ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 4, Winter 1981

THE REAL SEMI-CHARACTERISTIC OF A HOMOGENEOUS SPACE

BY

J. C. BECKER[1]

1. Introduction

The *real Kervaire semi-characteristic* of a closed orientable manifold of dimension $4s + 1$ is defined to be

$$k(M) = \sum_i \dim (H^{2i}(M, R)) \quad \text{mod } 2.$$

The main purpose of this paper is to give a formula for the semi-characteristic of a homogeneous space G/H along the lines of Hopf and Samelson's formula for the Euler characteristic [4].

Recall that the Weyl group of a compact Lie group G (not necessarily connected) is $W(G) = N_G(T)/C_G(T)$, where $N_G(T)$ and $C_G(T)$ are respectively the normalizer and centralizer of a maximal torus T of the identity component of G. Hopf and Samelson's theorem states that the Euler characteristic of a connected homogeneous space G/H is given by

$$E(G/H) = \begin{cases} |W(G)|/|W(H)|, & \text{rank } (H) = \text{rank } (G), \\ 0, & \text{rank } (H) < \text{rank } (G). \end{cases}$$

For a connected orientable homogeneous space G/H of dimension $4s + 1$ we will show that

$$k(G/H) = \begin{cases} |W(G)|/|W(H), & \text{rank } (H) = \text{rank } (G) - 1, \\ 0, & \text{rank } (H) < \text{rank } (G) - 1, \end{cases}$$

as integers mod 2 (see Corollary (5.1)).

The similarity in the statement of these two results is also present in their method of proof which in each case involves analyzing vector fields on G/H. The Euler characteristic arises as an obstruction to finding a non-zero vector field on G/H, whereas Atiyah and Dupont [2] have shown that the semi-characteristic arises as an obstruction to extending a non-zero vector field to a field of 2-frames on G/H.

Received November 6, 1979.

[1] Partially supported by a National Science Foundation grant.

It is a pleasure to thank R. Schultz for several helpful conversations during the course of this work.

2. The characteristic of a k-field

It is well known that a compact smooth manifold M has an associated "Gauss map" whose degree is the Euler characteristic of M. To be precise, choose an embedding $c\colon M \to R^s$ with normal bundle ν. Let τ denote the tangent bundle of M and $\dot{M}$ the boundary of M. The restriction of the inclusion i: $M^\nu \to M^{\tau\oplus\nu}$ to $\dot{M}^\nu$ is null homotopic by $v_x \to tN_x \oplus v_x$, $0 \leq t \leq \infty$, where N is the outward normal vector field on $\dot{M}$. Applying the homotopy extension property we have $\tilde{i}$: $(M, \dot{M})^\nu \to M^{\tau\oplus\nu}$. Then the degree of the map

$$S^s \xrightarrow{c_\#} (M, \dot{M})^\nu \to M^{\tau\oplus\nu} \xrightarrow{\tilde{i}} S^s$$

is the Euler characteristic of M.

There is an interesting generalization of this construction due to E. Y. Miller [5]. Suppose that $\Delta_1, \ldots, \Delta_k$ are linearly independent vector fields on M which are also tangent on $\dot{M}$. Let $\Delta\colon M \times R^k \to \tau$ denote the associated injection. The restriction of $\Delta \oplus 1\colon M^{R^k\oplus\nu} \to M^{\tau\oplus\nu}$ to $\dot{M}^{R^k\oplus\nu}$ is again canonically null homotopic so we obtain

$$\widetilde{\Delta \otimes 1}\colon (M, \dot{M})^{R^k\oplus\nu} \to M^{\tau\oplus\nu}.$$

The map

$$S^k \wedge S^s \xrightarrow{1 \wedge c_\#} S^k \wedge (M, \dot{M})^\nu = (M, \dot{M})^{R^k\oplus\nu} \xrightarrow{\widetilde{\Delta\oplus 1}} M^{\tau\oplus\nu} \to S^s$$

defines an element

$$\chi_k(M, \Delta_1, \ldots, \Delta_k) \in \pi_k(S^\circ). \tag{2.1}$$

It depends only on the homotopy class of the k-field $\{\Delta_1, \ldots, \Delta_k\}$ and its vanishing is a necessary condition that there exist a vector field $\bar{N}$ on M which extends the outward normal N on $\dot{M}$ and such that $\Delta_1, \ldots, \Delta_k, \bar{N}$ are linearly independent. Of course $\chi_0(M) \in \pi_0(S^\circ) = Z$ is the Euler characteristic $E(M)$.

