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THE LINDELÖF PROPERTY IN MI-SPACES

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1. Introduction

There are several unresolved questions about the existence of MI-spaces with some additional features. For example, it is not known whether there are any connected regular MI-spaces. This and several similar problems have been around ever since Hewitt [3] introduced the concept of resolvability. A recent paper by Bankston [1] has renewed interest in some of these problems; in this paper, we settle one of the problems raised in [1] by showing that the cardinality of a κ-compact MI-space is always less than κ. Assuming the continuum hypothesis, we also prove the existence of a paracompact MI-space of any infinite cardinality.

2. Main results

A topological space X is called *submaximal* [2] provided every dense subset of X is open. An *MI-space* is a dense-in-itself submaximal space. For a subset A of a topological space X, $\bar{A}$ and A^0 are respectively the closure and the interior of A in X. CH stands for the continuum hypothesis and ω_1 denotes the first uncountable ordinal number. The properties regular, completely regular and paracompact, as used here, imply the T_2 axiom, unless stated otherwise explicitly. The cardinal number of a set X is denoted by $|X|$.

We will now list some well-known [2] and rather easy-to-prove properties of the classes of spaces we have already defined.

(a) A topological space X is submaximal if and only if for each subset A of X, the subspace $\bar{A} - A^0$ is closed and discrete; and that happens if and only if for each subset A of X, the set $A - A^0$ is closed and discrete.

(b) If an MI-space has a collection of cardinality κ of pairwise disjoint non-empty open subsets, then X has a closed discrete subspace of cardinality κ.

(c) A subset G of a topological space X is said to be *regular open* provided G is the interior of a closed subset of X. The regular open subsets of a topological space (X, τ) form a base for a topology τ_∂ on X; τ_∂ is called the *semi-regularization* of τ. A topological space (X, τ) is said to be *semi-regular*

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provided $\tau = \tau_s$. It is easy to show that for any topological space (X, τ), the space (X, τ_s) is always semi-regular. For a topology τ on a set X, we let $[\tau]$ denote the collection of all topologies on X which have the same semiregularization as τ. This partitions the collection of all topologies on X into disjoint classes. Each of these class is known as an *S-class.* Any S-class contains a unique semi-regular topology and this topology is the smallest member (under inclusion) of that class. Each member of an S-class is contained in a maximal member of the class; and the maximal members of a class are precisely the submaximal topologies belonging to that class. If two topologies τ_1 and τ_2 on a set X belong to the same S-class, then the continuous mappings on (X, τ_1) and (X, τ_2) into any regular space are the same. In particular, if some member of an S-class is connected then all the topologies in that S-class are connected.

(d) There is a standard procedure used to construct MI-spaces. It is generally known as the Bourbaki Construction and proceeds as follows: Let (X, τ) be a topological space and let $\mathscr{F}$ be a maximal element of the collection of filters of dense subsets of X. Let $\tau(\mathscr{F})$ denote the topology on X generated by $\tau \cup \mathscr{F}$. Then $(X, \tau(\mathscr{F}))$ is submaximal and τ and $\tau(\mathscr{F})$ are in the same S-class. Furthermore every maximal member of $[\tau]$ containing τ can be obtained in this manner. If τ is a dense-in-itself Hausdorff topology on X, then a topology $\tau_1 \in [\tau]$ is MI if and only if τ_1 is a maximal member of $[\tau]$.

(e) We have already seen how the topology of a dense-in-itself Hausdorff space can be expanded to obtain an MI-space. However, we must note that the Bourbaki-Construction has a severe limitation, in that it cannot be used to produce a regular MI-space unless we start with one in the first place. A regular topology is necessarily semi-regular and therefore it has to be smallest member of its S-class.

(f) A countably infinite connected Hausdorff MI-space can be produced from a countably infinite connected Haudorff space via the Bourbaki Construction.

Let κ be an infinite cardinal number. A topological space X is said to be *κ-compact* if and only if each open cover of X has a subcover of cardinality $<\kappa$.

THEOREM 1. *Let κ be an infinite cardinal number. Then each κ-compact subspace of an* MI*-space must have cardinality less than κ.*

Proof. If possible, suppose some MI-space X has a subspace Y such that Y is κ-compact and $|Y| \geq \kappa$. Letting α be a partition of Y into at least κ sets, each of cardinality $\geq \kappa$, we have that for each $A \in \alpha$, $A - A^0$ is nowhere dense in X and therefore it is a closed discrete subspace of X. Since Y is κ-compact, $|A - A^0| < \kappa$ and therefore $|A^0| \geq \kappa$. Therefore we can pick one point from A^0 for each $A \in \alpha$, to form a subset Z of Y. Clearly $|Z| = |\alpha| \geq \kappa$ and so Z

cannot be closed discrete in X. On the other hand, X is MI and so Z is clearly closed and discrete, a contradiction. Hence the theorem is proved.

The following consequence of the above theorem settles a problem posed by Bankston [1].

COROLLARY. *Every Lindelöf* MI-*space is countable.*

We have already seen that Hausdorff MI-spaces abound and we can even have countable, connected Hausdorff MI-spaces. To the best of our knowledge it is not known whether a paracompact MI-space can be constructed without any extra set-theoretic assumptions.

