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QUILLEN'S $\mathcal{K}$-THEORY AND ALGEBRAIC CYCLES ON ALMOST NON-SINGULAR VARIETIES

BY

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Introduction

Let X denote an irreducible quasi-projective variety defined over an algebraically closed field, x_0 a distinguished closed point of X. We say that (X, x_0) is almost non-singular if $X - x_0$ is non-singular, and make this assumption in the following discussion.

Let X_i be the set of points (i.e., irreducible cycles) of codimension i in X and let $X_i^* = \{x \in X_i : x_0 \notin \bar{x}\}$. Set

$$C^i = \coprod_{x \in X_i} \mathbf{Z}_x \qquad \text{and} \qquad C^{*i} = \coprod_{x \in X^*_i} \mathbf{Z}_x.$$

Define R^i to be the subgroup of C^i which is generated by the elements of the form (s, f), where s is in X_{i-1}, f is an element of $k(s)^*$, the group of invertible elements in the function field of s, and (s, f) denotes the cycle $((f)_0 - (f)_\infty)$ computed on X. We refer to the elements of R as "relations". The group $C^i/R^i = CH^i(X)$ is the ith graded part of the covariant Chow group (cf. [2]).

Quillen [5] has associated sheaves $\mathcal{K}_{iX}$ with any scheme X, and proved that if X is a non-singular quasi-projective variety then

$$(0.1) \qquad CH^i(X) \simeq H^i(X, \mathcal{K}_{iX}).$$

If X is any variety, $H^1(X, \mathcal{K}_{1X})$ still has a geometric interpretation, indeed $\mathcal{K}_{1X} = \mathcal{O}_X^*$; therefore $H^1(X, \mathcal{K}_{1X}) = \operatorname{Pic}(X)$. It is a natural question to inquire about the geometrical meaning of the groups $H^i(X, \mathcal{K}_{iX})$.

Define R^{*i} to be the subgroup of C^{*i} generated by the relations (s, f) with the further requirement $s \in X^*_{i-1}$, i.e., by the relations which avoid the distinguished point. Set $CH^i(X, x_0) = C^{*i}/R^{*i}$. Our interpretation is:

(0.2) THEOREM. *If X is almost non-singular then*

$$CH^i(X, x_0) \simeq H^i(X, \mathcal{K}_{iX}) \quad i > 1.$$

Note that if X is non-singular, (0.1) and (0.2) together provide a highbrow proof that $CH^i(X) \simeq CH^i(X, x_0)$, $i > 1$.

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When X is non-singular one introduces a topological filtration in the group K_0X of vector bundles on X; let $G^i(K_0X)$ denote the associated graded groups. It is a consequence of Riemann–Roch (cf. [6, XIV]), that

(0.3) $$CH^2(X) \simeq G^2(K_0X)$$

(0.4) $$CH^i(X) \simeq G^i(K_0X) \text{ mod torsion}, \quad i > 2.$$

If X is almost non-singular we also introduce a filtration of topological nature on K_0X and still let $G^i(K_0X)$ denote the associated graded groups. Our next interpretation is:

(0.5) THEOREM. (a) $CH^2(X, x_0) \simeq G^2(K_0X)$.
(b) $CH^i(X, x_0) \simeq G^i(K_0X)$ *mod torsion*, $i > 2$.

If X is an affine surface this result appears in [4].

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1. Plan of work

We keep the notations of the introduction and assume that (X, x_0) is almost non-singular. Let $\mathscr{M}(X)$ be the category of finitely generated coherent modules on X, $\mathscr{P}_\infty(X)$ the exact subcategory of $\mathscr{M}(X)$ with objects the modules of finite projective dimension, $\mathscr{P}(X)$ the category of locally free sheaves, $\mathscr{M}(X, x_0)$ the Serre subcategory of $\mathscr{M}(X)$ with objects the modules M which are torsion at x_0, namely $M_{x_0} = 0$. Note that $\mathscr{M}(X, x_0)$ is also a subcategory of $\mathscr{P}_\infty(X)$, because $X - x_0$ is non-singular. On $\mathscr{M}(X, x_0)$ there is a decreasing filtration by codimension of the support:

(1.1) For $i \geq 0$, let $\mathscr{M}_i(X, x_0)$ be the full subcategory of $\mathscr{M}(X, x_0)$ whose objects are the modules M such that codim (supp M, $X) \geq i + 1$.

