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BOREL MEASURES ON COMPACT GROUPS ARE MEAGER

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One may ask whether it is possible that the Haar measure m on an infinite compact group G is a category measure in the sense of Oxtoby [9], [10]. That is, is it possible to find a group G such that if F is a meager subset then $m(F) = 0$? Any positive Borel measure on G with this property with respect to meager sets is called a *residual* Borel measure on G [3]. Those residual Radon measures on G with support equal to G are category measures. The signed residual Borel measures ν (with $|\nu|$ residual) form a band in $M(G)$ the vector lattice of Borel measures of bounded variation of G. The complementary band in $M(G)$ consists of those ν such that $|\nu|$ assigns full mass to a set of the first category [13]. This band will be called the band of *meager* Borel measures. Since the bands of residual and meager Borel measures are translation invariant the greatest meager minorant of Haar measure is either 0 or a multiple of Haar measure. In the first case Haar measure is a category measure and in the second case it is a meager measure. We will show that Haar or any other Borel measure is always meager.

If X_1 and X_2 are compact spaces and ν is an element of $M(X_1 \times X_2)$ then ν_{X_1} denotes the *marginal* measure of ν on X_1 given by the formula $\nu_{X_1}(A) = \nu(A \times X_2)$ for A a Borel set in X_1. If there is a meager $F \subset X_1$ with $|\nu_{X_1}|(F) = |\nu_{X_1}|(X_1)$ then $F \times X_2$ is meager in $X_1 \times X_2$ with $|\nu|(F \times X_1) = |\nu|(X_1 \times X_2)$. Thus we have the following lemma.

LEMMA 1. *Let X_1 and X_2 be compact.*

(a) *If ν is a Borel measure on $X_1 \times X_2$ with meager marginal ν_{X_1} then ν is meager.*

(b) *If all Borel measures on X_1 are meager then all Borel measures on $X_1 \times X_2$ are meager.*

Any separable perfect space X has the property that all Borel measures are meager, [3, page 78], or [9]. If G is an infinite compact group which is separable then all Borel measures on G are meager. If G is non-separable it has an infinite separable closed subgroup H. Since G is (topologically) the product $H \times (G/H)$ we have the following proposition as a consequence of Lemma 1.

PROPOSITION 2. *All elements of $M(G)$ are meager when G is an infinite compact group.*

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Remark. When H is a separable closed subgroup of G then we may replace G with G/H in Proposition 2 with little difficulty. To obtain this for any H (in Proposition 4) we must do a little more work.

We recall [12] that a compact Hausdorff space is Stonian or extremally disconnected iff it is the Stone space of a complete Boolean algebra or equivalently its clopen sets are a complete Boolean algebra and form a base of the topology. The Stone space X_Σ of the Boolean algebra Σ of regular open sets in the compact space X is called the Gleason space of X [1], [3]. X_Σ admits an irreducible continuous surjection $\pi_\Sigma: X_\Sigma \to X$ so that if $Y \subset X_\Sigma$ is closed with $\pi_\Sigma(Y) = X$ then $Y = X_\Sigma$. If Z is Stonian and $\pi: Z \to X$ is a continuous surjection then a continuous surjection $\pi': Z \to X_\Sigma$ exists such that $\pi = \pi_\Sigma \circ \pi'$. Thus, X_Σ is the minimal Stonian space admitting X as a continuous image. A Stonian space X is said to be hyperstonian iff every non-empty open subset contains the support of a *normal* measure of Dixmier (which is defined to be a residual Radon measure). There is a bijective correspondence between residual Radon (Borel) measures on X and on X_Σ so X_Σ is hyperstonian iff every nonempty open set in X contains the support of a residual Radon measure. Every measure on X_Σ is meager iff the same is true for X. If X admits a category measure ν so does X_Σ and X_Σ is homeomorphic with the maximal ideal space, Z_ν, of $L^\infty(X, \nu)$. If ν is a Radon measure on X there is, a la Hahn-Banach, a measure ν_Σ on X_Σ whose image under π_Σ is ν. One has supp $(\nu) = X$ iff supp $(\nu_\Sigma) = X_\Sigma$. In this case there is a continuous surjection π_ν from Z_ν onto X dual to the inclusion $\mathscr{C}(X) \subset L^\infty(X, \nu)$. There is a continuous surjection $\pi': Z_\nu \to X_\Sigma$. If $Y_\nu \subset Z_\nu$ is a minimal closed subset of Z_ν so that $\pi'(Y_\nu) = X_\Sigma$ (hence π' and π_ν are irreducible on Y_ν) there is a measure $\tilde{\nu}$ on Y_ν whose image under π' is ν_Σ. Since π' is irreducible π' is a homeomorphism, [12, 24.2.10]. Thus, X_Σ may be considered as a subset of Z_ν. Any $f \in \mathscr{C}(X)$ may be regarded as an element of $L^\infty(X, \nu) = \mathscr{C}(Z_\nu)$. We consider ν_Σ as a Radon measure with support X_Σ.

