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DIRICHLET-FINITE FUNCTIONS AND HARMONIC MAJORANTS

BY

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1. Introduction

For a differentiable real function u defined in the disk $U = \{|z| < 1\}$ in the complex plane, the integral

$$D_\alpha(u) = \iint_U (1 - |z|)^\alpha |\text{grad } u(z)|^2 \, dx \, dy \tag{1.1}$$

is called the α-weighted Dirichlet integral of u, where $0 \le \alpha < \infty$, $z = x + iy$, and $|\text{grad } u|^2 = u_y^2$; the Dirichlet integral of u is precisely $D_0(u)$. Notice that $D_\alpha(u) \le D_\beta(u)$ if $0 \le \beta \le \alpha < \infty$. A prototype of our main theme is the following well-known result:

THEOREM A. *Let u be a harmonic function in U with $D_0(u) < \infty$. Then, for each p, $0 < p < \infty$, the function $|u|^p$ admits a harmonic majorant in U.*

The last sentence means that there is a harmonic function v in U such that $|u|^p \le v$ in U. A function-theoretic proof is as follows. The holomorphic function f in U with $\operatorname{Re} f = u$ and $f(0) = u(0)$, satisfies $\iint_U |f'(z)|^2 \, dx \, dy < \infty$. It then follows from the familiar result (see [2, Exercise 7, p. 106]; see [4, Theorem] and [5, Theorem 1] for a further generalization) that f is of Hardy class H^p for each p, $0 < p < \infty$, which, together with $|u|^p \le |f|^p$, proves Theorem A.

Let Q be a nonnegative constant, and consider the Euler–Lagrange differential equation

$$\Delta u \equiv u_{xx} + u_{yy} = Qu \quad \text{in} \quad U, \tag{1.2}$$

of the variational problem to which the energy integral

$$\iint_U (|\text{grad } u|^2 + Qu^2) \, dx \, dy \quad (\ge D_0(u))$$

leads; see [1, p. 258 ff.]. From the familiar fact on the elliptic differential operators, each solution u of (1.2) is C^∞ in U.

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THEOREM 1. *Let u be a solution of* (1.2) *with $D_\alpha(u) < \infty$ for an α, $0 < \alpha \leq 1$. Then $|u|^{2/\alpha}$ admits a harmonic majorant in U.*

The above-mentioned Theorem A is the harmonic case $Q = 0$ of the following

COROLLARY. *Let u be a solution of* (1.2) *with $D_0(u) < \infty$. Then, for each p, $0 < p < \infty$, the function $|u|^p$ admits a harmonic majorant in U.*

In effect, $D_{2/p}(u) \leq D_0(u) < \infty$ for all p, $2 \leq p < \infty$, so that $|u|^p$ admits a harmonic majorant in U. Since $|u|^q < |u|^p + 1$ for $0 < q < 2 \leq p < \infty$, one obtains the corollary.

We cannot replace the condition $D_\alpha(u) < \infty$ in Theorem 1 by $D_{\alpha+\varepsilon}(u) < \infty$ for any $\varepsilon > 0$ in the harmonic case $Q = 0$. Actually, consider $u = \operatorname{Re} f$ in U, where $f(z) = (1 - z)^{-\alpha/2}$, $f(0) = 1$. Since $1 - |z| \leq |1 - z|$, it follows that

$$D_{\alpha+\varepsilon}(u) = \iint_U (1 - |z|)^{\alpha+\varepsilon} |f'(z)|^2 \, dx \, dy < \infty.$$

If $|u|^{2/\alpha}$ would admit a harmonic majorant in U, then it follows from M. Riesz's theorem [2, Theorem 4.1, p. 54] that $f \in H^{2/\alpha}$, being a contradiction.

In Section 3 we shall consider a sufficient condition for a holomorphic function in U to be of Hardy class H^q for $2 \leq q < \infty$ in terms of its Taylor coefficients, as an application of Theorem 1 in the specified case $Q = 0$. Our Theorem 2 may be compared with the known sufficient condition [2, Theorem 6.3, p. 97].

In Section 4 we shall consider the n-dimensional case for $n \geq 3$; Theorem 3 asserts that if u is a harmonic function with the finite, n-dimensional, α-weighted Dirichlet integral in the unit ball B_n, $0 \leq \alpha \leq 1$, then $|u|^p$ for $p = (2n - 2)/(n + \alpha - 2)$, admits a harmonic majorant in B_n.