Similarly one introduces filtrations on $\mathscr{M}(X)$ and $\mathscr{P}_\infty(X)$

(1.2) $$\mathscr{M}_0^*(X) = \mathscr{M}(X), \mathscr{M}_1^*(X) = \mathscr{M}(X, x_0), \ldots, \mathscr{M}_{i+1}^*(X) = \mathscr{M}_i(X, x_0).$$

(1.3) $$F^0\mathscr{P}_\infty(X) = \mathscr{P}_\infty(X), \ldots, F^{i+1}\mathscr{P}_\infty(X) = \mathscr{M}_i(X, x_0).$$

We recall the standard notations $K_iX = K_i(\mathscr{P}(X))$, $K'_iX = K_i(\mathscr{M}(X))$ and the isomorphism $K_iX \simeq K_i(\mathscr{P}_\infty(X))$. The natural functors

$$F^i\mathscr{P}_\infty(X) \to \mathscr{P}_\infty(X), \qquad \mathscr{M}_i^*(X) \to \mathscr{M}(X)$$

induce homomorphisms

$$a\colon K_j(F^i\mathscr{P}_\infty(X)) \to K_jX, \qquad b\colon K_j(\mathscr{M}_i^*(X)) \to K'_jX.$$

For later reference we set

(1.4) $$S^iK_jX = \text{image } (a), \qquad S^iK'_jX = \text{image } (b),$$

(1.5) $$G^iK_jX = S^iK_jX/S^{i+1}K_jX, \qquad G^iK'_jX = S^iK'_jX/S^{i+1}K'_jX.$$

Let X_{x_0} denote spec $(\mathcal{O}_{X,x_0})$. Then the category $\mathcal{M}(X_{x_0})$ is equivalent to the quotient category $\mathcal{M}(X)/\mathcal{M}(X, x_0)$. By Theorem 5 of [5], there is the exact sequence of localization

(1.6) $$\cdots \to K_i\,(\mathcal{M}_1^*(X)) \to K_i'X \to K_i'X_{x_0} \to K_{i-1}(\mathcal{M}_1^*(X)) \to \cdots.$$

Similarly, for $p > 0$,

$(1.7)_p$
$$\cdots \to K_i(\mathcal{M}_{p+1}^*(X)) \to K_i(\mathcal{M}_p^*(X)) \to \coprod_{x \in X^*_p} K_i k(x) \to K_{i-1}(\mathcal{M}_{p+1}^*(X)) \to \cdots$$

where we have used the isomorphism

$$K_i(\mathcal{M}_p^*(X)/\mathcal{M}_{p+1}^*(X)) \simeq \coprod_{x \in X^*_p} K_i k(x),$$

which follows from Theorem 4, Corollary 1 of [5]. For KX, we produce, in Section 3, an exact sequence

(1.8) $$\cdots \to K_i(F^1\mathcal{P}_\infty(X)) \to K_iX \to K_iX_{x_0} \to K_{i-1}(F^1\mathcal{P}_\infty(X)) \to \cdots$$

while (1.7) can be rewritten as

$(1.9)_p$ $$\cdots \to K_i(F^{p+1}\mathcal{P}_\infty(X)) \to K_i(F^p\mathcal{P}_\infty(X)) \to \cdots.$$

By a standard process (cf. proof of (5.4) in Section 7 [5]), the exact sequences (1.8), (1.9) give rise to a spectral sequence

$$E_1^{pq}(X) \Rightarrow K_{-n}X, \quad p \geq 0, p + q \leq 0, n \leq 0.$$

Similarly (1.6), (1.7) give rise to

$$E_1'^{pq}(X) \Rightarrow K'_{-n}X,$$

where

$$E_1^{0q} = K_{-q}X_{x_0}, \quad E_1'^{0q} = K'_{-q}X_{x_0}, \quad E_1^{pq} = E_1'^{pq} = \coprod_{x \in X^*_p} K_{-p-q}k(x).$$

Using the preceding notations we have $E_\infty^{pq} = G^pK_{-p-q}X$, $E_\infty'^{pq} = G^pK'_{-p-q}X$. Following [5] we next identify E_2^{pq}.

(1.10). THEOREM. $E_2^{pq} = H^p(X, \mathcal{K}_{-q})$, $E_2'^{pq} = H^p(X, \mathcal{K}'_{-q})$.

Our procedure is to produce exact sequences of sheaves,

(1.11) $$0 \to \mathcal{K}_{iX} \to \mathcal{K}_i(X_{x_0}) \to \mathcal{K}_{i-1}(X, x_0) \to 0,$$

(1.11)′ $$0 \to \mathcal{K}'_{iX} \to \mathcal{K}'_i(X_{x_0}) \to \mathcal{K}_{i-1}(X, x_0) \to 0,$$

where $\mathcal{K}_{i-1}(X, x_0)$ is obtained from $K_{i-1}(\mathcal{M}(X, x_0))$ by means of a sheafifying process, while $\mathcal{K}_i(X_{x_0})$ shall be conveniently defined. Now the sheaves $\mathcal{K}_i(X,$

x_0) have the following exact resolution by flabby sheaves, which we call the Gersten resolution:

$$\text{(GR)}\quad 0 \to \mathscr{K}_n(X, x_0) \to \coprod_{x \in X^{*}_1} (i_x)_* K_n k(x) \to \coprod_{x \in X^{*}_2} (i_x)_* K_{n-1} k(x) \to \cdots.$$

Therefore

$$\text{(GR*)}\qquad 0 \to \mathscr{K}_{iX} \to \mathscr{K}_i(X_{x_0}) \to \coprod_{x \in X^{*}_1} (i_x)_* K_{i-1} k(x) \to \cdots$$

is also exact.