We list now some of the properties of this element. In what follows, by a *k-field* on M (always assumed compact) we will mean k linearly independent vector fields on M which are also tangent on $\dot{M}$.

(2.2) *Multiplicativity. Suppose that* $\Delta_1, \ldots, \Delta_p$ *is a p-field on M and* $\delta_1, \ldots, \delta_q$ *is a q-field on N. Define* Δ'_j *on* $M \times N$, $1 \leq j \leq p$, *by* $\Delta'_j(x, y) = i_{y*}\Delta_j(x)$, *where* i_y: $M \to M \times N$ *is the inclusion* $x \to (x, y)$, *and define* δ'_j, $1 \leq j \leq q$, *similarly. Then* $\Delta'_1, \ldots, \Delta'_p, \delta'_1, \ldots, \delta'_q$ *is a* $(p+q)$*-field on* $M \times N$ *and*

$$\chi_{p+q}(M \times N, \Delta'_1, \ldots, \Delta'_p, \delta'_1, \ldots, \delta'_q) = \chi_p(M, \Delta_1, \ldots, \Delta_p)\chi_q(N, \delta_1, \ldots, \delta_q).$$

($M \times N$ has the product smooth structure which involves straightening the angle along $\dot{M} \times \dot{N}$ if both $\dot{M}$ and $\dot{N}$ are non empty.)

Suppose now that $M = M_1 \cup M_2$ where M_1 and M_2 are topological n-submanifolds of the smooth n-manifold M such that $M_1 \cap M_2 = \dot{M}_1 \cap \dot{M} = M_{12}$ say, and M_{12} is a smooth submanifold with boundary $\dot{M}_{12} = M_{12} \cap \dot{M}$. Then M_1 and M_2 inherit a smooth structure from M by straightening the angle along $\dot{M}_{12}$. If Δ is a 1-field on M with the additional property that $\Delta^{12} = \Delta | M_{12}$ is tangent on M_{12}, it is easy to check that Δ induces a 1-field Δ^j on M_j, $j = 1, 2$, uniquely determined by the condition that $\Delta^j | M_j - \dot{M}_{12} = \Delta | M_j - \dot{M}_{12}$.

(2.3) *Excision. Suppose that* $\Delta_1, \ldots, \Delta_k$ *is a* k*-field on* M *such that* $\Delta_i^{12} = \Delta_i | M_{12}$ *is tangent on* M_{12}, $1 \leq i \leq k$. *Then* $\Delta_1^j, \ldots, \Delta_k^j$ *is a* k*-field on* M_j, $j = 1, 2$, *and*

$$\chi_k(M, \Delta_1, \ldots, \Delta_k)$$
$$= \chi_k(M_1, \Delta_1^1, \ldots, \Delta_k^1) + \chi_k(M_2, \Delta_1^2, \ldots, \Delta_k^2) - \chi_k(M_{12}, \Delta_1^{12}, \ldots, \Delta_k^{12}).$$

The proofs of (2.2) and (2.3) are routine and will be omitted.

(2.4) THEOREM. *Let* M *be closed, orientable, and odd dimensional. Let* Δ *be a* 1*-field on* M. *Then* $\chi_1(M, \Delta) \in \pi_1(S^\circ) = Z_2$ *is independent of* Δ *and is given by*

$$\chi_1(M, \Delta) = \begin{cases} k(M), & \dim(M) \equiv 1 \bmod 4, \\ 0\ , & \dim(M) \equiv 3 \bmod 4, \end{cases}$$

where $k(M)$ *is the real Kervaire semi-characteristic of* M.

This is implicit in the work of Atiyah and Dupont [2]. It is simply a matter of relating $\chi_1(M, \Delta)$ with the index defined there. Since the Hurewicz map

$$\pi_1(S^\circ) = \pi^\circ(S^1) \to \widetilde{KO}^\circ(S^1)$$

is an isomorphism we can work with the image of $\chi_1(M, \Delta)$ in $\widetilde{KO}^\circ(S^1)$ which we again denote by $\chi_1(M, \Delta)$. Now Atiyah and Dupont define an element

$$\operatorname{Ind} \alpha_{M,2}^s \in \widetilde{KO}^s(P_{s+1}/P_{s-1}),$$

where $0 \leq s \leq 3$ and $\dim(M) + s \equiv 0$ (4). We have an exact sequence

$$\widetilde{KO}^\circ(S^1) = \widetilde{KO}^s(P_{s+1}/P_s) \xrightarrow{j^*} \widetilde{KO}^s(P_{s+1}/P_{s-1}) \to \widetilde{KO}^s(P_s/P_{s-1}) = Z,$$

and, on comparing definitions, it can be shown that $j^*(\chi_1(M, \Delta)) = \operatorname{Ind} \alpha_{M,2}^s$. From the calculation of $\widetilde{KO}^s(P_{s+1}/P_{s-1})$ given in [2, Section 3] we see that j^* is injective and therefore $\chi_1(M, \Delta)$ is independent of Δ. The main theorem of [2] then gives the stated value for $\chi_1(M, \Delta)$.

