}(f, L)$ is the correspondence of Proposition 3.9, between Reidemeister classes and Nielsen classes, then we set $N_\rho = \Gamma(\rho)$. Also, we set

$I(\rho) = I(N_\rho)$, the index of the corresponding Nielsen class. Of course, if $\Gamma(\rho) = \phi$, we set $I(\rho) = 0$.

We can only give a short proof of Theorem 5.1.

Proof of Theorem 5.1. By Lemma 5.6,

$$\left\langle c''(f, L), \sum_s h(g_s)s \right\rangle = (-1)^m \sum_j I(N_j(f, L))\tau_j^{-1} \in \mathbf{Z}[\pi].$$

Passing to Reidemeister classes on the right, we obtain

$$\langle o(f, L), \underline{\mu}(L) \rangle = (-1)^m \sum_\rho I(\rho)\rho \in \mathbf{Z}R[i_L, \varphi_L].$$

COROLLARY 5.9. *Let f: $U \to M$ be compactly fixed and let $K = \coprod L_j$, a finite disjoint union of connected submanifolds with boundary. Then under the isomorphism*

$$\begin{array}{ccc} \sum_j \langle \cdot, \underline{\mu}(L_j) \rangle : H^m(K, \partial K; \mathscr{B}(f, K)) & \approx & \sum_j H^m(L_j, \partial L_j; \mathscr{B}(f, L_j)) \\ & & \downarrow \\ \mathbf{Z}R[i_K, \varphi_K] & \approx & \sum_j \mathbf{Z}R[i_{L_j}, \varphi_{L_j}], \end{array}$$

we have

$$\langle o(f, K), \sum \underline{\mu}(L_j) \rangle = \sum_j \sum_{\rho \in R_j} I(\rho)\rho$$

where $R_j = R[i_{L_j}, \varphi_{L_j}]$.

COROLLARY 5.10. (Global case). *Let f: $M \to M$ denote a self map of a compact, connected manifold with boundary ∂M such that* (Fix f) $\cap$ $\partial M = \phi$. *Then the global obstruction index*

$$o(f) \in H^m(M, \partial M; \mathscr{B}(f))$$

is given by

$$\langle o(f), \underline{\mu}(M) \rangle = \sum_{\rho \in R} I(\rho)\rho$$

where $R = R[\mathrm{id}, \varphi]$ and $\varphi = f_*$: $\pi \to \pi = \pi_1(M)$.

COROLLARY 5.11. *Let f: $U \to M$ be compactly fixed. Suppose K is a compact submanifold with boundary such that $K \subset U$,* Fix $f \subset$ int K *and the Nielsen classes $\mathscr{N}(f, U)$ and $\mathscr{N}(f, K)$ are identical. (The existence of such a K is guaranteed by Proposition* 3.14). *Then $o(f) = 0$ if, and only if, $o(f, K) = 0$.*

Proof. The "if part" is obvious. On the other hand suppose $o(f) = 0$. Then for some K', $K \subset K' \subset U$ we have $o(f, K') = 0$ and hence

$$0 = \langle o(f, K'), \underline{\mu}(K') \rangle = \sum_{\rho \in R'} I(\rho)\rho;$$

thus, $I(\rho) = 0$ for all Reidemeister classes in $R' = R[i_{K'}, \varphi_{K'}]$. Consequently, all the Nielsen classes in K' have index 0. This forces all the Nielsen classes of f relative to K to be inessential and thus

$$\langle o(f, K), \underline{\mu}(K)\rangle = \sum_{\rho \in R} I(\rho)\rho = 0$$

therefore, $o(f, K) = 0$.

THEOREM 5.12. *Suppose $f\colon U \to M$ is compactly fixed with U connected. Then, under the isomorphism*

$$\langle \cdot, \underline{\mu}(U)\rangle\colon H^m_c(U; \mathscr{B}(f)) \to \mathbf{Z}R[i_U, \varphi_L]$$

we have

$$\langle o(f), \underline{\mu}(U)\rangle = \sum_{\rho \in R} I(\rho)\rho$$

where $R = R[i_U, \varphi_U]$.

Proof. Choose a connected K satisfying the condition of Corollary 5.11. Let

$$h^U_K\colon R' = R[i_K, \varphi_K] \to R[i_U, \varphi_U] = R$$

denote the correspondence in Section 3. Then,

$$\langle o(f), \underline{\mu}(U)\rangle = h^U_K \langle o(f, K), \underline{\mu}(K)\rangle = h^U_K\left(\sum_{\rho \in R'} I(\rho)\rho\right) = \sum_{\rho \in R} I(\rho)\rho.$$

COROLLARY 5.13. *Suppose $f\colon U \to M$ is compactly fixed. Then f is deformable, via a compactly fixed homotopy, to a fixed point free map $g\colon U \to M$ if, and only if, the local Nielsen number $n(f, U) = 0$.*