The associated complex of global sections can be written as

$$\text{(C)}\qquad 0 \to H^0(X, \mathscr{K}_{iX}) \to E_1^{0-i}(X) \to E_1^{1-i}(X) \to E_1^{2-i}(X) \to \cdots.$$

The differential in (C) turns out to be the differential d_1 in the spectral sequence, hence E_2^{p-i} is the pth cohomology group of (C). On the other hand we shall prove that $\mathscr{K}_i(X_{x_0})$ is acyclic, hence GR* is an acyclic resolution of $\mathscr{K}_{iX}$. Therefore $E_2^{p-i} = H^p(X, \mathscr{K}_{iX})$. The same argument works for $\mathscr{K}'_{iX}$. Furthermore by explicitly identifying the differential d_1 in $E_1^{i-1,-i} \to E_1^{i-i}$ of (C) one gets $E_1^{i-i}(X) = C^{*i}$, image $d_1 = R^{*i}$, if $i > 1$. Hence

$$\text{(0.1)}\qquad H^i(X, \mathscr{K}_{iX}) \simeq CH^i(X, x_0). \quad i > 1.$$

$H^i(X, \mathscr{K}'_{iX}) \simeq CH^i(X, x_0)$, $i > 1$; we chose formulation (0.1) because both functors are contravariant in the category or pointed almost non-singular varieties.

2. The sheaf $\mathscr{K}_n(X, x_0)$

Let Y denote a constructible subset of X, $\mathscr{M}(Y)$ the category of finitely generated coherent modules on Y, $\mathscr{M}(Y, x_0)$ the exact subcategory of $\mathscr{M}(Y)$ with objects modules M having the property that x_0 does not belong to the closure in X of the support of M. Note that if Y is closed and x_0 is not in Y then $\mathscr{M}(Y, x_0) = \mathscr{M}(Y)$.

(2.1) LEMMA. *$\mathscr{M}(Y, x_0)$ is a Serre subcategory of $\mathscr{M}(Y)$.*

Proof. If $0 \to M \to M' \to M'' \to 0$ is exact then

$$\operatorname{supp}(M') = \operatorname{supp}(M) + \operatorname{supp}(M''),$$

hence $\operatorname{cl}(\operatorname{supp}(M')) = \operatorname{cl}(\operatorname{supp}(M)) + \operatorname{cl}(\operatorname{supp}(M''))$. Using this remark it is straightforward to check that $\mathscr{M}(Y, x_0)$ is closed under subobjects, quotients and extensions.

(2.2) Given an open set U in X we denote $K_n(U, x_0) = K_n(\mathscr{M}(U, x_0))$. Filtering $\mathscr{M}(U, x_0)$ by codimension of the support in X we obtain categories $\mathscr{M}_p(U, x_0)$ defined as in (1.1). $K_n(U, x_0)$ is filtered by the images of the groups

$K_n(\mathcal{M}_p(U, x_0))$, which we denote by $S^pK_n(U, x_0)$. By means of the localization theorem of [5] one gets long exact sequences which provide a spectral sequence

$$(2.3) \qquad E_1^{*pq}(U) \Rightarrow K_{-n}(U, x_0), \quad p \geq 0, \, p + q \leq 0, \, -n \geq 0,$$

where

$$E_1^{*pq}(U) = \coprod_{x \in U^*_{p+1}} K_{-p-q}k(x)$$

and U^*_{p+1} denotes the set of points of codimension $(p + 1)$ in U which have the property that x_0 is not in $\bar{x}$. By looking at the construction of the spectral sequence we find an augmented complex

$$(2.4) \qquad 0 \to K_n(U, x_0) \xrightarrow{e} \coprod_{x \in U^*_1} K_n k(x) \xrightarrow{d} \coprod_{x \in U^*_2} K_{n-1} k(x) \to \cdots.$$

We sheafify the presheaf $K_n(\quad, x_0)$ and let $\mathcal{K}_n(X, x_0)$ be the corresponding sheaf. Note that

$$\mathcal{K}_n(X, x_0)_{x_0} = 0, \quad \mathcal{K}_n(X, x_0)_x = \lim K_n(V, x_0) \text{ for } x \in V.$$

Complex (2.4) yields a complex of sheaves (GR) which we have written in (1).

(2.5) PROPOSITION. *Sequence* (GR) *is exact.*

(2.6) COROLLARY. $E_2^{*pq}(X) = H^p(X, \mathcal{K}_{-q}(X, x_0))$.

(2.7) COROLLARY. $H^p(X, \mathcal{K}_p(X, x_0)) \simeq CH^{p+1}(X, x_0)$.

Proof of (2.5). We imitate the proof of exactness for the Gersten resolution of $\mathcal{K}_{nX}$ when X is non-singular (cf. [5]) and indicate only the variations needed for the present case.