Suppose now that $p: E \to B$ is a vector bundle over a closed manifold B. Let $D(E)$ and $S(E)$ denote the unit disk and sphere bundles (relative to some metric).

(2.5) LEMMA. *Suppose that $\delta_1, \ldots, \delta_k$ is a k-field on $D(E)$ and $\Delta_1, \ldots, \Delta_k$ is a k-field on B such that $p_* \delta_1 = \Delta_i p$, $1 \leq i \leq k$. Then $\chi_k(D(E), \delta_1, \ldots, \delta_k) = \chi_k(B, \Delta_1, \ldots, \Delta_k)$.*

Proof. There is the natural inclusion $p^*(E) \to \tau(D(E))$ and we have

$$\tau(D(E)) \simeq p^*(\tau(B)) \oplus p^*(E).$$

Write $\delta_i(e) = \delta_i'(e) \oplus \delta_1''(e)$, where $\delta_i'(e) \in p^*(\tau(B))$ and $\delta_i''(e) \in p^*(E)$. Since δ_i is homotopic to δ_i' and $\delta_i'(e) = (e, \Delta p(e))$, we may assume that $\delta_i(e) = (e, \Delta p(e))$, $1 \leq i \leq k$.

Let $s\colon B \to D(E)$ denote the zero section and observe that if θ is any vector bundle over B the following is homotopy commutative:

$$\begin{array}{ccc}
(D(E)), S(E)^{R^k \oplus p^*(\theta)} & \xrightarrow{\widetilde{\delta \oplus 1}} & D(E)^{\tau(D(E)) \oplus p^*(\theta)} = D(E)^{p^*(\tau(B) \oplus E \oplus \theta)} \\
\Big\downarrow {\scriptstyle s_\#} & & \Big\uparrow {\scriptstyle s} \\
B^{R^k \oplus E \oplus \theta} & \xrightarrow{\Delta \oplus 1} & B^{\tau(B) \oplus E \oplus \theta}
\end{array}$$

In fact we may take

$$s(\Delta \oplus 1)s_\#(v_b, x, w_b) = \left(\theta_b, \Delta(b, x), \frac{1}{1 - |v_b|} v_b, w_b\right),$$

$v_b \in D(E)$, $x \in R^k$, $w_b \in \theta$. And since the outward normal on $S(E)$ is given by $v_b \to (v_b, v_b) \in p^*(E)$, we may take

$$\widetilde{\delta \oplus 1}(v_b, x, w_b) = \left(v_b, \Delta(b, x), \frac{1}{1 - |v_b|} v_b, w_b\right).$$

It is clear now that $\widetilde{\delta \oplus 1} \simeq s(\Delta \oplus 1)s_\#$.

Now choose an embedding $c'\colon E \to R^s$ with normal bundle ν'. Let

$$c = c's\colon B \to R^s \quad \text{and} \quad \nu = s^*(\nu').$$

Then $\nu' = p^*(\nu)$ and by the remarks above,

$$\begin{array}{ccccccc}
S^{k+s} & \xrightarrow{1 \wedge c'_\#} & (D(E), S(E))^{R^k \oplus p^*(\nu)} & \xrightarrow{\widetilde{\delta \oplus 1}} & D(E)^{\tau(D(E)) \oplus p^*(\nu)} & \longrightarrow & S^s \\
& \searrow {\scriptstyle 1 \wedge c_\#} & \Big\downarrow {\scriptstyle s_\#} & & \Big\uparrow {\scriptstyle s} & \nearrow & \\
& & B^{R^k \oplus E \oplus \nu} & \xrightarrow{\Delta \oplus 1} & B^{\tau(B) \oplus E \oplus \nu} & &
\end{array}$$

is homotopy commutative. The lemma follows.

3. G-manifolds

We shall eventually be dealing with both left and right G-spaces so we will adopt the standard notation for the orbit space: $G\backslash X$ if X is a left G-space and X/G if X is a right G-space.