Suppose now that $f\colon U \to M$ as usual, $L = \coprod_j L_j \subset K \subset U$ such that $\operatorname{Fix} f \subset \coprod_j (\operatorname{int} L_j)$, and L_j, K are connected submanifolds with boundary. We now want to describe how $o(f, L)$ in $H^m(L, \partial L; \mathscr{B}(f, L))$ "coalesces" to $o(f, K)$ in $H^m(K, \partial K; \mathscr{B}(f, K))$ thus yielding the appropriate "additivity property" for our generalized local index. We make use of the correspondences (Section 3)

$$h^K_{L_j}\colon R[i_{L_j}, \varphi_{L_j}] \to R[i_K, \varphi_K].$$

LEMMA 5.14. *If $\rho \in R[i_K, \varphi_K]$ and $I(\rho)$ is its numerical index, then*

$$I(\rho) = \sum_j \sum_{\beta \in P_j} I(\beta)$$

where $P_j = \{\beta\colon h^K_{L_j}(\beta) = \rho\}$.

Proof. Let $N_\beta(L_j, f)$ denote the Nielsen class in $\mathscr{N}(L_j, f)$ corresponding to $\beta \in P_j$, and $N(\rho)$ the Nielsen class in $\mathscr{N}(K, f)$ corresponding to ρ. It suffices to prove that

$$\coprod_j \coprod_{\beta \in P_j} N_\beta(L_j, f) = N(\gamma).$$

Recall (Section 3) that given a fixed point $x \in N(\rho)$, the Reidemeister class ρ is determined by the element $\alpha \in \pi$ subject to the condition

$$(\tilde{f}_K \alpha)(\tilde{x}) = \tilde{\imath}_K(\tilde{x})$$

where $\tilde{x} \in \eta_K^{-1}(x)$. Such an x belongs to some L_j and hence to some Nielsen class $N_\beta(L_j, f)$ where β is the L_j-Reidemeister class belonging to $N_\beta(L_j, f)$. We need to show that $h_{L_j}^K(\beta) = \rho$. Or, equivalently that α also represents β. Choose $\tilde{x} = \tilde{\imath}_{L_j}^K(\tilde{y})$ and then

$$(\tilde{f}_L \alpha)(\tilde{y}) = (\tilde{\imath}_{L_j}^K \tilde{f}_K \alpha)\tilde{y} = \tilde{\imath}_L(\tilde{y});$$

thus α does represent β, and hence

$$N(\gamma) \subset \coprod_j \coprod_{\beta \in P_j} N_\beta(L_j, f).$$

The reverse inclusion has a similar argument and is omitted.

The following theorem is a consequence of Lemma 5.14.

THEOREM 5.15 (Additivity). *Let $f: U \to M$ be compactly fixed and suppose $V = \coprod_j V_j$ is a disjoint union of open sets in U covering* Fix f. *We identify*

$$o(f, U) \equiv \sum_{\rho \in R} I(\rho)\rho, \quad o(f, V_j) \equiv \sum_{\beta \in R_j} I(\beta)\beta$$

where $R = R[i_U, \varphi_U]$, $R_j = R[i_{V_j}, \varphi_{V_j}]$. Then, under the correspondence

$$h_V^U: R[i_V, \varphi_V] \to R[i_U, \varphi_U],$$

we have

$$o(f, V) \equiv \sum_j \sum_{\beta \in R_j} I(\beta)\beta \to \sum_{\rho \in R} \left(\sum_j \sum_{\beta \in P_j(\rho)} I(\beta) \right) \rho \equiv o(f, U)$$

where $P_j(\rho) = \{\beta: h_{V_j}^U(\beta) = \rho\}$.

Remark 5.16. When M is 1-connected, Theorem 5.15 reduces to

$$I(f, K) = \sum_j I(f, L_j)$$

the "addivity property" of the classical (numerical) local index.

The next result is another application of Theorem 5.1.

THEOREM 5.17. *Suppose $f: M \to M$ is a compactly fixed map on a connected manifold with boundary such that* $(\text{Fix} f) \cap \partial M = \phi$. *Suppose K is a connected submanifold with boundary and* $\text{Fix} f \subset \text{int } K$. *If $i_K: \pi(K) \to \pi$ is surjective then*

(a) *$h_K^M: R[i_K, \varphi_K] \to R[i_M, \varphi_M]$ is bijective,*
(b) *$\mathcal{N}(f, K) \equiv \mathcal{N}(f, M) = \mathcal{N}(f)$,*
(c) *$n(f, K) = n(f, M) = n(f)$,*
(d) *$o(f, K) = 0$ if, and only if, $o(f, M) = 0$.*

Proof. Part (a) is a simple exercise which establishes a one-one correspondence between Nielsen classes relative to K and Nielsen classes relative to M. Then (d) is an immediate consequence of Theorem 5.1.

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