PROPOSITION 3. *Let X be a compact Hausdorff space and let ν be a positive Radon measure with* supp $(\nu) = X$. *The Gleason space X_Σ is a subset of Z_ν. There is a measure ν_Σ on X_Σ such that if $\tilde{\nu}$ is the usual measure on Z_ν corresponding to ν, considering ν as an element of $L^{\infty *}(X, \nu)$, then $\int_{X_\Sigma} f\, d\nu_\Sigma = \int_{Z_\nu} f\, d\tilde{\nu}$ for all $f \in \mathscr{C}(X)$ regarded as elements of $\mathscr{C}(Z_\nu)$.*

Suppose that $Z_\nu = X_\Sigma$ in Proposition 2. Then, $\nu_\Sigma = \tilde{\nu}$ so ν_Σ is a category measure on X_Σ and ν is a category measure on X. On the other hand, if ν is a category measure on X then the ideals of meager Borel sets and of ν-negligible Borel sets are identical. $\mathscr{C}(X_\Sigma)$ is identified in [3] and [12] as the Banach lattice of bounded Borel functions modulo Borel functions with meager support hence with $L^\infty(X, \nu)$. That is, the inclusion $\mathscr{C}(X_\Sigma) \subset L^\infty(X, \nu)$ dual to π' is a Banach lattice isomorphism. Thus, π' is a homeomorphism so we may write $X_\Sigma = Z_\nu$. We summarize these remarks, which are folklore, in this corollary.

COROLLARY 3.1. *If ν is a positive Radon measure with* supp $(\nu) = X$ *the following are equivalent.*

(a) *ν is a category measure.*

(b) $Z_\nu = X_\Sigma$.

(c) *X_Σ is hyperstonian.*

If m is Haar measure on the compact group G it induces the marginal measure $m_{G/H}$ on the coset space G/H of a closed group H under the quotient mapping. We have supp $(m_{G/H}) = G/H$.

Efimov in [4] has shown that the Gleason space of the quotient G/H of a compact group G with a closed subgroup H is the Gleason space of $\{0, 1\}^{\mathscr{T}}$ for a cardinal number $\mathscr{T}$ which is infinite if G/H is infinite. In [3], it is shown that the Gleason space of $\{0, 1\}^{\mathscr{T}}$ admits no residual Borel measures. Thus, we have the following proposition and corollary.

PROPOSITION 4. *If H is a closed subgroup of the compact group G and G/H is infinite then G/H has no nontrivial residual Borel measures.*

COROLLARY 4.1. *$(G/H)_\Sigma$ is a Stonian, non-hyperstonian subset of $Z_{m_{G/H}}$.*

Since $(G/H)_\Sigma$ isn't hyperstonian neither is G/H. In fact G isn't Stonian or even basically disconnected [6] (i.e. ω-extremally disconnected [12]). On page 76 of [3] it is shown that if X and Y are infinite compact Hausdorff spaces then $X \times Y$ isn't basically disconnected. Thus, if $X \times Y$ is basically disconnected one factor is finite and the other basically disconnected.

PROPOSITION 5. *No infinite compact group G is basically disconnected.*

Proof. Let $G = G_0$ be a basically disconnected compact group. By Theorem 7.5 of [8] there is a compact open proper normal subgroup G_1 of G_0. Since G_1 is infinite G_0/G_1 is finite and G_1 is basically disconnected. If $G_0, \ldots, G_n$ have been defined so that G_k is a proper clopen normal subgroup of G_{k-1} with G_{k-1}/G_k finite for $k \leq n$, find G_{n+1}, a proper clopen normal subgroup of G_n with G_{n+1}/G_n finite. Thus the sequence $\{G_n\colon n = 0, 1, 2 \ldots\}$ is defined inductively. Let G_∞ be $\bigcap_{n=1}^{\infty} G_n$. G_∞ is a closed normal subgroup of G with G/G_∞ infinite. Thus, G/G_∞ is basically disconnected and G_∞ is finite. We may write G as the topological product $G_\infty \times \prod_{i=1}^{\infty} (G_i/G_{i-1})$. Since this may in turn be written as $X_1 \times X_2$ with X_1 and X_2 infinite, we have a contradiction. Thus, G isn't basically disconnected. ■

If G is a compact group and H is a closed subgroup with G/H infinite is it possible that G/H be basically disconnected? If this is possible it may be assumed that whenever H' is a proper closed subgroup of G properly containing H then H is not normal in H' nor is H' normal in G. Furthermore one may assume that G is countably generated over H. In this case it may be assumed that either G is singly generated over H so is the smallest closed subgroup, $H(g)$, containing H and g in $G\backslash H$ or that there is a sequence $\{g_n: n \in N\}$ so that, if $H(g_1, \ldots, g_n)$ is the closed subgroup generated by H and $(g_1, \ldots, g_n)$, then $g_{n+1} \notin H(g_1, \ldots, g_n)$, $H(g_1, \ldots, g_{n+1})/H(g_1, \ldots, g_n)$ is finite for all n and $G = H(g_1, \ldots, g_n, \ldots)$. If there were an H and G with G/H Stonian then it would follow that there is a unique invariant, finitely additive, residual, Borel probability measure on G/H. This is obtained by restricting $m_{G/H}$ to the clopen algebra of G/H and then extending to the Borel algebra by assigning measure 0 to meager Borel sets. This follows from Proposition 12 of [1] and the uniqueness of the measure $m_{G/H}$ invariant under the action of G on G/H.