Suppose that M is a smooth G-manifold having no isotropy subgroup of maximal rank. Let T be a maximal torus of G. A choice of a generator t of T determines a l-field Δ_t on M as follows: t defines a 1-parameter subgroup $R \subset T$ and we have $\tau_0(R) \subset \tau_1(T)$. Let $v \in \tau_1(T)$ denote the image of the canonical generator of $\tau_0(R)$ and define $\Delta_t(x) = \omega_{x^*}(v)$, where $\omega_x\colon T \to M$ is the evaluation map $s \to sx$, $s \in T$.

If H is a subgroup of G let (H) denote its conjugacy class, let $M_{(H)}$ denote the set of points of M having isotropy subgroup in (H) and let $\mathring{M}_{(H)}$ denote the one-point compactification of $M_{(H)}$.

(3.1) THEOREM. *If M is a G-manifold having no isotropy subgroup of maximal rank then*

$$\chi_1(M, \Delta_t) = \sum E(G\backslash\mathring{M}_{(H)}, \infty)\chi_1(G/H, \Delta_t),$$

the sum taken over all conjugacy classes of isotropy subgroups of M.

Proof (Cf. [3, Theorem (4.2)].) We proceed by induction on the dimension of M and on the number of handles in an equivariant handle decomposition of M as in [7]. The theorem holds vacuously for 0-dimensional manifolds.

Consider first the case of the unit disk bundle $D(V)$ of a Riemannian G-vector bundle V over an orbit G/H with rank $(H) <$ rank (G). By Lemma (2.5),

$$\chi_1(D(V), \Delta_t) = \chi_1(G/H, \Delta_t). \tag{3.2}$$

If K is an isotropy subgroup of $D(V)$ then some conjugate of K lies in H. Consider the case $(K) = (H)$. Then $V_{(H)}$ is a subbundle W of V, hence $D(V)_{(H)} = D(W)$. Since $p\colon D(W) \to G/H$ is a G-homotopy equivalence, $E(G\backslash\mathring{D}(W), \infty) = E(G\backslash D(W)) = 1$.

If K is a proper subgroup of H then

$$D(V)_{(K)} = S(V)_{(K)} \times [0, 1)$$

since $v \in D(V)_{(K)}$ implies that $\lambda v \in D(V)_{(K)}$, $\lambda \neq 0$. Therefore

$$G\backslash D(V)_{(K)} = G\backslash S(V)_{(K)} \times [0, 1)$$

and it follows that $E(G\backslash\mathring{D}(V)_{(K)}) = 0$. Therefore

$$\sum E(G\backslash\mathring{D}(V)_{(K)}, \infty)\chi_1(G/K, \Delta_t) = \chi_1(G/H, \Delta_t) \tag{3.3}$$

The result for $D(V)$ now follows from (3.2) and (3.3).

Suppose now that M is obtained from N by attaching a G-handle; $M = N$

$\bigcup_F \mathscr{H}$ where $\mathscr{H} = D(V) \times_{G/H} D(W)$, V and W Riemannian G-vector bundles over an orbit G/H. By (2.3),

$$\chi_1(M, \Delta_t) = \chi_1(N, \Delta_t) + \chi_1(\mathscr{H}, \Delta_t) - \chi_1(N \cap \mathscr{H}, \Delta_t).$$

We may assume by induction on the number of handles that the result holds for N and by induction on dimension that the result holds for $N \cap \mathscr{H}$. Since $\mathscr{H} = D(V \oplus W)$ is a smooth manifold we have from above that the theorem holds for $\mathscr{H}$. It is now easy to check that the theorem also holds for M.

Given an action of a torus T on M, define the *circle point set* of M to be

$$\Sigma(M) = \{x \in M \mid \dim (T/T_x) = 1\}. \tag{3.4}$$

(3.5) COROLLARY. *If T acts on M without fixed points then*

$$\chi_1(M, \Delta_t) \equiv E(T\backslash\Sigma(M)) \mod 2.$$

Proof. First observe that

$$\chi_1(T, \Lambda_t) = \begin{cases} 1, & \dim (T) = 1, \\ 0, & \dim (T) > 1. \end{cases}$$

If T' is a subgroup of T let $t' \in T/T'$ denote the image of t. Since T/T' is again a torus

$$\chi_1(T/T', \Delta_t) = \chi_1(T/T', \Delta_{t'}) = \begin{cases} 1, & \dim (T/T') = 1, \\ 0, & \dim (T/T') > 1. \end{cases}$$

Hence we have

$$\chi_1(M, \Delta_t) \equiv \sum E(T\backslash \mathring{M}_{(T')}, \infty) \mod 2$$

where the sum is taken over all isotropy subgroups T' such that $\dim (T/T') = 1$. It is easy to see that this sum is equal to $E(T\backslash\Sigma(M))$.