2. Proof of Theorem 1

We may suppose that u is non-constant. Set $p = 2/\alpha$. Since $p \geq 2$, and since

$$\Delta(|u|^p) = p(p-1)|u|^{p-2}|\operatorname{grad} u|^2 + pQ|u|^p \geq 0, \tag{2.1}$$

the function $|u|^p$ is subharmonic in U. Set $g(z) = -\log|z|$, and $\delta(z) = 1 - |z|$, $z \in U$. Since

$$\lim_{|z| \to 1} \delta(z)^{-1} g(z) = 1, \tag{2.2}$$

it follows that there exists r_0, $0 < r_0 < 1$, such that

$$g(z) \leq 2\delta(z), \quad r_0 \leq |z| < 1. \tag{2.3}$$

Set $g_r(z) = g(z) - g(r)$ for $r_0 < r < 1$ and $z \in U$, and apply the Green formula to the functions g_r and $|u|^p$ on the annulus $R(r) = \{r_0 < |z| < r\}$; we shall make use of $\partial/\partial\nu$ as the derivatives along the direction of the radii. Then, observing that $g_r(z) \equiv 0$ on $|z| = r$, and $\Delta g_r(z) \equiv 0$ in $R(r)$, one obtains

$$\begin{aligned}
K(r) &\equiv \iint_{R(r)} g_r(z)\Delta(|u(z)|^p)\,dx\,dy \\
&= -\int_{|z|=r} |u(z)|^p \frac{\partial}{\partial\nu} g_r(z)\,|dz| - \int_{|z|=r_0} g_r(z)\frac{\partial}{\partial\nu}|u(z)|^p\,|dz| \\
&\quad + \int_{|z|=r_0} |u(z)|^p \frac{\partial}{\partial\nu} g_r(z)\,|dz| \\
&= -\int_{|z|=r} |u(z)|^p \frac{\partial}{\partial\nu} g(z)\,|dz| + I_2(r) - \int_{|z|=r_0} |u(z)|^p r_0^{-1}\,|dz| \\
&\equiv I_1(r) + I_2(r) + I_3(r).
\end{aligned}$$

Note that $I_1(r) = 2\pi V_r(0)$, where V_r is the least harmonic majorant of the subharmonic function $|u|^p$ in $|z| < r$, being the Poisson integral of $|u|^p$ on the circle $|z| = r$. If $I_1(r)$ is observed to be bounded for $r_0 < r < 1$, then $|u|^p$ admits the least harmonic majorant $V = \lim_{r\to 1} V_r$ in U. To be more precise, V_r's are increasing as r increases, so that V is harmonic in U by the Harnack theorem. This is the principal idea of the proof.

To prove that $I_1(r)$ is bounded, it suffices to show that $K(r)$, $I_2(r)$, and $I_3(r)$ are bounded for $r_0 < r < 1$. Apparently, $I_3(r)$ is a constant. As for $I_2(r)$ we observe that $|u(z)|^p$ is smooth since $p \geq 2$ and therefore $(\partial/\partial\nu)|u(z)|^p$ is bounded on the circle $|z| = r_0$. Since $g_r(z) - g(z) = -g(r) \to 0$ as $r \nearrow 1$ on $|z| = r_0$, it follows that $I_2(r)$ has the limit as $r \nearrow 1$. To prove that $K(r)$ is bounded, we note that

$$\begin{aligned}
(2.4) \qquad 0 &\leq K(r) \\
&\leq \iint_{R(r)} g(z)\Delta(|u(z)|^p)\,dx\,dy \\
&\leq 2 \iint_{r_0<|z|<1} \delta(z)\Delta(|u(z)|^p)\,dx\,dy \\
&\equiv 2A
\end{aligned}$$

by means of (2.3). Therefore, the remaining point which requires verification for the boundedness of $I_1(r)$ is that $A < \infty$.