G/H is a homogeneous space hence is topologically homogeneous. If G/H is infinite and separable, the minimum cardinality, w, of the base for the topology of G/H is between $\aleph_0$ and $2^{\aleph_0}$ [4, page 246]. If G/H were extremally disconnected then, by Corollary 6 of [4], $w = w^{\aleph_0}$ so $w = 2^{\aleph_0}$. Since G/H is topologically homogeneous, Theorem 5 of [4], due to Arhangelskii, implies that $w = 2^{\aleph_0}$ is impossible. Thus G/H, if separable, can't be extremally disconnected.

PROPOSITION 6. *Let H be a closed subgroup of the compact group G with G/H extremally disconnected and infinite.*

(a) *G/H is not separable.*

(b) *H is not separable.*

(c) *There are closed subgroups $H_1 \subset H_2$ of G so that $H \subset H_1$; H_1/H and G/H_2 are finite; when g is in $G\backslash H_1$, the closed group, $H_1(g)$, generated by g and H_1 has $H_1(g)/H_1$ infinite; and there is a $g \in H_2\backslash H_1$ with $H_2 = H_1(g)$.*

Proof. (a) This is already established.

(c) If there is no $g \in G\backslash H$ such that $H(g)/H$ is finite, set $H_1 = H$ and $H_2 = H(g)$ for some $g \in G\backslash H$. Otherwise assume that $\{g_1, \ldots, g_n\} \subset G\backslash H$ have been found so that if $H(g_1, \ldots, g_j)$ is the closed group generated by H and $\{g_1, \ldots, g_j\}$ then

$$H(g_1, \ldots, g_j) \neq H(g_1, \ldots, g_{j+1}) \quad \text{and} \quad H(g_1, \ldots, g_{j+1})/H(g_1, \ldots, g_j)$$

is finite for $j < n$. If there is a $g_{n+1} \in G\backslash H(g_1, \ldots, g_n)$ with the property that

$$H(g_1, \ldots, g_{n+1})/H(g_1, \ldots, g_n)$$

is infinite, set

$$H_1 = H(g_1, \ldots, g_n) \quad \text{and} \quad H_2 = H(g_1, \ldots, g_{n+1}).$$

Otherwise pick $g_{n+1} \in G\backslash H(g_1, \ldots, g_n)$ with $H(g_1, \ldots, g_{n+1})/H(g_1, \ldots, g_n)$ finite. This inductive process must terminate with the definition of H_1 and H_2 at some finite stage. If not let H_3 be the closure of $\bigcup_{n=1}^{\infty} H(g_1, \ldots, g_n)$. H_3 is a closed group with H_3/H infinite, hence extremally disconnected, and with H_3/H separable, hence (by (a)) not extremally disconnected. This contradiction shows that the inductive procedure terminates hence establishes (c).

(b) If H were separable then H_1 would also be separable. Since H_2 is generated by H_1 and a $g \in G \backslash H_1$, it must be separable. Thus, H_2/H_1 is separable and infinite which contradicts (a) since H_2/H_1 would be extremally disconnected. Thus, H is not separable. ■

Remark. (1) The density character d of G/H and d' of H must satisfy $2^d > 2^{\aleph_0} < 2^{d'}$.

(2) Frolik [5] has established that if the continuum hypothesis is true, or if $c^+ \neq 2^c$ where $c = 2^{\aleph_0}$, then no infinite homogeneous compact space is extremally disconnected.

(3) In results related to Proposition 4, Karel Prikry has shown that all Borel measures on a perfect metric space are meager (private communication). Fishel and Papert [5] have shown that on a locally compact group, any residual Borel measure, inner regular with respect to the closed subsets of the group, is absolutely continuous with respect to Haar measure. Thus, if Haar measure is meager so are regular Borel measures. Fishel and Papert also show that on a perfect locally connected topological space all regular Borel measures are meager. The results of [3] imply that when real-valued measurable cardinals don't exist then all residual Borel measures are regular. Thus, if real-valued measurable cardinals don't exist the regularity hypotheses in Fishel and Papert's results may be dropped. Proposition 2 then is a corollary of the results of Fishel and Papert when Haar measure is meager.

(4) Thanks are due to Bruce Mericle for pointing out the results of Fishel and Papert.

Added in Proof. Proposition 6 has been improved by the author in his paper *Borel measures on compact groups are meager*, Proc. 1980 Conference, Northern Illinois University, G. A. Goldin and R. F. Wheeler, editors, DeKalb, Illinois, 1981, pp. 141–144, to show that if G/H is infinite then it is not basically disconnected.

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