4. Homogeneous spaces

In this section we evaluate $\chi_1(G/H, \Delta_t)$. We assume that G is connected but H need not be connected.

If rank $(H) =$ rank $(G) - 1$ let $I_G(H) = C_G(T')/T'$ where T' is a maximal torus of the identity component of H. Since $I_G(H)$ is a connected compact Lie group of rank 1 it is either S^1, $SO(3)$, or S^3.

(4.1) THEOREM. *If* rank $(H) <$ rank $(G) - 1$,

$$\chi_1(G/H, \Delta_t) = 0.$$

If rank $(H) =$ rank $(G) - 1$,

$$\chi_1(G/H, \Delta_t) \equiv |W(G)|/|W(H)| \mod 2.$$

Moreover, if $I_G(H)$ *is* $SO(3)$ *or* S^3 *then* $|W(G)|/|W(H)| \equiv 0 \bmod 2$, *hence*

$$\chi_1(G/H, \Delta_t) = 0.$$

Proof. Fix a maximal torus T' of the identity component of H and a maximal torus T of G such that $T' \subset T$. By Corollary (3.5),

$$\chi_1(G/H, \Delta_t) \equiv E(T\backslash\Sigma(G/H)) \mod 2, \tag{4.2}$$

where $\Sigma(G/H)$ is the circle point set of G/H relative to the left action of T. If rank $(H) <$ rank $(G) - 1$, the circle point set is empty and we are done. Assume then, from now on, that rank $(H) =$ rank $(G) - 1$. Let

$$N_G(T', T) = \{g \in G \mid gT'g^{-1} \subset T\} \tag{4.3}$$

and define

$$\phi\colon N_G(T', T) \to \Sigma(G/H) \tag{4.4}$$

by $\phi(g) = gH$. To see that ϕ is well defined note that the T-isotropy subgroup of gH is $T \cap gHg^{-1}$. Then $g \in N_G(T', T)$ implies that $gT'g^{-1} \subset T \cap gHg^{-1}$ and therefore dim $(T/T \cap gHg^{-1}) = 1$.

Since ϕ is T-equivariant we have

$$\psi = T\backslash\phi\colon T\backslash N_G(T', T) \to T\backslash\Sigma(G/H). \tag{4.5}$$

Now $U(H) = N_H(T')/T'$ acts on the right of $T\backslash N_G(T', T)$ by

$$(Tg)(hT') = Tgh.$$

This action is well defined since $hT' = T'h$ and $gT' \subset Tg$.

(4.6) ψ *is* $U(H)$*-invariant and induces a homeomorphism*

$$T\backslash N_G(T', T)/U(H) \to T\backslash\Sigma(G/H)$$

To prove (4.6) we first show that

$$\phi\colon N_G(T', T) \to \Sigma(G/H)$$

is onto. If $gH \in \Sigma(G/H)$ its isotropy subgroup $T \cap gHg^{-1}$ has maximal rank in gHg^{-1}. Hence $g^{-1}Tg \cap H$ has maximal rank in H. Let $T'' \subset g^{-1}Tg \cap H$ be a maximal torus of the identity component H_0 of H and let $h \in H_0$ be such that $hT'h^{-1} = T''$. Then $hT'h^{-1} \subset g^{-1}Tg$ and we have $ghT'h^{-1}g^{-1} \subset T$. Therefore $gh \in N_G(T', T)$ and $\phi(gh) = gH$.

It follows that the orbit map

$$\psi\colon T\backslash N_G(T', T) \to T\backslash\Sigma(G/H)$$

is onto. Obviously ψ is $U(H)$-invariant so it remains to show that if $\psi(Tg) = \psi(T\bar{g})$ there is $h \in N_H(T')$ such that $Tg = T\bar{g}h$. Since $\psi(Tg) = \psi(T\bar{g})$ we have $Tgh = T\bar{g}H$, hence there is $h \in H$ such that $Tg = T\bar{g}h$. We will show that $h \in N_H(T')$. $h = g^{-1}sg$ for some $s \in T$ so

$$h^{-1}T'h = \bar{g}^{-1}s^{-1}\bar{g}T'\bar{g}^{-1}sg \subset g^{-1}T'g,$$

since $\bar{g}T'\bar{g}^{-1} \subset T$. Hence

$$h^{-1}T'h \subset g^{-1}Tg \cap H_0.$$

Now $g^{-1}Tg \cap H_0 = T'$ since $gT'g^{-1} \subset T$ implies that $T' \subset g^{-1}Tg \cap H_0$. This completes the proof of (4.6).