For this purpose we first claim that $W = |\text{grad}\, u|^2$ is subharmonic in U because

$$\Delta W = 2(u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2) + 2QW \geq 0.$$

Therefore, for each $z \in U$,

$$\begin{aligned}
(2.5)\quad W(z) &\leq \pi^{-1}2^2\,\delta(z)^{-2} \iint_{|\zeta - z| < 2^{-1}\delta(z)} W(\zeta)\,d\xi\,d\eta \quad (\zeta = \xi + i\eta)\\
&\leq \pi^{-1}2^{2+\alpha}\,\delta(z)^{-2-\alpha} \iint_{|\zeta - z| < 2^{-1}\delta(z)} (1 - |\zeta|)^{\alpha} W(\zeta)\,d\xi\,d\eta\\
&\leq c_1\,\delta(z)^{-2-\alpha} D_\alpha(u)\\
&\leq c_2\,\delta(z)^{-2-\alpha};
\end{aligned}$$

hereafter, $c_j (j = 1, \ldots, 21)$ are positive constants. It then follows from (2.5) that $|\text{grad } u(z)| \leq c_3\,\delta(z)^{-1-1/p}$, whence

$$(2.6)\qquad |u(z)| \leq |u(0)| + c_4 L(z), \quad \text{where } L(z) = \delta(z)^{-1/p}, \quad z \in U.$$

Remembering the inequality $(X + Y)^\lambda \leq 2^\lambda(X^\lambda + Y^\lambda)$ for $X, Y \geq 0, \lambda > 0$, one obtains the following estimates from (2.6):

$$\begin{aligned}
|u(z)|^{p-2} &\leq c_5 L(z)^{p-2} + c_6 = c_5\,\delta(z)^{-1+2/p} + c_6,\\
|u(z)|^{p} &\leq c_7 L(z)^{p} + c_8 = c_7\,\delta(z)^{-1} + c_8, \quad z \in U.
\end{aligned}$$

Notice that $|u(z)|^{p-2} \equiv 1$ in the case $p = 2$. In view of (2.1) one now evaluates

$$\begin{aligned}
\delta(z)\,\Delta(|u(z)|^p) &\leq \delta(z)^{1-2/p}[c_9\,\delta(z)^{-1+2/p} + c_{10}]\,\delta(z)^\alpha |\text{grad } u(z)|^2\\
&\quad + c_{11}\,\delta(z)\,\delta(z)^{-1} + c_{12}\\
&\leq c_{13}\,\delta(z)^\alpha |\text{grad } u(z)|^2 + c_{14}.
\end{aligned}$$

Therefore $A < \infty$, which completes the proof.

3. Taylor coefficients and Hardy class

THEOREM 2. *Let f be a function holomorphic in U with the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in U$. Suppose that*

$$(3.1)\qquad \sum_{n=1}^{\infty} n^{1-2/q} |a_n|^2 < \infty$$

for a certain $q, 2 \leq q < \infty$, Then $f \in H^q$. The constant $2/q$ in (3.1) *is sharp in the sense that we cannot replace $2/q$ by $2/q + \varepsilon$ for any $\varepsilon > 0$.*

It is known [2, Theorem 6.3, p. 97] that if

$$(3.2)\qquad \sum_{n=1}^{\infty} n^{q-2} |a_n|^q < \infty$$

for a certain q, $2 \le q < \infty$, then $f \in H^q$. To investigate the relations between (3.1) and (3.2) is, therefore, interesting. It appears, however, to be impossible to prove (3.1)$\Rightarrow$(3.2) or (3.2)$\Rightarrow$(3.1) by the Hölder inequality.

One merit of the condition (3.1) might be that, (3.1) is satisfied for each q, $2 \le q < \infty$, provided that

$$\pi \sum_{n=1}^{\infty} n|a_n|^2 = \iint_U |f'(z)|^2 \, dx \, dy < \infty; \tag{3.3}$$

actually the sequence $\{n^{-2/q}\}$ is bounded. Therefore we have another proof of the fact that f is of Hardy class H^p for all $p > 0$ if (3.3) is satisfied.

Proof of Theorem 2. Since the case $q = 2$ is well known we assume that $q > 2$. Since $0 < 2/q < 1$, it follows from the familiar result [3, pp. 8–9] that (3.1) implies

$$\iint_U (1 - |z|)^{2/q} |f'(z)|^2 \, dx \, dy < \infty.$$

Let $f = u + iv$ in U. By Theorem 1, both $|u|^q$ and $|v|^q$ have harmonic majorants in U, whence the same is true of $|f|^q \le 2^q(|u|^q + |v|^q)$. Therefore $f \in H^q$. To prove the sharpness of $2/q$ we set

$$h(z) = (1 - z)^{-1/q} = \sum_{n=0}^{\infty} b_n z^n \quad (b_0 = 1) \text{ in } U.$$

Since $|b_n| = O(n^{1/q - 1})$ as $n \to \infty$, it follows that

$$n^{1-(2/q+\varepsilon)} |b_n|^2 = O(n^{-1-\varepsilon}) \quad \text{as } n \to \infty,$$

whence (3.1) holds on replacing $2/q$ and a_n by $2/q + \varepsilon$ and b_n, respectively. Apparently, $h \notin H^q$.

4. Harmonic functions in the n-dimensional space

Let B_n be the unit ball $|x| = (\sum_{j=1}^n |x_j|^2)^{1/2} < 1$ in the n-dimensional Euclidean space, $n \ge 3$.

THEOREM 3. *Let u be a harmonic function in B_n $(n \ge 3)$ such that*

$$D_\alpha(u) \equiv \int_{B_n} (1 - |x|)^\alpha |\text{grad } u(x)|^2 \, dx < \infty$$

for an α, $0 \le \alpha \le 1$, where $dx = dx_1, \ldots, dx_n$ and $|\text{grad } u|^2 = \sum_{j=1}^n u_{x_j}^2$. Then for $p = (2n-2)/(n + \alpha - 2)$, the function $|u|^p$ admits a harmonic majorant in B_n.

The sharpness of the constant $p = (2n-2)/(n + \alpha - 2) \ge 2$ remains open even if $\alpha = 0$.

The proof is in spirit the same as that of Theorem 1, and we shall give a sketch. The equality (2.1) is valid for the present p (even for $Q > 0$) which we shall denote (2.1)'. We set $g(x) = (n-2)^{-1}|x|^{2-n}$ to obtain

$$\lim_{|x|\to 1} \delta(x)^{-1}g(x) = 1, \tag{2.2)'}$$

where $\delta(x) = 1 - |x|$, $x \in B_n$. Thus the analogy (2.3)' of (2.3) is again true in the present case. By the similar argument of making use of the Green formula after (2.3), one finds that the principal point to be proved is that

$$A \equiv \int_{r_0<|x|<1} \delta(x)\,\Delta(|u(x)|^p)\,dx < \infty. \tag{4.1}$$

Again, $W = |\mathrm{grad}\, u|^2$ is subharmonic in B_n because $\Delta W = 2\sum_{j,k=1}^n u_{x_jx_k}^2 \geq 0$. Therefore (2.5) is turned to be

$$W(x) \leq c_{15}\,\delta(x)^{-n-\alpha}D_\alpha(u) \leq c_{16}\,\delta(x)^{-n-\alpha}, \quad x \in B_n, \tag{2.5)'}$$

because the volume of the ball $\{y;\, |y-x| < 2^{-1}\,\delta(x)\}$ is $c_{17}\,\delta(x)^n$. Consequently,

$$|u(x)| \leq |u(0)| + c_{18}\mathscr{L}(x), \quad x \in B_n, \tag{2.6)'}$$

where $\mathscr{L}(x) = \delta(x)^\beta$, $\beta = 1 - (n+\alpha)/2$. Considering (2.1)' with $Q = 0$, one now evaluates

$$\begin{aligned}\delta(x)\,\Delta(|u(x)|^p) &\leq \delta(x)^{1-\alpha}[c_{19}\mathscr{L}(x)^{p-2} + c_{20}]\,\delta(x)^\alpha|\mathrm{grad}\, u(x)|^2 \\ &\leq c_{21}\,\delta(x)^\alpha|\mathrm{grad}\, u(x)|^2, \quad x \in B_n,\end{aligned}$$

because $1 - \alpha + \beta(p-2) = 0$. Thus, (4.1) holds, which completes the proof of Theorem 3.

REFERENCES

1. S. BERGMAN and M. SCHIFFER, *Kernel functions and elliptic differential equations in mathematical physics*, Pure and Applied Mathematics, vol. 4, Academic Press, New York, 1953.
2. P. L. DUREN, *Theory of H^p spaces*, Pure and Applied Mathematics, vol. 38, Academic Press, New York, 1970.
3. V. I. GAVRILOV, *On the theorems of Beurling, Carleson and Tsuji on exceptional sets*, Math. USSR Sbornik, vol. 23 (1974), Np. 1, pp. 1–12 (Mat. Sbornik, vol. 94 (136) (1974), No. 1, pp. 1–15).
4. L. J. HANSEN, *The Hardy class of a function with slowly-growing area*, Proc. Amer. Math. Soc., vol. 45 (1974), pp. 409–410.
5. L. J. HANSEN and W. K. HAYMAN, *On the growth of functions omitting large sets*, J. Analyse Math., vol. 30 (1976), pp. 208–214.

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