Let A denote the local ring $\mathcal{O}_{X,x}$, $\mathcal{M}_p(A, x_0)$ the category of finitely generated modules M on A such that (i) $x_0 \notin$ cl (supp M) and (ii) codim (cl (supp M), $X) \geq p + 1$. Note that $\mathcal{K}_n(X, x_0)_x = K_n(\mathcal{M}_0(A, x_0))$. By the same argument as in [5], the proposition holds if we prove that the inclusion

$$\mathcal{M}_{p+1}(A, x_0) \to \mathcal{M}_p(A, x_0), \quad p \geq 0,$$

induces the zero map on K-groups. If $x = x_0$ then $\mathcal{M}_p(A, x_0)$ is the zero category and everything is trivial, so take $x \neq x_0$. Since X is quasi-projective there is an affine open subspace of X, say spec (R), to which both x and x_0 belong; without restriction we can assume $X =$ spec (R). By Section 2, (9) of [5],

$$K_n(\mathcal{M}_{p+1}(A, x_0)) = \lim K_n(\mathcal{M}_{p+1}(R_f, x_0))$$

where f runs over the regular elements of R for which $f(x) \neq 0$. We need to show that $\mathcal{M}_{p+1}(R_f, x_0) \to \mathcal{M}_p(A, x_0)$, $p \geq 0$, induces zero on K-groups.

For a constructible subscheme Z in X let $\mathscr{M}_p(Z)$ be the full subcategory of $\mathscr{M}(Z)$ with objects modules M such that codim (supp M, Z) $\geq p + 1$; note the shift in indexes. With this notation,

$$K_n(\mathscr{M}_{p+1}(R_f, x_0)) = \lim K_n(\mathscr{M}_p(R_f/tR_f)),$$

where t runs over the regular elements of R with $t(x_0) \neq 0$. Given f and t, it suffices to show that there is a multiple $f' = fh$ with $f'(x) \neq 0$ for which (*) the functor $M \to M_{f'}$ from $\mathscr{M}_p(R_f/tR_f)$ to $\mathscr{M}_p(R_{f'}, x_0)$ induces zero on K-groups.

Set $Z = \text{spec}\ (R/t)$, $Z_f = \text{spec}\ (R_f/t)$ and note that $x_0 \notin Z$ because $t(x_0) \neq 0$. With M as in (*) above, let $W = \text{cl}\ (\text{supp}\ M)$. Then

$$(+) \qquad x_0 \notin W \text{ and } \text{codim}\ (W, Z) \geq p + 1 > 0.$$

(2.9) LEMMA. *Let Z be a divisor in X =* spec (R), *W a proper subvariety of Z as in* $(+)$. *Suppose that X is regular at x and let r =* dim Z. *There is a morphism u: $X \to A^r$, where A^r is the affine space, so that* (i) $u/_Z$: $Z \to A^r$ *is finite,* (ii) *u is smooth at x and* (iii) $u(x_0) \notin u(W)$.

Proof. Say X is embedded in the affine space A^n. The set of linear maps from A^n to A^r is itself an affine space $A^{\cdot}$; it is standard to check that (i), (ii) and (iii) each impose open, non-empty conditions on $A^{\cdot}$, hence there is a linear map of the required type.

Take u as in the lemma and build a cartesian diagram

$$\begin{array}{ccc} \text{spec}\ (R^+) = X^+ & \xrightarrow{\ b\ } & X \\ \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle u} \\ Z & \xrightarrow{\ u/_Z\ } & A^r. \end{array}$$

For any Z-module M there is an exact sequence of X-modules

$$(++) \qquad 0 \to \text{Kernel} \to a^*M \to M \to 0.$$

If supp $M \subseteq W$, then by (iii) of (2.9), $x_0 \notin \text{supp}\ (a^*M)$, hence $(++)$ is a sequence of functors from $\mathscr{M}(W)$ to $\mathscr{M}_p(X, x_0)$. By the same argument as in [5, p. 50] we can take a function $f' = fh$ in R with $f'(x) \neq 0$ such that (i) $X^+_{f'}$ is flat over Z and (ii) sequence $(++)$ becomes

$$(\text{s}) \qquad 0 \to I_{f'} \otimes_Z M \to a^*M_{f'} \to M_{f'} \to 0$$

where $I_{f'}$ is isomorphic to $R^+_{f'}$ as an $R_{f'}$-module. Sequence (s) is therefore an exact sequence of exact functors from $\mathscr{M}(W)$ to $\mathscr{M}_p(R_{f'}, x_0)$; this allows us to conclude that the functor from $\mathscr{M}(W_f)$ to $\mathscr{M}_p(R_{f'}, x_0)$ induces the zero map on K-groups. To complete the proof we remark that $K_n(\mathscr{M}_p(R_f/tR_f) = \lim$

$K_n(\mathcal{M}(W_f))$ where W_f runs over the set of subschemes of Z_f of codimension at least $p+1$ in Z_f (cf. (5.1) [5]).

Proof of (2.6). The proof of Proposition 5.8 in [5] applies.

Proof of (2.7). The proof of Theorem 5.19 of [5] applies, One should recall that if $x_0 \notin Z$, Z closed, then $\mathcal{M}(Z, x_0) = \mathcal{M}(Z)$.

3. The sheaves $\mathcal{K}_n(X_{x_0})$ and $\mathcal{K}'_n(X_{x_0})$

Consider the morphism i: Sp $R \to X$ where $R = \mathcal{O}_{X,x_0}$. Let $\mathcal{K}_n(X_{x_0}) = i_*(\mathcal{K}_{n,\mathrm{Sp}\,R})$, $\mathcal{K}'_n(X_{x_0}) = i_*(\mathcal{K}'_{n,\mathrm{Sp}\,R})$. Our aim is to produce exact sequences

$$(3.1) \qquad 0 \to \mathcal{K}'_{nX} \to \mathcal{K}'_n(X_{x_0}) \to \mathcal{K}_{n-1}(X, x_0) \to 0,$$

$$(3.2) \qquad 0 \to \mathcal{K}_{nX} \to \mathcal{K}_n(X_{x_0}) \to \mathcal{K}_{n-1}(X, x_0) \to 0.$$

We start with the first one. For V open in X let $K'_n(V_{x_0})$ denote the group $K_n(\mathcal{M}(V)/\mathcal{M}(V, x_0))$. Observe that this notation is coherent with our convention $X_{x_0} = \mathrm{Sp}\,R$, because $K'_n(X_{x_0}) = K'_n(\mathrm{Sp}\,R)$.

LEMMA. $\mathcal{K}'_n(X_{x_0})_y = \lim_{y \in V} K'_n(V_{x_0})$.

Proof. Let U be an open set containing x_0, $D = X - U$. By the localization theorem,

$$K'_{n+1}(U \cap V) \to K'_n(D \cap V) \to K'_n V \to K'_n(U \cap V)$$

is exact, Taking limit over the U's gives

$$(a) \qquad K'_{n+1}(V_{x_0}) \to K'_n(V, x_0) \to K'_n V \to K'_n(V_{x_0}).$$

The right hand side of the equation in the lemma is then the limit of $K'_n(U \cap V)$, where U and V vary as indicated above. Now $\mathcal{K}'_n(X_{x_0})_y$ is also the limit of the same family.

Sequence (a) gives rise to a long exact sequence of sheaves

$$(b) \qquad \mathcal{K}'_n(X, x_0) \to \mathcal{K}'_{nX} \to \mathcal{K}'_n(X_{x_0}) \to \mathcal{K}'_{n-1}(X, x_0).$$

(3.3) PROPOSITION. *Sequence* (b) *splits in short exact sequences of type* (3.1).

Proof. By looking at stalks in (b) it suffices to show that the functor $\mathcal{M}(X_x, x_0) \to \mathcal{M}(X_x)$ induces the zero map on K-groups. If $x = x_0$ then $\mathcal{M}(X_{x_0}, x_0)$ is the zero category and everything is clear. If $x \neq x_0$ then the above map factors into

$$\mathcal{M}(X_x, x_0) \to \mathcal{M}_1(X_x) \xrightarrow{a} \mathcal{M}(X_x).$$

By (5.10) of [5] we know that a induces the zero map on K-groups because X_x is regular.

In order to find (3.2) write the diagram with exact rows

$$\begin{array}{ccccc} 0 \to \mathscr{K}_{nX} & \longrightarrow & \mathscr{K}_n(X_{x_0}) & & \\ \downarrow & & \downarrow & & \\ 0 \to \mathscr{K}'_{nX} & \longrightarrow & \mathscr{K}'_n(X_{x_0}) & \to \mathscr{K}_{n-1}(X, x_0) & \to 0. \end{array}$$

We know that the vertical maps are isomorphisms except possibly for the stalk at x_0; moreover

$$\mathscr{K}_{n-1}(X, x_0)_{x_0} = 0, \ \mathscr{K}_{nX,x_0} = \mathscr{K}_n(X_{x_0})_{x_0}.$$

Therefore (3.2) is exact.

An alternative way of finding (3.2) is to imitate what we have done for $\mathscr{K}'_n(X_{x_0})$. We do not give the complete argument but produce only the global localization sequence which we promised in (1.8).

Since $\mathscr{P}_\infty(X)$ is not an abelian category we cannot use the localization theorem used previously. In [3] we find the following result.

(3.4) For any affine open subscheme U of X there is an exact sequence

$$(+) \qquad K_{q+1}U \to K_qH \to K_qX \to K_qU \to \cdots,$$

where H is the category of quasi-coherent sheaves on X which are zero on U and admit a resolution of length one by vector bundles on X.

Building on (3.4) we shall recover the exact sequence we want. Set $D = X - U$ and assume furthermore that x_0 is in U and that D is a divisor. Let $\mathscr{M}(D)$ be the category of coherent modules on D; note that any object in $\mathscr{M}(D)$ admits a finite projective resolution by vector bundles on X, since $X - x_0$ is non-singular.

(3.5) LEMMA. $K_qH \simeq K_q(\mathscr{M}(D))$.

Proof. Apply Theorem 3 of [5] to the pair of exact categories (H_n, H_{n+1}), $n > 0$, where H_n denotes the subcategory of $\mathscr{M}(D)$ whose objects are modules M of X-homological dimension at most n. A routine argument shows that condition (i) of the theorem holds. To prove that condition (ii) is satisfied, for M'' in H_{n+1} we produce a resolution $0 \to M' \to M \to M'' \to 0$ with M in H_n. There is a resolution in $\mathscr{M}(X)$,

$$(\mathrm{r}) \qquad 0 \to K \to P \to M'' \to 0,$$

where P is projective. Since $x_0 \notin D$ then $\mathscr{O}(-D)$ is invertible and the restriction $P_D = P \otimes O_D$ is in H_1, hence in H_n. Now tensoring (r) with O_D gives $0 \to M' \to P_D \to M'' \to 0$.

Sequence $(+)$ of (3.4) can be written as

$$(3.6) \qquad \cdots \to K_{q+1}U \to K'_qD \to K_qX \to K_qU \to \cdots.$$

Recall that $X_{x_0} = \lim_{x_0 \in U} U$, where U runs over the family of affine open neighborhoods of x_0. Then by Proposition 2.2 of [5], $K_q(X_{x_0}) = \lim K_q U$; by [5, (9) p. 20], $K_q(X, x_0) = K_q(\lim \mathscr{M}(D)) = \lim K'_q D$. Taking direct limits, (3.6) gives the exact sequence

$$(3.7) \qquad \cdots \to K_{q+1} X \to K_{q+1} X_{x_0} \to K_q(X, x_0) \to K_q X \to \cdots .$$

In Section 5 below we need the following result.

(3.8) LEMMA. *$K_n X_{x_0} \to H^0(X, \mathscr{K}_n(X_{x_0}))$ is an isomorphism; similarly for K'_n.*

Proof.

$$\begin{array}{c} H^0(X, \mathscr{K}_n(X_{x_0})) = H^0(X, i_* \mathscr{K}_{n,\mathrm{Sp}\, R}) = H^0(\mathrm{Sp}\, R, \mathscr{K}_{n,\mathrm{Sp}\, R}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \simeq \big\downarrow (*) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad K_n(R) \end{array}$$

we have the isomorphism (*) because any open set containing the closed point of $X_{x_0} = \mathrm{Sp}\, R$ must contain all of $\mathrm{Sp}\, R$.

4. $\mathscr{K}_n(X_{x_0})$ and $\mathscr{K}'_n(X_{x_0})$ are acyclic sheaves

We prove this only for $\mathscr{K}_n$; the proof for $\mathscr{K}'_n$ is similar. The global sections functor on Sp R is exact, so sheaves there have no higher cohomology. Thus to show that $i_*(\mathscr{K}_{n,\mathrm{Sp}\, R})$ is acyclic, it suffices to show that the higher derived images are zero, i.e.,

$$(+) \qquad R^m i_*(\mathscr{K}_{n,\mathrm{Sp}\, R}) = 0, \quad m > 0.$$

Looking at stalks we see that equality holds trivially at x if $x \in \mathrm{Sp}\, R$. If $x \notin \mathrm{Sp}\, R$, then to prove (+) at x amounts to proving that the Gersten–Quillen resolution is exact for the ring $R(x, x_0)$, the quotient ring of $R = \mathcal{O}_{x_0}$ obtained by inverting all the functions which do not vanish at x.

Remark. This last statement depends on the property that the G–Q resolution of the sheaf $\mathscr{K}_{nU}$ is exact if $x_0 \notin U$, U open in Sp R, U a regular scheme.

LEMMA. *If x is a non-singular point of X, the Gersten–Quillen resolution of $K_n(R(x, x_0))$ is exact.*

Proof. We assume $x \notin \mathrm{Sp}\, R$, the other case being obvious because $R(x, x_0) = \mathcal{O}_x$ if $x \in \mathrm{Sp}\, R$. Following Quillen we need to prove that for any $p \geq 0$, the inclusion $\mathscr{M}_{p+1}(R(x, x_0)) \to \mathscr{M}_p(R(x, x_0))$ induces zero on K-groups. Let $Z' \neq \phi$ be a divisor in Sp $R(x, x_0)$ of equation $t' = 0$. It suffices to show that the functor $\mathscr{M}_p(Z') \to \mathscr{M}_p(R(x, x_0))$ induces zero on K-groups. We may assume that X is affine, say $X = \mathrm{Sp}\, C$, and take t to be an element of C which localizes to t'. The divisor Z on X with equation $t = 0$ restricts to Z' on Sp $R(x, x_0)$, hence x and x_0 both belong to Z. One has $K_*(\mathscr{M}_p(Z')) = \lim K_*(\mathscr{M}_p(Z_{fg}))$, where g runs over the elements of C which do not vanish at x_0 and f runs over the elements of C which do not vanish at x. Therefore it suffices to show:

(+ +) The functor $\mathscr{M}_p(Z_{fg}) \to \mathscr{M}_p(R(x, x_0))$ induces zero on K-groups.

We denote by G the divisor cut on Z by the equation $g = 0$, by F the divisor cut on Z by $f = 0$. Without restriction we may assume that no irreducible component of Z is contained in F or G (otherwise take Z to be the original Z minus the components contained either in F or G), so that F and G are proper divisors in Z. By the same argument of Lemma (2.9) we have a diagram

$$\text{(d)} \qquad \begin{array}{ccc} X^+ & \xrightarrow{\ b\ } & X = \operatorname{Sp} C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{\ u|Z\ } & A^r \end{array}$$

where (i) $u|_Z\colon Z \to A^r$ is finite, (ii) u is smooth at x, (iii) $u(x_0) \notin u(G)$ and (iv) $u(x) \notin u(F)$. Now take ϕ to be a function in A^r vanishing along $u(F)$ but not vanishing at $u(x)$, take γ to be a function vanishing along $u(G)$ but not vanishing at $u(x_0)$. Localizing diagram (d) at $\phi\gamma$ we have

$$\begin{array}{ccc} X^+_{\phi\gamma} & \longrightarrow & X_{\phi\gamma} \\ {\scriptstyle a_{\phi\gamma}}\downarrow & & \downarrow \\ Z_{\phi\gamma} & \longrightarrow & A^r_{\phi\gamma}. \end{array}$$

To prove (+ +) we replace Z_{fg} by $Z_{\phi\gamma}$; we may do this because $Z_{\phi\gamma} \hookrightarrow Z_{fg}$ and $\phi(x) \neq 0$, $\gamma(x_0) \neq 0$. For any $Z_{\phi\gamma}$-module M we have an exact sequence of $X_{\phi\gamma}$ modules

$$\text{(s)} \qquad 0 \to \text{Kernel} \to a^*_{\phi\gamma} M \to M \to 0.$$

Returning for a moment to diagram (d) we recall that by the same argument as in [5, p. 50] there is a function h in C, not vanishing at x, such that (i) X^+_h is flat over Z and (ii) the ideal I_h of $(Z \cap X^+_h)$ in X^+_h is principal. Localizing sequence (s) at h we have

$$0 \to I_h \otimes_Z M \to (a^*_{\phi\gamma} M)_h \to M_h \to 0$$

which is now an exact sequence of exact functors from $\mathscr{M}_p(Z_{\phi\gamma})$ to $\mathscr{M}_p(X_{\phi\gamma h})$. We conclude as in [5].

5. Another interpretation of $CH^i(X, x_0)$

From (1.4) and (1.10) it follows that

$$SK_i(X) = \text{Ker}\,(K_i X \to H^0(X, \mathscr{K}_{iX})), \tag{5.1}$$

$$S^2K_{i-1}(X) = \text{Ker}\,(SK_{i-1}(X) \to H^1(X, \mathscr{K}_{iX})). \tag{5.2}$$

The groups $S^jK_i(X, x_0), j = 1, 2$, have a similar description. From (3.7), using (3.8), we find the top row in

$$\begin{array}{ccccccccc} K_i X & \longrightarrow & K_i X_{x_0} & \longrightarrow & K_{i-1}(X, x_0) & \longrightarrow & SK_{i-1}(X) & \longrightarrow & 0 \\ \downarrow\simeq & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to H^0(X, \mathscr{K}_{iX}) & \to & H^0(X, \mathscr{K}_i(X_{x_0})) & \to & H^0(X, \mathscr{K}_{i-1}(X, x_0)) & \to & H^1(X, \mathscr{K}_{iX}) & \to & 0. \end{array} \tag{5.3}$$

By chase, one has

$$0 \to SK_i X \to K_i X \to H^0(X, \mathscr{K}_{iX}) \to SK_{i-1}(X, x_0) \to S^2K_{i-1}(X) \to 0. \tag{5.4}$$

When $i = 1$, the sequence splits because of the following result.

LEMMA. *$K_1 X \to H^0(X, \mathscr{K}_{1X})$ is surjective.*

Proof. If X is projective or affine the result is clear, since $\mathscr{K}_{1X} = \mathcal{O}_X^*$. A proof for the general case when X is quasi-projective can be given as follows. Consider the diagram

$$\begin{array}{ccc} X_{x_0} & \xrightarrow{\ i\ } & X \\ & \searrow^{a} & \downarrow{\scriptstyle b} \\ & & Z = \text{Sp}\, B \end{array}$$

where $B = H^0(X, \mathcal{O}_X)$. Since Z is affine, $K_1 Z = R^* \oplus SK_1 Z$ and clearly $R^* = H^0(X, \mathcal{O}_X^*)$. By functoriality, $a^*K_1 Z \hookrightarrow i^*K_1 X$ in $K_1 X_{x_0}$, hence

$$a^*(H^0(X, \mathcal{O}_X^*)) \hookrightarrow i^*K_1 X.$$

A look at (5.3) completes the proof.

(5.5) PROPOSITION. $SK_0(X, x_0) \simeq S^2K_0 X$.
Then one has isomorphisms of the graded groups

$$E_\infty^{i+1, -i-1}(X) \simeq G^{i+1}K_0 X \simeq G^iK_0(X, x_0) \simeq E_\infty^{*i, -i}(X), \quad i \geq 1.$$

Our interpretation is:

(5.6) THEOREM. (a) $CH^2(X, x_0) \simeq G^2K_0 X$.
(b) *if $j > 2$ then $CH^j(X, x_0) \simeq G^jK_0 X$ mod torsion.*

Proof. (a) From the spectral sequence E^* (cf. (2.3)), we have

$$E_2^{*1,-1} = E_\infty^{*1,-1} = G^1K_0(X, x_0) = G^2K_0X;$$

moreover

$$E_2^{*1,-1} = H^1(X, \mathscr{K}_1(X, x_0)) \simeq H^2(X, \mathscr{K}_{2X}) \simeq CH^2(X, x_0).$$

(b) Recall the isomorphism $CH^j(X, x_0) \simeq E_2^{*j-1,-j+1} = E_2^{j,-j}$. From the spectral sequence there is a surjective morphism $\sigma: E_2^{j,-j} \to E_\infty^{j,-j}$. We show that σ is injective modulo torsion. Let $a' = \sum m_i z_i$, $z_i \in X_j^*$, represent an element a in $CH^j(X, x_0)$ such that $\sigma(a) = 0$. By the construction of σ we know that $\sigma(a)$ is represented in $E_\infty^{j,-j} = G^jK_0X$ by $\sum m_i\gamma(z_i)$, where $\gamma(z_i) = \text{class}\ (\mathcal{O}_{z_i})$ in K_0X. Since $\sigma(a) = 0$, $\sum m_i\gamma(z_i)$ is contained in $S^{j+1}K_0X$. In other words

$$(+) \qquad \sum m_i\gamma(z_i) = \sum n_s\gamma(w_s) \quad \text{in } K_0X,$$

where w_s belongs to X_{j+t}^*, $t > 0$. This equality holds in S^jK_0X, hence it holds in $S^{j-1}K_0(X, x_0)$ by (5.5). Therefore $(+)$ is true in $K_0(X, x_0)$ also.

From the definition of $K_0(X, x_0)$ it follows that there is a closed subscheme S of X, S not necessarily irreducible, so that (i) $x_0 \notin S$, (ii) z_i, w_s, are points of S and (iii) equation $(+)$ holds in $K_0'S$. At this point we need a basic result from [1]. Let $CH(S)$ be the group $A(S)$ in [1]; $CH(S)$ is the Chow covariant group graded by dimension. Then there is an isomorphism

$$(++) \qquad \tau: K_0'S \simeq CH(S) \quad \text{mod torsion}$$

with the property that if $t = \gamma(T)$ then

$$\tau(t) = \text{class}\ (T) + \text{terms of lower degree}.$$

Applying τ to equation $(+)$ one gets

$$\text{class}\ (\Sigma m_j z_j) = \text{terms in lower degree} \quad \text{mod torsion};$$

hence class $(\Sigma m_j z_j) = 0$ in $CH(S)_Q$. From the definition of $CH(S)$ we have a natural map $CH(S) \to CH(X, x_0)$, hence the above equality holds also in $CH(X, x_0)_Q$.

(5.7) *Remark.* Since $K_1'X \to H^0(X, \mathscr{K}'_{1X})$ is not surjective in general, (5.6) does not hold for the group $K_0'X$.

6. Final remarks

(6.1) If X is non-singular there are two filtrations for the groups K_nX, the topological filtration used in [5] and the one we introduced in (1.4). We want to show that the two filtrations coincide.

In Section 1 we produced a spectral sequence with the property that $E_\infty^{p-q,q} = G^{p-q}K_{-q}X$, the graded groups associated with the filtration (1.4).

Our proof was inspired by [5], where the same result is proved for the topological filtration. In a standard way one finds a natural map from our spectral sequence to Quillen's one. In both cases $E_2^{pq} = H^p(X, \mathscr{K}_{-q,X})$ (cf. (1.10) and [5]), hence the two spectral sequences coincide from the E_2^{pq} terms on. Consequently the two filtrations on $K_n X$ coincide.

(6.2) We now assume that X contains finitely many singular points $x_1, \ldots, x_n$. By analogy to what is done above, one can define groups $CH^i(X, x_1, \ldots, x_n)$, abbreviated $CH^i(X, x.)$. Similarly sheaves $\mathscr{K}_n(X, x.)$ can be introduced. Everything we proved in Sections 1, 2, 3 above can be proved again by the same arguments properly adapted. The results in Section 4 do not extend. Let X denote the theta divisor inside the Jacobian variety of a general curve of genus 4; algebraic geometers know that X is a threefold with exactly two singular points. We have computed $H^2(X, K_{1X}) = \mathbf{Z}$, hence $\mathscr{K}_1(X_{x.})$ is not acyclic. Details will appear elsewhere.

The results in Section 5 do not depend on Section 4, in particular Theorem (0.5) holds for the case of X with finitely many singular points.

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