By (4.2) and (4.6) we have

$$(4.7) \qquad \chi_1(G/H, \Delta_t) \equiv E(T\backslash N_G(T', T)/U(H)) \mod 2.$$

In order to compute this Euler characteristic we first determine the $U(H)$-isotropy subgroups of $T\backslash N_G(T', T)$.

(4.8) *The $U(H)$-isotropy subgroup of Tg is $g^{-1}Tg \cap H/T'$.*

Suppose $Tgh = Tg$. Then $h \in g^{-1}Tg$ and therefore $h \in g^{-1}Tg \cap H$. Conversely, if $h \in g^{-1}Tg \cap H$ then $Tgh = Tg$. Write $h = g^{-1}sg$, $s \in T$. Then, since $gT'g^{-1} \subset T$,

$$hT'h^{-1} = g^{-1}sgT'g^{-1}s^{-1}g = g^{-1}Tg$$

and therefore $hT'h^{-1} \subset g^{-1}Tg \cap H_0 = T'$. So $h \in N_H(T')$.

Let $I(H) = C_H(T')/T'$. Then $I(H)$ is a finite subgroup of $I_G(H) = C_G(T')/T'$. From (4.8) the $U(H)$-isotropy subgroups of $T\backslash N_G(T', T)$ are precisely the subgroups of $I(H)$ of the form $T'' \cap H/T'$ where T'' is a maximal torus of G such that $T' \subset T''$. Note that $T'' \cap H/T'$ is cyclic since it is a subgroup of T''/T'. It is easy to see that the situation may be rephrased as follows.

(4.9) *The $U(H)$-isotropy subgroups of $T\backslash N_G(T', T)$ are the cyclic subgroups of $I(H)$ having the form $S \cap I(H)$ where S is a maximal torus (circle) of $I_G(H)$.*

(4.10) *If A is a $U(H)$-isotropy subgroup then* $E(\text{Fix}\,(A)) = |W(G)|$.

Let $A = T'' \cap H/T'$ as above. Then $T'' \cap H$ is an abelian extension of the torus T' by the cyclic group and therefore A is topologically cyclic [1, P.80]. Let s be a generator of $T'' \cap H$. We will now apply a standard argument. For $x \in G$ define θ_x: $T\backslash G \to T\backslash G$ by $\theta_x(Tg) = Tgx$. In particular for θ_s: $T\backslash G \to T\backslash G$ we see that

$$\text{Fix}\,(\theta_s) \subset T\backslash N_G(T', T) \quad \text{and} \quad \text{Fix}\,(\theta_s) = \text{Fix}\,(A).$$

Since θ_s is an isometry relative to a G-invariant metric, the Lefschetz number $\Lambda(\theta_s)$ of θ_s is equal to $E(\text{Fix}\,(\theta_s))$. We now have

$$E(\text{Fix}\,(A)) = E(\text{Fix}\,(\theta_s)) = \Lambda(\theta_s) = \Lambda(\theta_e) = E(T\backslash G) = |W(G)|,$$

where $e \in G$ is the identity. This proves (4.10).

(4.11) *Let A be a $U(H)$-isotropy subgroup and $h \in N_H(T')$. If $A \neq hAh^{-1}$ then* $\text{Fix}\,(A) \cap \text{Fix}\,(hAh^{-1}) = \Phi$.

Suppose $x \in \text{Fix}\,(A) \cap \text{Fix}\,(hAh^{-1})$. If B is the isotropy subgroup of x then $A \subset B$ and $hAh^{-1} \subset B$. Since B is cyclic, $A = hAh^{-1}$.

To cut down on notation write $Z = T\backslash N_G(T', T)$. Let Z_A denote the set of points having isotropy subgroup A and, as before, let $Z_{(A)}$ denote the set of points having isotropy subgroup a conjugate of A. Now

$$E(Z/U(H)) = \sum \frac{|A|}{|U(H)|} E(\mathring{Z}_{(A)}, \infty)$$

and from (4.11), $\mathring{Z}_{(A)} = \bigvee \mathring{Z}_{A'}$, $A' \in (A)$. Therefore

$$\chi_1(G/H, \Delta_t) = E(Z/U(H)) = \frac{1}{|U(H)|} \sum |A| E(\mathring{Z}_A, \infty), \tag{4.12}$$

the sum taken over subgroups $A \subset I(H)$ of the form $S \cap I(H)$, S a circle of $I_G(H)$.