THEOREM 2. *Assuming* CH, *there exists a countable regular* MI-*space.*

Proof. Let (X, τ) be the rational line. Using CH, we list the collection of all dense subsets of (X, τ) in a transfinite sequence $\{D_\alpha: \alpha < \omega_1\}$.

We will define a transfinite sequence $\{\tau_\alpha: \alpha < \omega_1\}$ of topologies on X satisfying the following conditions:

(i) $\alpha < \beta < \omega_1$ implies $\tau_\alpha \subset \tau_\beta$.

(ii) For each $\alpha < \omega_1$, the space (X, τ_α) is dense-in-itself, regular and first countable (hence metrizable).

(iii) For each $\alpha < \omega_1$, the set D_α is either open in (X, τ_α) or else it is not dense in (X, τ_α).

Assuming that we have already defined such a sequence, we let τ^* be the topology on X generated by the base $\bigcup \{\tau_\alpha: \alpha < \omega_1\}$. Clearly (X, τ^*) is dense-in-itself. Furthermore, since the sup of a collection of regular topologies on a set is regular, (X, τ^*) is regular. It is also clear that every dense subset of (X, τ^*) is open. Consequently (X, τ^*) is a countable regular MI-space.

The role of CH in the construction is to assure that each (X, τ_α) is first countable. The first countability of (X, τ_α) will make it possible to carry out the induction.

Now we set out to define the sequence $\{\tau_\alpha: \alpha < \omega_1\}$ of topologies on X. For convenience, we postpone defining τ_0. Suppose there is a $\gamma < \omega_1$ such that we have already defined τ_α for each $\alpha < \gamma$ in such a way that conditions (ii) and (iii) hold for each $\alpha < \gamma$ and condition (i) holds for all α, $\beta < \gamma$ with $\alpha < \beta$. Let τ_γ^* be the toplogy generated by $\bigcup \{\tau_\alpha: \alpha < \gamma\}$. Clearly (X, τ_γ^*) is dense-in-itself and regular. Moreover, since τ_γ^* is a sup of only a countable number of second countable topologies, it is itself second countable, and, in particular, it is first countable. Now if D_γ is not dense in (X, τ_γ^*), or if D_γ is open in (X, τ_γ^*) we simply let $\tau_\gamma = \tau_\gamma^*$. If D_γ is dense but not open in (X, τ_γ^*), we let E denote the interior of D_γ in (X, τ_γ^*). Clearly E is dense-in-itself and first countable and so, by a result of Hewitt [3], it can be written as a union of two disjoint dense subsets E_1 and E_2 (if E is void, we simply let $E_1 = E_2 = \phi$). We let $F_1 = E_1 \cup$

$(X - D_\gamma)$ and $F_2 = X - F_1$. Clearly F_1 and F_2 are disjoint dense subsets of (X, τ_γ^*) and $F_1 \cup F_2 = X$. Now we define τ_γ to be the toplogy generated by τ_γ^* together with the sets F_1 and F_2. It is easily seen that (X, τ_γ) is dense-in-itself. Furthermore, since τ_γ is the sup of the regular toplogy τ_γ^* and the regular non-Hausdorff topology $\{\phi, X, F_1, F_2\}$; the space (X, τ_γ) is regular.

Finally we note that we start the induction by letting τ_0^* be τ itself and then we define the topology τ_0 by disposing of the set D_0 as outlined in the induction above.

It should be noted that if D_γ is dense but not open in (X, τ_γ^*), then we cannot simply let τ_γ to be the topology generated by $\tau_\gamma^* \cup \{D_\gamma\}$, because the latter topology is not regular (in fact, it is not even semi-regular).

THEOREM 3. *Assuming* CH, *there is a paracompact* MI-*space of any infinite cardinality.*

Proof. Assuming CH, there is a countably infinite paracompact MI-space. Since a disjoint union of paracompact MI-spaces is a paracompact MI-space, the theorem is proved.

THEOREM 4. *Let (X, τ) be a regular space and suppose the set X can be partitioned into sets $\{A_\alpha: \alpha \in \Lambda\}$ such that each A_α is a countable, first countable, dense-in-itself subspace. Then, assuming* CH, *there is a topology τ^* containing τ such that (X, τ^*) is a paracompact* MI-*space.*

Recall that the *dispersion character*, $\Delta(X)$, of a topological space X is defined to be the smallest cardinal number such that X has a non-empty open subset of that cardinality. A topological space X is said to have a *uniform dispersion* provided each element of X has a neighborhood of cardinality $\Delta(X)$.

Using Theorem 4, we can easily prove the following result.

THEOREM 5. *Let (X, τ) be a dense-in-itself, separable metrizable space having a uniform dispersion. Then, assuming* CH, *there exists an expansion τ^* of τ such that (X, τ^*) is a paracompact* MI-*space.*

The following result is an immediate consequence of this theorem.

COROLLARY. *Let (X, τ) be a dense-in-itself, separable, homogeneous metrizable space. Then, assuming* CH, *there is an expansion τ^* of τ such that (X, τ^*) is a paracompact* MI-*space.*

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REFERENCES

1. PAUL BANKSTON, *The total negation of a topological property*, Illinois J. Math., vol. 23 (1979), pp. 241–252.
2. N. BOURBAKI, *General Topology, Part I*, Addison Wesley, Reading, Massachusetts, 1966.
3. E. HEWITT, *A problem in set-theoretic topology*, Duke J. Math., vol. 10 (1943), pp. 309–333.

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