To compute this sum we consider the three possibilities for $I_G(H)$ separately.

Case 1. $I_G(H) = S^1$. Then the only subgroup of $I(H)$ that meets the requirement is $I(H)$ itself. We then have $E(\mathring{Z}_{I(H)}, \infty) = E(\text{Fix}\ (I(H))) = |W(G)|$, and

$$\chi_1(G/H, \Delta_t) = \frac{1}{|U(H)|} |I(H)|\,|W(G)| = \frac{|W(G)|}{|W(H)|}.$$

Case 2. $I_G(H) = SO(3)$. Then $I(H)$ is a finite group of rotations of R^3. Since each rotation fixes a line and a rotation that fixes two distinct lines is the identity, we easily deduce:

(a) A subgroup of $I(H)$ of the form $I(H) \cap S$, S a circle of $SO(3)$, is either maximal cyclic or the trivial subgroup $\{1\}$.

(b) If A and A' are distinct maximal cyclic subgroups then $A \cap A' = \{1\}$.

Now let $A_1, \ldots, A_n$ denote the maximal cyclic subgroups of $I(H)$. Then

$$E(\mathring{Z}_{A_i}, \infty) = E(\text{Fix}\ (A_i)) = |W(G)|$$

and

$$\begin{aligned} E(\mathring{Z}_{\{1\}}, \infty) &= E(\text{Fix}\ (\{1\})/\bigcup_1^m \text{Fix}\ (A_i)) \\ &= E(\text{Fix}\ (\{1\})) - \sum_1^m E(\text{Fix}\ (A_i)) \\ &= |W(G)|(1 - n). \end{aligned}$$

Hence

$$\chi_1(G/H, \Delta_t) = \frac{|W(G)|}{|U(H)|}\left[\left(\sum_1^n |A_i|\right) + (1 - n)\right].$$

Since each element of $I(H)$ lies in some A_i and $A_i \cap A_j = \{1\}$, $i \neq j$,

$$\sum_1^n |A_i| = |I(H)| + (n-1).$$

Therefore

$$\chi_1(G/H, \Delta_t) = \frac{|W(G)|}{|U(H)|} |I(H)| = \frac{|W(G)|}{|W(H)|}.$$

Case 3. $I_G(H) = S^3$. Using the double cover $\pi: S^3 \to SO(3)$ we deduce that $I(H)$ is either cyclic of odd order or $I(H) = \pi^{-1}(\Gamma)$ where $\Gamma \subset SO(3)$ [8; P.88].

If $I(H)$ is cyclic of odd order the subgroups of the form $S \cap I(H)$, S a circle of S^3, are $I(H)$ and $\{1\}$. Then

$$E(\mathring{Z}_{I(H)}, \infty) = E(\text{Fix }(I(H))) = |W(G)|$$

and

$$E(\mathring{Z}_{\{1\}}, D) = E(\text{Fix }(\{1\})) - E(\text{Fix }(I(H)) = 0.$$

It follows that $\chi_1(G/H, \Delta_t) = |W(G)|/|W(H)|$.

In the case where $I(H) = \pi^{-1}(\Gamma)$, $\Gamma \subset SO(3)$, we see that:

(a) A subgroup of $I(H)$ of the form $I(H) \cap S$, S a circle of S^3, is either maximal cyclic or the subgroup $\{+1, -1\}$.

(b) If A and A' are distinct maximal cyclic subgroups then $A \cap A' = \{+1, -1\}$.

The calculation of the right hand side of (4.12) now proceeds as in the $SO(3)$ case so we will omit the details. Once again we obtain $\chi_1(G/H, \Delta_t) = |W(G)|/|W(H)|$.

To complete the proof of Theorem (4.1) we will show that $|W(G)|/|W(H)|$ is even if $I_G(H)$ is $SO(3)$ or S^3. We have a fiber bundle

$$C_G(T')/T \to G/T \to G/C_G(T').$$

Let $S = T/T'$. Then $C_G(T')/T = I_G(H)/S$ so that $E(C_G(T')/T) = |W(I_G(H)| = 2$. Thus

$$W(G) = 2E(G/C_G(T')),$$

and to show that $|W(G)|/|W(H)|$ is even we will show that $|W(H)|$ divides $E(G/C_G(T'))$. Now $W(H) = N_H(T')/C_H(T')$ may be regarded as a subgroup of $N_G(T')/C_G(T')$ so that

(a) $|W(H)|$divides $E(N_G(T')/C_G(T'))$.

We have a covering

$$N_G(T')/C_G(T') \to G/C_G(T') \to G/N_G(T')$$

so that

(b) $E(N_G(T')/C_G(T'))$ divides $E(G/C_G(T'))$.

From (a) and (b), $|W(H)|$ divides $E(G/C_G(T'))$.

5. The semi-characteristic

The previous Theorem (4.1) together with (2.4) leads to the following result concerning the real semi-characteristic of a homogeneous space.

(5.1) COROLLARY. *Let G/H be a connected orientable homogeneous space of dimension* $4s+1$. *Then, as integers* mod 2,

$$k(G/H) = \begin{cases} |W(G)|/|W(H)|, & \text{rank } (H) = \text{rank } (G) - 1, \\ 0, & \text{rank } (H) < \text{rank } (G) - 1. \end{cases}$$

Moreover, if $I_G(H)$ is $SO(3)$ or S^3 then $|W(G)|/|W(H)| \equiv 0 \bmod 2$, *hence* $k(G/H) = 0$.

If dim $(G/H) = 4s - 1$ then Theorems (4.1) and (2.4) imply that $|W(G)|/|W(H)| \equiv 0 \mod 2$ when G/H is orientable and rank $(H) =$ rank $(G) - 1$. However if G/H is not orientable this is not necessarily the case. Consider the space $U_n/S_{n-1} \int T^{n-1}$ where $S_{n-1} \int T^{n-1}$ is the wreath product of the symmetric group S_{n-1} with the $(n-1)$-torus T^{n-1} embedded in the usual way. We have

$$|W(U_n)| \Big/ \left| W\left(S_{n-1} \int T^{n-1}\right)\right| = n$$

and

$$\dim \left(U_n/S_{n-1} \int T^{n-1}\right) = n^2 - n + 1.$$

Thus when $n - 1 = 2$ (odd) we see that

$$\dim \left(U_n/S_{n-1} \int T^{n-1}\right) \equiv -1 \bmod 4$$

and

$$|W(U_n)|/\left|W\left(S_{n-1} \int T^{n-1}\right)\right| \equiv 1 \bmod 2.$$

As an example of a class of homogeneous spaces having non-zero semi-characteristic consider the spaces $U_n/U_s \times U_{n-s-1}$. We have

$$|W(U_n)|/|W(U_s \times U_{n-s-1})| = \frac{n!}{s!\,(n-s-1)!} = m\binom{n-1}{s}$$

Write $n - 1 = \sum \alpha_i 2^i$ and $s = \sum \beta_i 2^i$, $0 \leq \alpha_i, \beta_i \leq 1$. Using the well known rule for computing binomial coefficients mod 2 (cf. [6, P.5]) we see that $k(U_n/U_s \times U_{n-s-1}) = 1$ if (a) n is odd and (b) $\beta_i \neq 0$ implies $\alpha_i \neq 0$, for all i.

From Theorems (4.1) and (3.1) we obtain under certain conditions a formula relating the semi-characteristic of a G-manifold to its orbit structure, which is similar to the well known formula for the Euler characteristic of a G-manifold.

(5.2) COROLLARY. *Let M be an orientable G-manifold of dimension* $4s+1$ *having no isotropy subgroups of maximal rank. Then, as integers* mod 2,

$$k(M) = \sum E(G\backslash \mathring{M}_{(H)}, \infty) | W(G) | / | W(H) |,$$

the sum taken over all conjugacy classes of isotropy subgroups H such that rank $(H) =$ rank $(G) - 1$.

REFERENCES

1. J. F. ADAMS, *Lectures on Lie groups*, W. A. Benjamin, New York, 1969.
2. M. F. ATIYAH and J. L. DUPONT, *Vector fields with finite singularities*, Acta Math., vol. 128 (1971), pp. 1-40.
3. J. C. BECKER and R. E. SCHULTZ, *Fixed point indices and left invariant framings*, Lecture Notes in Math. vol. 657, Springer, New York, 1978, pp. 1-31.
4. H. HOPF and H. SAMELSON, *Ein Satz über die Winkungsräume geschlossener Lie'scher Gruppen*, Comment Math. Helv., vol. 13 (1940), pp. 240-251.
5. E. Y. MILLER, Letter to the author, 1974.
6. N. E. STEENROD, *Cohomology operations*, Princeton Univ. Press, Princeton, N. J., 1962.
7. A. G. WASSERMAN, *Equivariant differential topology*, Topology, vol. 8 (1969), pp. 127-150.
8. J. A. WOLF, *Spaces of constant curvature*, McGraw-Hill, New York, 1967.

PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA