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A GHASTLY GENERALIZED n-MANIFOLD[1]

BY

ROBERT J. DAVERMAN AND JOHN J. WALSH

1. Introduction

A standard source for generalized n-manifolds (finite dimensional ANR's X such that $H_*(X, X - pt; Z) \cong H_*(E^n, E^n - pt; Z)$) is decomposition spaces of cell-like but not necessarily cellular decompositions of n-manifolds. The generalized 3-manifolds that contain no 2-disk constructed by Bing-Borsuk [3] arise as decompositions of S^3 whose nondegenerate elements form a null sequence of noncellular arcs. The improvements produced by S. Singh [14, $n = 3$], and D. Wright [18, $n \geq 4$], in which generalized n-manifolds are constructed containing no proper ANR's of dimension greater than or equal to 2, also result from decompositions of S^n whose nondegenerate elements form a null sequence of noncellular arcs. Cannon-Daverman [5] constructed cell-like totally noncellular decompositions of n-manifolds and used these to build totally wild flows. We add to this list of generalized manifolds having properties not satisfied by honest manifolds and produce a generalized n-manifold X ($n \geq 3$) such that, for each map $F: B^2 \to X$ where $F | S^1$ is an embedding, $F(B^2)$ has nonempty interior in X. An interesting feature of these examples is that, while they arise from totally noncellular decompositions similar to those of [5], they exhibit properties similar to those exhibited in [3], [14], [15] and [18]. Clearly they contain no 2-disks nor ANR's of dimension strictly between 1 and n.

Prior to the description of these examples, this paper lays a broad theoretical groundwork for dealing with defining sequences. It begins by setting forth an axiom base for a general definition of defining sequence for an upper semicontinuous decomposition, more general than the *ad hoc* definition given in [5], adds another axiom for working with cell-like decompositions, introduces the notion of shrinkable defining sequences, and treats the naturality of each. To summarize the remaining contents of the paper, the examples themselves then are constructed in Section 5, their pathology is studied in Section 6, and the result that, with extra care about their construction, their product with E^1 is a manifold is established in Section 7.

Spaces are always assumed to be metrizable and are usually assumed to be locally compact. The general development of defining sequences for decompositions and the specialization to those yielding cell-like decompositions are

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valid for arbitrary metric spaces while the discussion of shrinkable defining sequences is limited to locally compact spaces.

2. Defining sequences and decompositions

The cell-like decompositions of manifolds mentioned in the literature, which often happen to be closed-0-dimensional ones (meaning that in the decomposition space the closure of the image of the nondegenerate elements is 0-dimensional), frequently are described by defining sequences. In this section we generalize this standard notion of defining sequence. Although one can find more restrictive generalizations elsewhere, such as those given in [5] and [12], and for technical reasons aimed at producing specific kinds of decompositions one might wish to vary our definition somewhat, ours serves as an all-encompassing definition, because every upper semicontinuous decomposition of a locally compact metric space arises from such a defining sequence (Theorem 2.4).

Let X be a space and $\mathcal{M}$ a collection of subsets of X. For an arbitrary subset A of X define its *star in* $\mathcal{M}$ as

$$st(A, \mathcal{M}) = A \cup (\cup\{M \in \mathcal{M} \mid M \cap A \neq \phi\})$$

and, recursively for any integer $k \geq 1$, define its kth-*star in* $\mathcal{M}$ as

$$st^k(A, \mathcal{M}) = st(st^{k-1}(A, \mathcal{M}), \mathcal{M}).$$

When $A = \{x\}$, $x \in X$, we write $st^k(\{x\}, \mathcal{M})$ simply as $st^k(x, \mathcal{M})$.

Now suppose X is a (locally) compact metric space. A *defining sequence* (in X) is a sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ satisfying the following axioms:

AXIOM 1. For each i the set $\mathcal{M}_i$ is a (locally) finite collection

$$\{M(i, 1), M(i, 2), \ldots, M(i, r(i))\}$$

of compact subsets of X having nonempty, pairwise disjoint interiors.

AXIOM 2. For each i and each $x \in X$, $st^3(x, \mathcal{M}_{i+1}) \subset \operatorname{Int} st^2(x, \mathcal{M}_i)$.

The decomposition G of X associated with a defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is the relation prescribed by the rule: for any $x \in X$, $G(x)$ is the subset of X consisting of all $y \in X$ such that for every integer $i > 0$, $y \in st^2(x, \mathcal{M}_i)$.

Such a relation G on X obviously is reflexive and symmetric. The next lemma aids in showing that it is transitive.

LEMMA 2.1. *Suppose $\{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is a defining sequence on X. For each $k > 0$ and each $x \in X$,*

$$st^4(x, \mathcal{M}_{k+1}) \subset st^3(x, \mathcal{M}_k).$$

Proof. Certainly each $x' \in st^4(x, \mathcal{M}_{k+1})$ satisfies $x' \in st(x^*, \mathcal{M}_{k+1})$ for some $x^* \in st^3(x, \mathcal{M}_{k+1})$. By Axiom 2,

$$x^* \in st^3(x, \mathcal{M}_{k+1}) \subset \text{Int } st^2(x, \mathcal{M}_k).$$

Name $M^* \in \mathcal{M}_{k+1}$ for which $x^* \in M^* \subset st^3(x, \mathcal{M}_{k+1})$. By Axiom 1, M^* has non-empty interior, and the local finiteness of $\mathcal{M}_k$ yields a point $x^{**} \in M^*$ such that some $M \in \mathcal{M}_k$ contains x^{**} in its interior. The uniqueness of such an M, guaranteed by Axiom 1, indicates that $M = st(x^{**}, \mathcal{M}_k)$. Thus, the fact that $x^{**} \in M^* \subset st^2(x, \mathcal{M}_k)$ implies $M \subset st^2(x, \mathcal{M}_k)$, and the observation that $x' \in st^2(x^{**}, \mathcal{M}_{k+1})$ then implies

$$x' \in st^2(x^{**}, \mathcal{M}_{k+1}) \subset st^2(x^{**}, \mathcal{M}_k) = st(M, \mathcal{M}_k) \subset st^3(x, \mathcal{M}_k).$$

LEMMA 2.2. *The relation G associated with the defining sequence $\{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is transitive.*

Proof. Suppose $y \in G(x)$ and $z \in G(y)$. It follows that for all $i > 0$ $z \in st^4(x, \mathcal{M}_{i+2})$. By Lemma 2.1 and Axiom 2,

$$z \in st^4(x, \mathcal{M}_{i+2}) \subset st^3(x, \mathcal{M}_{i+1}) \subset st^2(x, \mathcal{M}_i).$$

Hence, $z \in G(x)$.

THEOREM 2.3. *The decomposition G associated with a defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is upper semicontinuous.*

Proof. To see this directly from the definition of upper semicontinuity, consider a neighborhood U of some $g \in G$, where $g = \bigcap_i st^2(x, \mathcal{M}_i)$. Lemma 2.1 and the compactness of g imply the existence of an integer $k > 0$ such that $st^4(x, \mathcal{M}_k) \subset U$. Now define V as

$$U - \bigcup \{M \in \mathcal{M}_k \mid M \cap st^2(x, \mathcal{M}_k) = \phi\}.$$

Then, for each $M \in \mathcal{M}_k$ satisfying $M \cap V \neq \phi$, $M \subset st^3(x, \mathcal{M}_k)$, and

$$st^2(V, \mathcal{M}_k) = V \cup st^4(x, \mathcal{M}_k) \subset U.$$

According to the definition of G, any $g' \in G$ intersecting V is contained in $st^2(V, \mathcal{M}_k) \subset U$.

THEOREM 2.4. *If G is an upper semicontinuous decomposition of the locally compact metric space X, then G is the decomposition associated with some defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$.*

Proof. Let X/G denote the associated decomposition space and $\pi: X \to X/G$ the induced map. For the moment, assume that $\mathcal{S} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots\}$ is a defining sequence in X/G for the trivial decomposition consisting of single points; further, assume that each $\mathcal{P}_k$ is a cover and that

$$\text{Int } (\pi^{-1}(\text{Fr } (A))) = \phi \quad \text{for each } A \in \mathcal{P}_k.$$

Let $\mathcal{M}_k = \{\pi^{-1}(A): A \in \mathcal{P}_k\}$; it is an easy matter to verify that $\{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is a defining sequence and that G is the associated decomposition.

It remains to produce the defining sequence $\mathcal{S}$. In particular, the inductive construction given below produces $\mathcal{P}_1$. Let $\mathcal{U}$ be a locally finite open cover of X/G satisfying:

(1) each element of $\mathcal{U}$ has compact closure and diameter less than $1/k$;
(2) for each $U \in \mathcal{U}$, Int $(\pi^{-1}(\mathrm{Fr}\ (U))) = \phi$;
(3) for each $y \in X/G$,

$$st^3(y, \bar{\mathcal{U}}) \subseteq st^2(y, \mathcal{P}_{k-1}) \quad \text{where } \bar{\mathcal{U}} = \{\bar{U}: U \in \mathcal{U}\}.$$

If a first choice of $\mathcal{U}$ does not satisfy condition (2), then perform the following modification. Let $\mathcal{V} = \{V(U): U \in \mathcal{U}\}$ be an open cover obtained by shrinking $\mathcal{U}$ [9, p. 152]. For each $U \in \mathcal{U}$, choose a map $f: \bar{U} \to [0, 1]$ with $f(\mathrm{Fr}\ (U)) = 0$ and $f(\overline{V(U)}) = 1$; for uncountably many $t \in (0, 1)$, $U' = f^{-1}((t, 1])$ has the property that Int $(\pi^{-1}(\mathrm{Fr}\ (U'))) = \phi$ ($\pi^{-1}(\bar{U})$ is compact and, hence, second countable). Modify the cover $\mathcal{U}$ by replacing each $U \in \mathcal{U}$ by such a U'.

Let $\{U_1, U_2, \ldots, U_\alpha, \ldots\}$ be a well-ordering of the elements of $\mathcal{U}$. For each α define $P_\alpha = \mathrm{Cl}\ (U_\alpha - \bigcup_{\beta<\alpha} \bar{U}_\beta)$. Finally, let $\mathcal{P}_k$ be the collection of those P_α's that are nonempty. The local finiteness of $\mathcal{U}$ insures that $\pi^{-1}(\mathrm{Fr}\ P_\alpha)$ is contained in a finite subcollection of

$$\{\pi^{-1}(\mathrm{Fr}\ U_\beta): \beta \leq \alpha\},$$

implying that $\pi^{-1}(\mathrm{Fr}\ U_\alpha)$ is empty.

For those who feel that, in case X is a PL n-manifold, a defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ for a decomposition G on X should involve PL n-manifolds M as the elements of $\mathcal{M}_i$, see Appendix 2 for a proof that G has a defining sequence of this type.

3. Defining sequences and cell-like decompositions

A sufficient condition that a decomposition G associated with a defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ be cell-like is that $\mathcal{S}$ satisfy the additional axiom:

AXIOM 3. For each i and each $x \in X$ there is a j with $st^3(x, \mathcal{M}_j)$ contractible in $st^2(x, \mathcal{M}_i)$.

Given any other defining sequence $\mathcal{S}' = \{\mathcal{M}'_1, \mathcal{M}'_2, \ldots\}$ for G, for each i and each $x \in X$, there is a j with $st^3(x, \mathcal{M}'_j) \subseteq st^3(x, \mathcal{M}_i)$; in particular, it follows that if Axiom 3 is satisfied for one defining sequence for G, then it is satisfied for every defining sequence for G.

PROPOSITION 3.1. *For X an* ANR, *Axiom 3 is a necessary and sufficient condition for the associated decomposition G to be cell-like.*

Proof. Assume that G is cell-like and let x and $i \geq 1$ be given. The element

$$G(x) = \bigcap st^2(x, \mathcal{M}_k) \subseteq \text{Int } (st^2(x, \mathcal{M}_i))$$

is contractible in Int $(st^2(x, \mathcal{M}_i))$. Since an open subset of X is an ANR, a given contraction extends to a contraction of a neighborhood N of $G(x)$. One can complete the verification of Axiom 3 by choosing $j \geq i$ such that $st^3(x, \mathcal{M}_j) \subset N$.

The proof that Axiom 3 implies cell-like is elementary.

4. Shrinkable defining sequences

R. H. Bing [1] and others have introduced concepts of a decomposition being shrinkable. No matter how it is defined, all such notions of shrinkability seem equivalent in compact spaces; however, they may differ in locally compact spaces. We find it convenient to say that a decomposition G of a locally compact space X is *shrinkable* provided that for each compact subset $C \subseteq X$, each locally finite G-saturated cover $\mathcal{W}$ of X by relatively compact open sets, and each open cover $\mathcal{U}$ of X, there is a homeomorphism $h: X \to X$ such that $h(x) \notin C$ for $x \notin st(C, \mathcal{W})$, such that $h(x) \in st(x, \mathcal{W})$ for $x \in st(C, \mathcal{W})$, and such that for $g \in G$ with $g \cap C \neq \phi$ there is a $U \in \mathcal{U}$ with $h(g) \subseteq U$. The following axion provides a notion of shrinkability for a defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ for G.

AXIOM 4. For each compact subset $C \subseteq X$, for each i, and for each open cover $\mathcal{U}$ of X, there is a j and a homeomorphism $h: X \to X$ such that $h(x) \notin C$ for $x \notin st^2(C, \mathcal{M}_i)$, such that $h(x) \in st^2(x, \mathcal{M}_i)$ for $x \in st^2(C, \mathcal{M}_i)$, and such that for $M \in \mathcal{M}_j$ with $M \cap C \neq \phi$ there is a $U \in \mathcal{U}$ with $h(M) \subseteq U$.

The next result records the equivalence of Axiom 4 and shrinkability of the associated decomposition for locally compact spaces.

PROPOSITION 4.1. *Let $\mathcal{S}$ be a defining sequence for a decomposition G of a locally compact metric space X. Then G is shrinkable if and only if $\mathcal{S}$ satisfies Axiom* 4.

Proof. First suppose that G is shrinkable. Let a compact subset $C \subseteq X$, an integer i, and an open cover $\mathcal{U}$ of X be given. Let C' be a G-saturated compact set with $st^2(C, \mathcal{M}_i) \subseteq C'$. Let $\mathcal{W}$ be a G-saturated open cover of X with $st(x, \mathcal{W}) \subseteq st^2(x, \mathcal{M}_i)$ for each $x \in X$. The "shrinking" homeomorphism for G chosen with respect to C', $\mathcal{U}$, and $\mathcal{W}$ is the necessary homeomorphism with the existence of j coming from the compactness of C'.

Conversely, suppose that $\mathcal{S}$ satisfies Axiom 4. Let a compact set $C \subseteq X$, a locally finite G-saturated cover $\mathcal{W}$ by relatively compact open sets, and an open cover $\mathcal{U}$ be given. Let $\mathcal{U}'$ be an open cover of X with

$$\{st^2(U, \mathcal{U}'): U \in \mathcal{U}'\}$$

refining $\mathscr{U}$. Let C' be a compact set with $st(C, \mathscr{W}) \subseteq C'$. Let i be such that $st^2(x, \mathscr{M}_i) \subseteq st(x, \mathscr{W})$ for $x \in C'$. The homeomorphism of Axiom 4 chosen with respect to C', i, and $\mathscr{U}'$ is the desired "shrinking" homeomorphism for G.

In a locally compact space which is also locally contractible, every shrinkable decomposition is cell-like.

COROLLARY 4.2. *Suppose that X is locally compact and locally contractible. Then a defining sequence of X that satisfies Axiom* 4 *also satisfies Axiom* 3.
The argument is left to the reader.

5. Defining sequences for totally noncellular decompositions

In this section we describe a defining sequence $\mathscr{S}$ for a cell-like but totally noncellular decomposition G of a closed connected, PL n-manifold T $(n > 2)$. Such sequences have also been described by Cannon-Daverman in [5], and we make explicit use of their techniques, with modifications ultimately yielding decomposition spaces having properties not possible with defining sequences purely of the form they set forth. As in [5], part of our strategy is to specify $\mathscr{S}$ so that each element of G is 1-dimensional and contains a wild Cantor set.

Before starting the construction, we point out an indispensible result about the formation of Cantor sets.

LEMMA 5.1. *Suppose Y is a Cantor set in the n-manifold T, S is a closed, orientable PL $(n-2)$-manifold with $S \times B^2 \subset T$, and $\varepsilon > 0$. Then there exists a finite collection $R_1, R_2, \ldots, R_t$ of closed, connected, PL $(n-2)$-manifolds in $(S \times \operatorname{Int} B^2) - Y$ such that the surfaces $R_1, \ldots, R_t$ have pairwise disjoint PL-product neighborhoods $R_1 \times B^2, \ldots, R_t \times B^2$ in $(S \times \operatorname{Int} B^2) - Y$, each of diameter less than ε, and such that whenever $f\colon H \to S \times B^2$ is an I-essential map of a disk with holes H, then $f(H) \cap (\bigcup R_i) \neq \phi$.*

Lemma 5.1 is due to Daverman-Edwards. Certain aspects of the proof are presented in Section 3 of [6], where additionally a map f of a disk with holes H to an n-manifold N is defined to be *I-essential* (an abbreviation of *interior-essential*) if $f(\partial H) \subseteq \partial M$ and there is no map $f'\colon H \to \partial M$ for which $f' | \partial H = f | \partial H$.

Serving as a guide for this construction of this section is a preordained denumerable collection $\mathscr{D}$ of thickened $(n-2)$-manifolds. In [5], only one such thickened object was employed.

The collection of thickened $(n-2)$-manifolds. Let

$$\mathscr{D} = \{L_1, L_2, \ldots, L_k, \ldots\}$$

denote a collection of compact PL n-manifolds in T, each of the form $L_k = S_k \times B^2$, where S_k denotes a closed, possibly disconnected, orientable, PL $(n-2)$-manifold. In addition, $\mathscr{D}$ is determined so that for any PL embedding $e: S \times B^2 \to T$, where S is a closed, orientable, PL $(n-2)$-manifold, and any $\varepsilon > 0$, there exists $L_k \in \mathscr{D}$ and there exists a PL homeomorphism e' of $S \times B^2$ onto L_k such that dist $(e', e) < \varepsilon$. This means, of course, that the elements of $\mathscr{D}$ are NOT required to be pairwise disjoint; however, if T is not simply connected, L_1 is required to lie interior to some n-cell in T.

Description of $\mathscr{M}_1$. Let $\mathscr{M}_1 = \{T\}$. Associate with $T \in \mathscr{M}_1$ the closed $(n-2)$-manifold $S(T)$ equal to the core $S_1 \times \{0\} \subset S_1 \times B^2$ of $L_1 \in \mathscr{D}$. In order to begin an inductive procedure, name a set $W(1, 1)$ as $W(1, 1) = L_1$.

At this spot one may wish to identify a simple closed curve J in T, where $J = pt \times \partial B^2$ in $S_1 \times B^2 = L_1$. The significance of J is the impossibility of contracting it in $T - S(T)$, an observation which will be used later to show that no element of G is cellular.

Inductive Hypothesis $(j-1)$. Suppose that $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_{j-1}$ (where $j > 1$) and closed, possibly disconnected, orientable, PL $(n-2)$-manifolds $S(A)$, $A \in \bigcup_{i=1}^{j-1} \mathscr{M}_i$, are given satisfying the following properties:

(1) For $1 \leq i \leq j-1$, $\mathscr{M}_i$ is a cover of T.

(2) $\mathscr{M}_1, \ldots, \mathscr{M}_{j-1}$ satisfy the appropriate features of Axioms 1 and 2 in the definition of defining sequence.

(3) For $1 \leq i < j-1$ and $x \in T$, the inclusion map

$$st^3(x, \mathscr{M}_{i+1}) \to st^2(x, \mathscr{M}_i)$$

is null homotopic.

(4) For $1 < i \leq j-1$ and $x \in T$, there exists a $(1/i)$-map of $st^2(x, \mathscr{M}_i)$ to a 1-complex.

(5) For $1 < i \leq j-1$, each $A \in \mathscr{M}_i$ is associated with a predecessor Pre $A \in \mathscr{M}_{i-1}$ where $A \cap$ Pre $A \neq \phi$ and $\{\text{Pre } A: A \in \mathscr{M}_i\} = \mathscr{M}_{i-1}$.

(6) For $1 \leq i \leq j-1$ and $A \in \mathscr{M}_i$, $S(A)$ has a PL-product neighborhood $S(A) \times B^2$ in Int A with components having diameters less than $1/i$ and with $S(A)$ corresponding to $S(A) \times \{0\}$.

(7) For $1 \leq k \leq j-1$, each $L_k \in \mathscr{D}$ contains pairwise disjoint, compact, PL n-manifolds $W(k, 1), \ldots, W(k, t(k))$ in Int L_k such that the only loops in $T -$ Int L_k null homotopic in $T - \bigcup_{i=1}^{t(k)} W(k, i)$ are those that are null homotopic in $T -$ Int L_k.

(8) For $1 \leq k \leq i \leq j-1$ and $A \in \mathscr{M}_i$, $S(A)$ possesses a closed and open subset (possibly empty) $S_{kq}(A)$ such that $S_{kq}(A) \times B^2 \subset W(k, q)$. In case $S_{kq}(A) \neq \phi$, then for any map $F: B^2 \to T$ such that $F^{-1}(W(k, q))$ has a component H for which $F|H: H \to W(k, q)$ is I-essential, $F(H) \cap S_{kq}(A) \neq \phi$.

(9) For $1 \leq k < i \leq j-1$ and $A \in \mathscr{M}_i$, $S_{kq}(A) \neq \phi$ if and only if

$$S_{kq}(\text{Pre } A) \neq \phi, \qquad S_{kq}(A) \times B^2 \subset S_{kq}(\text{Pre } A) \times B^2,$$

and for any map $F: B^2 \to T$ such that $F^{-1}(S_{kq}(\text{Pre } A) \times B^2)$ has a component H for which $F|H: H \to S_{kq}(\text{Pre } A) \times B^2$ is I-essential,

$$F(H) \cap S_{kq}(A) \neq \phi.$$

(10) For $1 \leq k < j-1$, each $W(k, q)$ contains a point w_{kq} such that every $A \in \mathscr{M}_{k+1}$ contained in $st^5(w_{kq}, \mathscr{M}_{k+1})$ has $S_{kq}(A) \neq \phi$.

Description of $\mathscr{M}_j$. Assuming Inductive Hypothesis $(j-1)$, we shall specify $\mathscr{M}_j$ and the associated $(n-2)$-manifolds $S(A)$, $A \in \mathscr{M}_j$, so that $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_j$ and these associated manifolds fulfill Inductive hypothesis j. Our specification involves two operations, the first a direct application of the methods of Cannon-Daverman [5], and the second a painstaking variation of the first. With this arrangement some of the traditionally messy details become quite manageable, for the methods of [5] used first permit controls governing all the epsilonic conditions arising in the Inductive Hypothesis.

The first operation. We dig far enough into the defining sequence constructed in [5] to extract a finite cover $\mathscr{N}_j$ of T by compact PL n-manifolds with pairwise disjoint interiors such that, in addition:

(2′) For all $x \in T$, $st^9(x, \mathscr{N}_j) \subset \operatorname{Int} st^2(x, \mathscr{M}_{j-1})$.

(3′) For all $x \in T$, the inclusion $st^9(x, \mathscr{N}_j) \to \operatorname{Int} st^2(x, \mathscr{M}_{j-1})$ is null homotopic.

(4′) For all $x \in T$, there exists a $(1/j)$-map of $st^6(x, \mathscr{N}_j)$ to a 1-complex.

(5′) Each $N \in \mathscr{N}_j$ is associated with a unique predecessor $\operatorname{Pre} N \in \mathscr{M}_{j-1}$ and $N \subset \operatorname{Pre} N$.

(6′) To each $N \in \mathscr{N}_j$ is associated a closed, possibly disconnected, orientable, PL $(n-2)$-manifold $S(N)$ having a PL product neighborhood $S(N) \times B^2$, with $S(N)$ corresponding to $S(N) \times \{0\}$, and with

$$S(N) \times B^2 \subset S(\operatorname{Pre} N) \times B^2.$$

(8′-9′) For $1 \leq k \leq j-1$ and $N \in \mathscr{N}_j$, $S(N)$ has a closed and open subset $S_{kq}(N)$ defined as

$$S_{kq}(N) = S(N) \cap (S_{kq}(\operatorname{Pre} N) \times B^2) \subset W(k, q),$$

where $S_{kq}(N) \neq \phi$ if and only if $S_{kq}(\operatorname{Pre} N) \neq \phi$. Moreover, for any map

$$F\colon B^2 \to T,$$

if $F^{-1}(S_{kq}(\operatorname{Pre} N) \times B^2)$ has a component H for which

$$F\,|\,H\colon H \to S_{kq}(\operatorname{Pre} N) \times B^2$$

is I-essential, then $F(H) \cap S_{kq}(N) \neq \phi$.

(10′) Each $W(j-1, q)$ contains a point w_q such that every $N \in N_j$ contained in $st^{15}(w_q, \mathscr{N}_j)$ satisfies $S_{j-1,q}(N) \neq \phi$.

By way of explanation we remark that, for any cover $\mathscr{M}$ of T by compact PL manifolds with pairwise disjoint interiors, the inductive procedure of [5] yields another cover $\mathscr{M}'$ of T such that, among other features,

$$st(x, \mathscr{M}') \subset st(x, \mathscr{M}) \quad \text{and} \quad st^3(x, \mathscr{M}') \subset \operatorname{Int} st^2(x, \mathscr{M})$$

for each $x \in T$. Conditions (2′), (3′), (4′), (5′) and (6′) follow routinely from properties of their procedure by selecting $\mathcal{N}_j$ as an iterate of that procedure applied to $\mathcal{M}_{j-1}$. By choosing $w_q \in \operatorname{Int} N'$, where $N' \in \mathcal{N}_j$ has Pre N' meeting Int $W(j-1, q)$ but where $N' \cap \partial(\text{Pre } N') = \phi$, one has that every $N \in \mathcal{N}_j$ included in $st^2(w_q, \mathcal{N}_j)$ satisfies $S_{j-1,q}(N) \neq \phi$; then by iterating again one also obtains Condition (10′). Condition (8′-9′) also follows, based upon the following fact describing how the I-essential property iterates through a nest. Although not quite the same result, the proof of Lemma 2.2 in [6] provides the bulk of the argument.

LEMMA 5.2. *Suppose $S_1, S_2, \ldots, S_t$ are compact, possibly disconnected, PL n-manifolds in T such that $S_{i+1} \subset \operatorname{Int} S_i$ and, for every I-essential map $f\colon H^* \to S_i$ of a disk with holes H^* in S_i $(i = 1, \ldots, t-1)$, $f(H^*) \cap S_{i+1} \neq \phi$. If $F\colon H \to S_1$ is an I-essential map of a disk with holes, then $F(H) \cap S_t \neq \phi$. Moreover, if $F\colon B^2 \to T$ is a map such that $F(\partial B^2) \subset T - S_1$ and $F \mid \partial B^2$ is not homotopic to a constant map in $T - S_1$, then $F(B^2) \cap S_t \neq \phi$.*

Ideally from our perspective the reader not only should be acquainted with the construction method of [5] but also should concur with the claim that this first operation can proceed as outlined above. In order to aid those who are unfamiliar with [5], we shall delineate it later on as a subprogram of the next operation.

The second operation. Ultimately with any $A \in \mathcal{M}_j$ we will associate a unique predecessor Pre* $A \in \mathcal{N}_j$ such that $\operatorname{Int} A \cap \operatorname{Int} \text{Pre*} A \neq \phi$, and then we will define Pre $A \in \mathcal{M}_{j-1}$, in the obvious fashion, to be Pre (Pre* A). Fix $N \in \mathcal{N}_j$ and let $S = S(N)$. It will suffice to describe the elements of $\mathcal{M}_j$ having N as predecessor. We will construct such elements of $\mathcal{M}_j$ to be n-manifolds $A_1, \ldots, A_s$ near N and almost covering N, together with their associated $(n-2)$-manifolds, in seven steps.

Step 1. Identifying a Cantor set Y_j with L_j. Since the collection

$$\{\operatorname{Int} st(x, \mathcal{N}_j)\colon x \in T\}$$

forms an open cover of T, there exists $\delta > 0$ such that every 2δ-set in T lies in some Int $st(x, \mathcal{N}_j)$. One should check that this implies that the δ-neighborhood U_z of any $z \in T$ satisfies

$$(*) \qquad U_z \subseteq \operatorname{Int} st^2(z, \mathcal{N}_j).$$

By Lemma 5.1, there exist closed, connected orientable PL $(n-2)$-manifolds $P_1, P_2, \ldots, P_{r(j)}$ in $S_j \times \operatorname{Int} B^2 = \operatorname{Int} L_j$ having pairwise disjoint, PL-product neighborhoods

$$P_1 \times B^2, P_2 \times B^2, \ldots, P_{r(j)} \times B^2 \quad \text{in } S_j \times \operatorname{Int} B^2,$$

each of diameter less than δ, such that if $f\colon H \to S_j \times B^2$ is an I-essential map of a disk with holes H, then $f(H) \cap (\bigcup P_i) \neq \phi$. Define L_{j1} as $\bigcup P_i$. Repeat this to obtain compact n-manifolds with boundary $L_{j1}, L_{j2}, \ldots$ such that each is a product of B^2 with a closed, orientable, PL $(n-2)$-manifold, $L_{j,i+1} \subset \operatorname{Int} L_{ji}$, the diameter of every component of L_{ji} is less than δ/i, and, whenever $f\colon H \to L_{ji}$ is an I-essential map, $f(H) \cap L_{j,i+1} \neq \phi$. Let Y_j denote the compact 0-dimensional set $\bigcap_{i=1}^{\infty} L_{ji}$.

Step 2. Splitting the surface $S(N)$. Again by Lemma 5.1, there exists a finite collection $R_1, \ldots, R_t$ of closed, connected, PL $(n-2)$-manifolds having pairwise disjoint PL-product neighborhoods

$$R_1 \times B^2, \ldots, R_t \times B^2 \quad \text{in } (S(N) \times \operatorname{Int} B^2) - Y_j$$

such that, whenever $f\colon H \to S(N) \times B^2$ is an I-essential map, $f(H) \cap (\bigcup R_i) \neq \phi$.

Step 3. Splitting the core $S_j \times \{0\}$ *of* L_j. Since Y_j misses each R_i above associated with (each) $N \in \mathcal{N}_j$, there exists an index k such that L_{jk} also misses R_i. Because $L_{jk} \subset L_j$, we find it convenient to think of k equalling 1 and to use all the notation introduced in Step 1 for referring to this L_{j1} and its components. While working near N, we will pay attention only to those components $P_1, \ldots, P_{r(N)}$ of L_{j1} that intersects N.

Step 4. Decomposing N into cells. Let $\sum_\partial$ denote a PL triangulation of ∂N. It defines a cell-decomposition of some (any) PL collar $\partial N \times [0, 1]$ on ∂N in N, which extends to a PL cell-decomposition $\sum_N$ of N. Shortening the collar and subdividing both $\sum_\partial$ and $\sum_N$, if necessary, we may assume that the collar $\partial N \times [0, 1]$ misses $S(N) \times B^2$ and that each of the n-cells $C_1, \ldots, C_s$ of $\sum_N$ is so small that there exists a point b_i in $\operatorname{Int} B^2$ such that C_i misses

$$\begin{aligned}(R_1 \cup \cdots \cup R_t) \times \{b_i\} \subset (R_1 \cup \cdots \cup R_t) \times B^2 \\ = (R_1 \times B^2) \cup \cdots \cup (R_t \times B^2).\end{aligned}$$

Furthermore, we may require general position features forcing $P_j = P_j \times \{0\}$ to contain a point of $\operatorname{Int} C_i$ $(i = 1, \ldots, s)$ whenever P_j intersects C_i. One should keep in mind the realignment completed in Step 3 guaranteeing that

$$[(R_1 \cup \cdots \cup R_t) \times B^2] \cap [(P_1 \cup \cdots \cup P_{r(j)}) \times B^2] = \phi.$$

Step 5. Ramifying the surfaces $(R_1 \cup \cdots \cup R_t) \cup (P_1 \cup \cdots \cup P_{r(N)})$. Pick distinct points $b_1, \ldots, b_s$ from $\operatorname{Int} B^2$, one for each n-cell C_i of $\sum_N$, as in the previous step, such that

$$C_i \cap [(R_1 \cup \cdots \cup R_t) \times \{b_i\}] = \phi.$$

In addition, pick distinct points p_{ik} from $\operatorname{Int} B^2$, one for each ordered pair $\langle i, k\rangle$ such that $\operatorname{Int} C_i$ meets P_k $(1 \leq k \leq r(N))$, close enough to $0 \in B^2$ that

Int C_i meets $P_k \times \{p_{ik}\}$, and so that the points $\{p_{ik}\}$ corresponding to the cells $\{C_i\}$ in $N \in \mathcal{N}_j$ differ from those corresponding to the cells $\{C'_i\}$ in any other $N' \in \mathcal{N}_j$. Now define

$$S(C_i) = [\bigcup_{k=1}^{t} (R_k \times \{b_i\})] \cup [\bigcup_{k=1}^{r(N)} (P_k \times \{p_{ik}\})].$$

One should interpret these sets $P_k \times \{p_{ik}\}$ to be nonempty only if $P_k \times \{0\}$ meets Int C_i.

Step 6. Connecting C_i to $S(C_i)$. Let $\alpha_1, \ldots, \alpha_s$ be pairwise disjoint sets, each the union of t disjoint PL arcs in Int N missing each of the $(n-2)$-manifolds $P_k \times \{p\}$, where p represents any of the points p_{ik} named above, not just corresponding to $N \in \mathcal{N}_j$ but to any $N' \in \mathcal{N}_j$, such that the t arcs of α_i irreducibly join the t components of

$$(R_1 \cup \cdots \cup R_t) \times \{b_i\} \subset S(C_i)$$

to C_i.

Step 7. Defining $A_1, \ldots, A_s \in \mathcal{M}_j$. After slight general position modifications, we may assume that $\alpha_i \cap S(C_m) = \phi$ whenever $i \neq m$ and that α_i and $S(C_i)$ meet each ∂C_m transversely. We choose a very fine triangulation $\sum$ of T subdividing each $\sum_N$, $N \in \mathcal{N}_j$, such that each C_i, each α_i, and each $S(C_i)$ is covered by a full subcomplex of $\sum$. By choosing $\sum$ sufficiently fine, we arrange things so that the sets

$$\alpha_1 \cup S(C_1), \ldots, \alpha_s \cup S(C_s)$$

have pairwise disjoint (simplicial) regular neighborhoods

$$U(\alpha_1 \cup S(C_1)), \ldots, U(\alpha_s \cup S(C_s))$$

in the first derived subdivision $\sum'$ of $\sum$, where the first derived neighborhood of α_i and of $R_i \times \{b_i\} \subset S(C_i)$ is contained in Int N and the first derived neighborhood of any $P_k \times \{p_{ik}\} \subset S(C_i)$ is a δ-set contained in Int $st(N, \mathcal{N}_j)$. (Recall condition (*) given in Step 1.) Define A_i to be $C_i \cup U(\alpha_i \cup S(C_i))$ minus the interiors of all the other $U(\alpha_m \cup S(C_m))$, for $\alpha_m \cup S(C_m)$ corresponding either to C_m contained in N or to an n-cell $C'_m \in \sum_{N'}$ contained in any other $N' \in \mathcal{N}_j$.

This essentially completes the inductive description of the defining sequence $S = \{\mathcal{M}_1, \ldots, \mathcal{M}_j, \ldots\}$. With the identification of $A_1, \ldots, A_s$ above, we have finished the construction of the elements of $\mathcal{M}_j$ whose predecessor (in $\mathcal{N}_j$) is N. Furthermore, in the course of that construction, we have specified $(n-2)$-manifolds $S(C_i)$ $(i = 1, \ldots, s)$ and now we designate the $(n-2)$-manifolds associated with $A_i \in \mathcal{M}_j$ as $S(A_i) = S(C_i)$ $(i = 1, \ldots, s)$. It should be clear that $S(A_i)$ has a PL product neighborhood $S(A_i) \times B^2 \subseteq$ Int A_i.

In a subsequent section we will need to make use of the observation that each $A \in \mathcal{M}_j$ is contained in the δ-neighborhood of Pre* $A \in \mathcal{N}_j$. (Recall from Step 1 that each P_k has diameter less than δ.)

Here is an appropriate place to explain that the first operation of this inductive construction, the one based on [5], proceeds like the second operation except: (i) the collection $\mathscr{D}$ and Steps 1 and 3 are completely left out, which entails leaving out all references to the $(n-2)$-manifolds P_k in the other steps as well; and (ii) size controls are added on the cells C_i in the decomposition of Step 4, on the manifolds $R_i \times B^2$ of Step 2, and on the thickenings of $C_i \cup \alpha_i \cup S(C_i)$ of Step 7 so as to yield a $(1/j)$-map of each A_i to a 1-complex and to insure $st^2(x, \mathscr{M}_j) \cap \mathrm{BdA}$ is small for any $x \in T$ and any $A \in \mathscr{M}_j$.

It remains to discuss why $\mathscr{M}_j$ and the associated $S(A)$, $A \in \mathscr{M}_j$, fulfill Inductive Hypothesis j, plus to specify

$$W(j, 1), \ldots, W(j, t(j))$$

corresponding to L_j in condition (7).

First, it should be clear from the foregoing description that conditions (1) and (5) of the Inductive Hypothesis are satisfied.

The key to conditions (2), (3), and (4) is the observation that, for each $x \in T$,

$$st(x, \mathscr{M}_j) \subset st^3(x, \mathscr{N}_j),$$

because the second operation yields

$$A_i \subset st^2(\mathrm{Int}\ C_i, \mathscr{N}_j) \subset st(N, \mathscr{N}_j)$$

for each $A_i \in \mathscr{M}_j$ and $N = \mathrm{Pre}^*\ A_i \in \mathscr{N}_j$. Then

$$st^3(x, \mathscr{M}_j) \subset st^9(x, \mathscr{N}_j) \subset \mathrm{Int}\ st^2(x, \mathscr{M}_{j-1})$$

and the total inclusion is null homotopic, by conditions (2′) and (3′) of the first operation. Similarly,

$$st^2(x, \mathscr{M}_j) \subseteq st^6(x, \mathscr{N}_j)$$

and the required $(1/j)$-map of the former to a 1-complex can be determined as the restriction of one on the latter, by condition (4′). It should be clear from the definition $S(A_i) = S(C_i)$ that condition (6) is satisfied.

For each $A_i \in \mathscr{M}_j$ define a set $S^*(A_i)$ by

$$S^*(A_i) = \bigcup_{k=1}^{r(N)} (P_k \times \{p_{ik}\})$$

(cf. Step 5), and regard $S^*(A_i) \times B^2$ as the obvious subset of $S(A_i) \times B^2$. List the components of all $S^*(A) \times B^2$ for $A \in \mathscr{M}_j$ as $W(j, 1), \ldots, W(j, t(j))$. Condition (7) follows from the definition of the $(n-2)$-manifolds P_k in Step 3 and from the fact that to each $P_k \times B^2$ there corresponds some C_i (anyone that intersects $P_k \times B^2$ will do) and hence some A_i such that a component of $S^*(A_i)$ equals $P_k \times \{p_{ik}\}$ in $P_k \times B^2$.

For $A \in \mathscr{M}_j$ define $S_{jq}(A)$ as $S^*(A) \cap W(j, q)$; notice that either $S_{jq}(A) = \phi$ or $S_{jq}(A) \times B^2 = W(j, q)$, making condition (8) trivially satisfied for $k = i = j$. For $1 \leq k < j$ define $S_{kq}(A)$ as

$$(S(A) - S^*(A)) \cap (S_{kq}(\mathrm{Pre}^*\ A) \times B^2);$$

the remainder of condition (8) results from Lemma 5.2, from conditions (8′-9′) in the first operation, and from the manner that those surfaces split from $S(\text{Pre* } A)$ in Step 2 are ramified in Step 5 to form $S(A) - S^*(A)$. Condition (9) follows for the same reasons.

Finally, according to (10′) each $W(j-1, q)$ contains a point w_q such that every $N \in \mathcal{N}_j$ included in $st^{15}(w_q, \mathcal{M}_j)$ has $S_{j-1,q}(N) = \phi$. Then

$$st^5(w_q, \mathcal{M}_j) \subset st^{15}(w_q, \mathcal{N}_j)$$

and, for each $A \in \mathcal{M}_j$ included in $st^5(w_q, \mathcal{M}_j)$, Pre* A is included in $st^{15}(w_q, \mathcal{N}_j)$, because Int $A \cap$ Int Pre* $A \neq \phi$. As a result

$$S_{j-1,q}(A) = (S(A) - S^*(A)) \cap (S_{j-1,q}(\text{Pre* } A) \times B^2)$$

is nonempty by construction, since $S_{j-1,q}(\text{Pre* } A) \neq \phi$.

THEOREM 5.3. *The sequence $S = \{\mathcal{M}_1, \ldots, \mathcal{M}_j, \ldots\}$ described above is a defining sequence for a cell-like upper semicontinuous decomposition G of T such that no element of G satisfies McMillan's Cellularity Criterion. Moreover, when $T = S^n$,* (A) *for any embedding f of ∂B^2 in T there exists a finite collection $\mathcal{V}$ of open subsets of T/G such that every map $F\colon B^2 \to T$ extending f has image $\pi F(B^2)$ in T/G that contains some $V \in \mathcal{V}$ and* (B) *for every compact subset Z of T/G that admits an essential map h to S^1 there exists an extension $h^*\colon U(Z) \to S^1$ of h over some neighborhood $U(Z)$ of Z and there exists a finite collection $\mathcal{V}$ of open sets in T/G such that, for any map $f\colon \partial B^2 \to U(Z)$ with $h^*f\colon \partial B^2 \to S^1$ essential, every extension $F\colon B^2 \to T/G$ of f has image $F(B^2)$ that contains some $V \in \mathcal{V}$.*

Proof. By condition (2) in the Inductive Hypothesis, $\mathcal{S}$ is a defining sequence; according to Theorem 2.3, G is upper semicontinuous. By condition (3), G satisfies Axiom 3; according to Proposition 3.1, G must be cell-like.

We claim that each $g \in G$ contains a compact subset C_g in Int $L_1 = W(1, 1)$ such that the simple closed curve J (identified immediately after the description of $\mathcal{M}_1$) cannot be contracted in $T - C_g$. To verify this, fix $x \in g$ and use the finiteness of the covers $\mathcal{M}_j$ to determine elements $M_i \in \mathcal{M}_i$ such that $M_i \subset st^2(x, \mathcal{M}_i)$ and $M_i = \text{Pre } M_{i+1}$. Then, by condition (9),

$$\phi \neq S_{11}(M_{i+1}) \times B^2 \subset S_{11}(M_i) \times B^2 \subset M_i.$$

Define C_g as $\bigcap_i S_{11}(M_i) \times B^2$. Since J cannot be contracted in

$$T - \text{Int } L_1 = T - \text{Int } (S_{11}(T) \times B^2),$$

it follows from Lemma 5.2 that any map $F\colon B^2 \to T$ sending ∂B^2 homeomorphically onto J satisfies

$$F(B^2) \cap (S_{11}(M_i) \times B^2) \neq \phi$$

for all i. Hence, $F(B^2) \cap C_g \neq \phi$.

By condition (4) in the Inductive Hypothesis, g has dimension ≤ 1. Consequently, J is homotopic in $T - C_g$ to another curve J' in $T - g$. If g did satisfy the Cellularity Criterion, J' would be contractible in $T - g$, and then J would be contractible in $T - C_g$, which was just shown to be impossible.

To establish the additional properties of this theorem, it is convenient to work with $T = S^n$, where linking theory involves no technical difficulty. The proofs of the two additional conclusions being quite similar, we shall discuss only conclusion (B).

To that end, suppose Z is a compact subset of T/G and $h\colon Z \to S^1$ is essential. There exists an extension $h^*\colon U(Z) \to S^1$ of h over a compact neighborhood $U(Z)$ of Z. It follows that

$$h^*\pi\colon \pi^{-1}(U(Z)) \to S^1$$

is also essential. Hence, by a construction of Singh [15, Lemma 2.3.1], there exists a PL embedding $e\colon S \times B^2 \to T - U(Z)$, where S denotes a closed, orientable, PL $(n-2)$-manifold, such that, for any loop

$$f^*\colon \partial B^2 \to \pi^{-1}(U(Z))$$

for which $h^*\pi f^*$ is essential, $f^*(\partial B^2)$ links $e(S \times \{0\})$; in particular, f^* is not null homotopic in $T - e(S \times \operatorname{Int} B^2)$. Because of the density of the guiding collection $\mathscr{D}$ of thickened $(n-2)$-manifolds, we can assume that $e(S \times B^2) = L_k \in \mathscr{D}$.

Associated with L_k there exist pairwise disjoint PL n-manifolds $W(k, 1), \ldots, W(k, t(k))$ in $\operatorname{Int} L_k$ satisfying condition (7) of the Inductive Hypothesis. Define the finite collection $\mathscr{V}$ of open sets in T/G as

$$\mathscr{V} = \{V_q = \text{interior in } T/G \text{ of } \pi(st^2(w_{kq}, \mathscr{M}_{k+1})) \mid q = 1, \ldots, t(k)\}.$$

Of course, the upper semicontinuity of G guarantees that $\pi(w_{kq}) \in V_q \in \mathscr{V}$ for each q.

For purposes of contradiction, suppose $f\colon \partial B^2 \to \operatorname{Int} U(Z)$ is a loop such that h^*f is essential and f has an extension $F\colon B^2 \to T/G$ that misses some $x_q \in V_q \in \mathscr{V}$, for $q = 1, \ldots, t(k)$. Then F "lifts" to $F^*\colon B^2 \to T$ with $F^*(B^2)$ PL and in general position with respect to the $W(k, q)$ and with πF^* so close to F that

$$\pi F^*(\partial B^2) \subset U(Z),$$

$h^*\pi F^* \mid \partial B^2\colon \partial B^2 \to S^1$ is essential, and $\pi F^*(B^2)$ misses each x_q. Condition (7) of the Inductive Hypothesis implies that $F^* \mid \partial B^2$ is not null homotopic in $T - \bigcup_{i=1}^{t(k)} W(k, i)$. Consequently, for some $q \in \{1, \ldots, t(k)\}$, there exists a component H^* of $(F^*)^{-1}(W(k, q))$ for which

$$F^* \mid H^*\colon H^* \to W(k, q)$$

is I-essential.

Set $g^* = \pi^{-1}(x_q)$. By the definition of $V_q \in \mathscr{V}$, there exists a point $x^* \in g^*$

with $x^* \in st^2(w_{kq}, \mathcal{M}_{k+1})$. We shall reach a contradiction by showing that $F^*(H^*) \cap g^* \neq \phi$. To achieve this, it suffices to show that $F^*(H^*) \cap st^2(x^*, \mathcal{M}_{k+j}) \neq \phi$ for $j = 1, 2, \ldots$. By condition (10), every $A \in \mathcal{M}_{k+1}$ included in $st^3(x^*, \mathcal{M}_{k+1})$ has $S_{kq}(A) \neq \phi$ because

$$st^3(x^*, \mathcal{M}_{k+1}) \subset st^5(w_{kq}, \mathcal{M}_{k+1}).$$

Every $A' \in \mathcal{M}_{k+j}$ included in $st^2(x^*, \mathcal{M}_{k+j})$ satisfies $A' \subset st^2(x^*, \mathcal{M}_{k+j-1})$ and hence

$$\text{Pre } A' \subset st(A', \mathcal{M}_{k+j-1}) \subset st^3(x^*, \mathcal{M}_{k+j-1}) \subset st^2(x, \mathcal{M}_{k+j-2}),$$

and, when $\text{Pre}^i A'$ is defined recursively as Pre $(\text{Pre}^{i-1} A')$, it follows by induction that

$$\text{Pre}^i A' \subset st^3(x^*, \mathcal{M}_{k+j-i}) \subset st^2(x^*, \mathcal{M}_{k+j-i-1}).$$

Since then $\text{Pre}^{j-1} A' \subset st^3(x^*, \mathcal{M}_{k+1})$, condition (9) insures that $S_{kq}(A') \neq \phi$, and the combination of condition (9) and Lemma 5.2 implies that $F^*(H^*) \cap A' \neq \phi$, yielding the desired contradiction.

Remark. We show in Appendix 1 that, by exercising supplemental controls, we can produce a defining sequence $\mathcal{S}$ for a decomposition G as in Theorem 5.3 such that T/G is a finite dimensional ANR.

6. Pathology of closed subsets of T/G

For a space X let $\mathcal{K}(X)$ be the collection of closed subsets K of X having empty interior and satisfying:

(*) some compact subset K^* of K admits an essential map to S^1 and, for every neighborhood W of K there exists a neighborhood W^* of K^* such that every loop in W^* contracts in W.

THEOREM 6.1. *If the locally compact and locally path connected space X satisfies conclusion* (B) *of Theorem* 5.3, *then* $\mathcal{K}(X) = \phi$.

Proof. Let K be a closed subset of X and K^* a compact subset of K satisfying condition (*). Name the essential map $h\colon K^* \to S^1$. Then there exists an extension $h^*\colon U(K^*) \to S^1$ over a neighborhood $U(K^*)$ of K^* and there exists a finite collection $\mathcal{V}$ of nonempty open sets in X satisfying conclusion (B) in Theorem 5.3.

Express K as the intersection of open sets $W_1, W_2, \ldots$ with $W_{i+1} \subset W_i$. Apply (*) to obtain corresponding neighborhoods $W_1^*, W_2^*, \ldots$ of K^* such that every loop in W_i^* is null homotopic in W_i. For the moment, assume that, for each i, there is a map $f_i\colon \partial B^2 \to W_i^* \cap U(K^*)$ such that $h^* f_i$ is essential, and let $F_i\colon B^2 \to W_i$ denote an extension of f_i. Then some $V \in \mathcal{V}$ must be contained in infinitely many $F_i(B^2) \subset W_i$ and hence in K, implying that $\mathcal{K}(X) = \phi$.

It remains to produce the f_i's. Using standard arguments, one obtains a cover $\mathscr{U}$ of K^* consisting of path connected open sets in $W_i \cap U(K^*)$, and, letting N denote the nerve of $\mathscr{U}$, maps $c: K^* \to N$ and $g: N \to S^1$ so that $g \circ c$ and h are "close" (in particular, are homotopic). It follows that g is essential. Further, since X is locally path connected, it can be arranged that the restriction of g to the 1-skeleton $N^{(1)}$ of N factors as $h^* \circ k$, where

$$k: N^{(1)} \to U(K^*) \cap W_i^*.$$

Since the higher homotopy groups of S^1 are zero, $h^* \circ k: N^{(1)} \to S^1$ is essential. Then there exists a map $m: S^1 \to N^{(1)}$ such that $h^* \circ k \circ m$ is essential, and $f_i = k \circ m$ is the desired map.

COROLLARY 6.2. *Suppose the locally compact and locally pathwise connected space X satisfies conclusion* (B) *of Theorem* 5.3, *and suppose K is a closed 2-dimensional subset of X satisfying:*

(**) *there exists $\delta > 0$ and a sequence of open sets $\{W_i \mid i = 1, 2, \ldots\}$ such that $K = \bigcap_i W_i$ and each map $f: \partial B^2 \to W_{i+1}$ for which* diam $f(\partial B^2) < \delta$ *extends to a map $F: B^2 \to W_i$.*
Then K has nonempty interior in X.

Proof. According to [11, p. 83], the hypothesis that dim $K \geq 2$ implies K has a compact subset K^* of diameter less than δ that maps essentially to S^1. Hence, (K, K^*) satisfies condition (*). But $\mathscr{K}(X) = \phi$ by Theorem 6.1, and the only explanation why $K \notin \mathscr{K}(X)$ is that Int K be nonempty.

Remark. In case X is an ANR, among its compact subsets that satisfy Condition (**) are those which are AR's, ANR's, FAR's (=cell-like sets), or FANR's [4] and those which satisfy Property 1-UV.

COROLLARY 6.3. *Let G denote the decomposition of $T = S^n$ described in Section* 5. *Then $\mathscr{K}(T/G) = \phi$.*

A *resolution* of a generalized n-manifold X is a proper cell-like map $f: M \to X$ of an n-manifold M onto X. Quinn [13] has shown that, for $n \geq 5$, every generalized n-manifold X has a resolution. The examples here, like those of Singh [14], [15], [16, 7.7] and Wright [18], show that some generalized n-manifolds do not admit cell-like maps onto n-manifolds.

COROLLARY 6.4. *Let G denote the decomposition of $T = S^n$ in Section* 5 *($n \geq 3$). There is no cell-like map $f: T/G \to M$ onto an n-manifold M.*

Proof. If $f: T/G \to M$ did exist, M would contain an uncountable family $\{K_\alpha\}$ of pairwise disjoint 2-cells, and the cell-likeness of f would imply that each element of $\{f^{-1}(K_\alpha)\}$ satisfies condition (*). Some set $f^{-1}(K_\alpha)$ would necessarily have no interior, contradicting Theorem 6.1.

7. The product with E^1

The combined work of [10], [13], and [7] shows that $X \times E^2$ is a manifold for every generalized n-manifold X $(n \geq 3)$; whether or not $X \times E^1$ is a manifold remains an open problem. When $n > 3$, R. D. Edwards' characterization [10] reduces the problem to deciding whether or not $X \times E^1$ satisfies the disjoint 2-disks property (DDP). Daverman [7] establishes that $X \times E^1$ satisfies the DDP for a large class of generalized n-manifolds X. In particular, spaces X arising as decomposition spaces from decompositions G of manifolds M of dimension at least 4 are in this class whenever G has a defining sequence $\mathscr{S}$ satisfying conditions (i), (ii) and (iii) in Section 2 of [5]; moreover, with an additional mild restriction on $\mathscr{S}$, Cannon and Daverman prove that $X \times E^1$ is homeomorphic to $M \times E^1$ when $\dim M \geq 3$ [5, Section 5]. The proof of the theorem below exploits these results.

Let $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \ldots\}$ be a defining sequence on a space T, with associated decomposition G. Let $\mathscr{T} = \{\mathscr{R}_1, \mathscr{R}_2, \ldots\}$ be a defining sequence on E^1 with each $\mathscr{R}_i$ a cover of E^1 consisting of closed intervals having diameters less than $1/i$, and let $q(1), q(2), \ldots$ be an increasing sequence of integers. The product decomposition $G \times E^1$ on $T \times E^1$ has for a defining sequence

$$\{\mathscr{M}_1 \times \mathscr{R}_{q(1)}, \mathscr{M}_2 \times \mathscr{R}_{q(2)}, \ldots\}.$$

The content of the next theorem is that an additional (minor) restriction on the construction of the defining sequence $\{\mathscr{M}_1, \mathscr{M}_2, \ldots\}$ in Section 5 insures that $T/G \times E^1$ is a manifold, where G is the associated decomposition.

THEOREM 7.1. *For $n \geq 3$ there exists a generalized n-manifold X such that $X \times E^1$ is a manifold and X satisfies conclusion* (B) *of Theorem* 5.3; *in particular, for each map $F\colon B^2 \to X$ that restricts to an embedding on ∂B^2,* $\operatorname{Int} F(B^2) \neq \phi$.

Proof. Suppose that $T \times E^1 = \bigcup_{i=1}^{\infty} D_i$ is expressed as an increasing union of compact sets $D_i = K_i \times I_i$ and suppose that the lengthy "Inductive Hypothesis $(j-1)$" in Section 5 includes the additional condition:

(11) For $2 \leq i \leq j-1$, there is an integer $q(i) > q(i-1)$ (where $q(1) = 1$) and a homeomorphism $h_i\colon T \times E^1 \to T \times E^1$ such that

$$h_i(x) \notin D_{i-1} \quad \text{for } x \notin st^2(D_{i-1}, \mathscr{M}_{i-1} \times \mathscr{R}_{q(i-1)}),$$

$$h_i(x) \in st^2(x, \mathscr{M}_{i-1} \times \mathscr{R}_{q(i-1)}) \quad \text{for } x \in st^2(D_{i-1}, \mathscr{M}_{i-1} \times \mathscr{R}_{q(i-1)}),$$

and $\operatorname{diam} h_i(A) < 1/i$ for $A \in \mathscr{M}_i \times \mathscr{R}_{q(i)}$ with $A \cap D_{i-1} \neq \phi$.

We leave to the reader to verify that condition (11) suffices to insure that the defining sequence $\{\mathscr{M}_1 \times \mathscr{R}_{q(1)}, \mathscr{M}_2 \times \mathscr{R}_{q(2)}, \ldots\}$ satisfies Axiom 4 and, in view of Proposition 4.1, that $T/G \times E^1$ is a manifold. We shall describe how to maintain condition (11) through the induction procedure of Section 5.

The first step in describing $\mathscr{M}_j$ in Section 5 is to extract $\mathscr{N}_j$ from a defining sequence constructed in [5], say $\mathscr{H} = \{\mathscr{P}_1, \mathscr{P}_2, \ldots\}$, with $\mathscr{P}_1 = \mathscr{M}_{j-1}$. The decomposition H associated with $\mathscr{H}$ has the property that its product with the trivial decomposition on E^1 is shrinkable. (This is shown directly in [5], provided additional mild constraints are placed on the $\mathscr{P}_i$'s and, for $n \geq 4$, this is shown in [7] by appealing to the DDP characterization in [10].) Proposition 4.1 implies that the defining sequence

$$\{\mathscr{P}_1 \times \mathscr{R}_{q(j-1)}, \mathscr{P}_2 \times \mathscr{R}_{q(j-1)+1}, \ldots\}$$

satisfies Axiom 4.

Let $\mathscr{U}$ be an open cover of $T \times E^1$ by sets of diameter less than $1/j$. Let k and $h_j\colon T \times E^1 \to T \times E^1$ be the integer and homeomorphism from Axiom 4 chosen with respect to the compact set D_{j-1}, the open cover $\mathscr{U}$, and $\mathscr{P}_1 \times \mathscr{R}_{q(j-1)}$. Since $\mathscr{P}_1 = \mathscr{M}_{j-1}$, we have that

$$h_j(x) \notin D_{j-1} \quad \text{for } x \notin st^2(D_{j-1}, \mathscr{M}_{j-1} \times \mathscr{R}_{q(j-1)}),$$

$$h_j(x) \in st^2(x, \mathscr{M}_{j-1} \times \mathscr{R}_{q(j-1)}) \quad \text{for } x \in st^2(D_{j-1}, \mathscr{M}_{j-1} \times \mathscr{R}_{q(j-1)}),$$

and diam $h_j(A) < 1/j$ for $A \in \mathscr{P}_k \times \mathscr{R}_{q(j-1)+k}$ with $A \cap D_{j-1} \neq \phi$. The proof of Proposition 4.1 shows that h_j can be chosen so that, for $k' \geq k$, $h_j(A) < 1/j$ for $A \in \mathscr{P}_{k'} \times \mathscr{R}_{q(j-1)+k'}$ with $A \cap D_{j-1} \neq \phi$.

The choice of $\mathscr{N}_j$ should include the stipulation that $\mathscr{N}_j = \mathscr{P}_{k'}$ for some $k' > k$. Now we set $q(j) = q(j-1) + k'$. The choice of $\delta > 0$ in the first step of "the second operation" used in describing $\mathscr{M}_j$ should include the conditions that the image under h_j of the δ-neighborhood of each $A \in \mathscr{N}_j \times \mathscr{R}_{q(j)}$ with $A \cap D_{j-1} \neq \phi$ have diameter less than $1/j$ and that the δ-neighborhood of K_j be contained in $st(K_j, \mathscr{N}_j)$. The latter condition insures that $A \times R \in \mathscr{M}_j \times \mathscr{R}_{q(j)}$ intersects D_{j-1} only if $(\text{Pre}^* A) \times R \in \mathscr{N}_j \times \mathscr{R}_{q(j)}$ does, and, in turn, the former condition insures that diam $h_j(A \times R) < 1/j$ for those $A \times R$ meeting D_{j-1}.

Appendix 1

A consequence of the result in Section 7 showing the product decomposition on $T \times E^1$ to be shrinkable is that the decomposition space T/G is finite dimensional for any decomposition G which has a defining sequence satisfying conditions (1)–(10) in Section 5 and condition (11) in Section 7. The proof of the next result contains a "direct" approach to showing the finite dimensionality of certain decomposition spaces. The proof is followed by a word about assuring that conditions (†) and (‡) are satisfied by the defining sequences constructed in Section 5.

PROPOSITION A.1. *Suppose that $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \ldots\}$ is a defining sequence on a locally compact space X with the elements of the associated decomposition G connected; suppose that each $A \in \mathscr{M}_{i+1}$ is associated with an element* Pre $A \in \mathscr{M}_i$ *such that* Int $A \cap$ Int Pre $A \neq \phi$*; and suppose that:*

(†) *for each* $\mathscr{H} \subseteq \mathscr{M}_i$, *there is an open set* $W(\mathscr{H})$ *containing* Bd H *and contained in* $st(\text{Bd } H, \mathscr{M}_i)$, *where* $H = \bigcup \{A \in \mathscr{H}\}$, *such that* $\overline{W(\mathscr{K})} \subseteq W(\mathscr{H})$ *where*

$$\mathscr{K} = \{A \in \mathscr{M}_{i+1}\colon \text{Pre } A \in \mathscr{H}\};$$

(‡) *for each* $\mathscr{H} \subseteq \mathscr{M}_i$ *and* $x \in X$, $st^2(x, \mathscr{M}_{i+1}) \cap \text{Bd } H$ *has diameter less than* $1/i$ (*whenever the intersection is not empty*).

Then $\dim X/G \leq \dim X + 1$.

Proof. Let $\pi\colon X \to X/G$ denote the induced map. The strategy is to identify closed subsets of X on which π is 1-1 and whose images contain boundaries of neighborhoods of points in X/G.

Let $y_0 \in X/G$ and let U be a compact neighborhood of y_0. Let $x_0 \in \pi^{-1}(y_0)$ and let k be such that $st^4(x_0, \mathscr{M}_k) \subset \pi^{-1}(U)$. Let

$$\mathscr{H}_k = \{A \in \mathscr{M}_k\colon A \subseteq st^3(x_0, \mathscr{M}_k)\} \quad \text{and} \quad W_k = W(\mathscr{H}_k)$$

and, recursively for $i \geq 1$, let

$$\mathscr{H}_{k+i} = \{A \in \mathscr{M}_{k+i}\colon \text{Pre } A \in \mathscr{H}_{k+i-1}\} \quad \text{and} \quad W_{k+i} = W(\mathscr{H}_{k+i}).$$

The nested W_{k+i}'s yield a closed set $C = \bigcap_{i=0}^{\infty} \bar{W}_{k+i}$ on which π is 1-1 by condition (‡).

Since $st^3(x_0, \mathscr{M}_{k+1}) \subset st^2(x_0, \mathscr{M}_k) \subset H_k - W_k$, each element of $\mathscr{M}_{k+1}$ contained in $st^3(x_0, \mathscr{M}_{k+1})$ is also an element of $\mathscr{H}_{k+1}$ and, therefore,

$$st^3(x_0, \mathscr{M}_{k+1}) \subset H_{k+1} - W_{k+1}.$$

The same argument shows that $st^3(x_0, \mathscr{M}_{k+i}) \subseteq H_{k+i} - W_{k+i}$ for each i. Since each W_{k+i} separates $\pi^{-1}(y_0)$ from $X - \pi^{-1}(U)$, the set C also performs this separation. Since the elements of G are connected, $\pi(C)$ separates y_0 from $X/G - U$; and since $\pi \mid C$ is an embedding, $\dim \pi(C) \leq \dim X$. In turn, we can conclude that $\dim X/G \leq \dim X + 1$. This completes the proof of the proposition.

Returning to the defining sequence $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \ldots\}$ defined in Section 5, we observe that the intermediate collection $\mathscr{N}_j$ constructed in the first operation can (easily) be required to satisfy the condition that

$$st^6(x, \mathscr{N}_j) \cap \text{Bd } (H)$$

has diameter less than $1/(j-1)$ (for $\mathscr{H} \subset \mathscr{M}_{j-1}$ and $H = \bigcup \{A \in \mathscr{H}\}$). Since $st^2(x, \mathscr{M}_j) \subset st^6(x, \mathscr{N}_j)$, condition (‡) will be satisfied. Suppose for inductive reasons that condition (†) is satisfied for $1 \leq i < j - 1$. The collection $\mathscr{N}_j$ has the property that, for $\mathscr{H} \subset \mathscr{M}_{j-1}$ and $H = \bigcup \{A \in \mathscr{H}\}$, Bd $(H) =$ Bd (K) where

$$\mathscr{K} = \{A \in \mathscr{N}_j\colon \text{Pre } A \in H\}$$

and $K = \bigcup \{A \in \mathscr{K}\}$; name an open set $W(\mathscr{K})$ such that

$$\mathrm{Bd}\,(K) \subseteq W(\mathscr{K}),\ \overline{W(\mathscr{K})} \subset W(\mathscr{H}),\quad \text{and}\quad \overline{W(\mathscr{K})} \subseteq \mathrm{Int}\ st(\mathrm{Bd}\ K, \mathscr{N}_j).$$

Each $A \in \mathscr{M}_j$ is contained in the δ-neighborhood of Pre* $A \in \mathscr{N}_j$ where δ is the number chosen in Step 1 of the second operation; for δ chosen sufficiently small, condition (†) will be satisfied for $i = j - 1$ by letting $W(\mathscr{K}) = W(\mathscr{K}')$ for $\mathscr{K} \subseteq \mathscr{M}_j$ where

$$\mathscr{K}' = \{\mathrm{Pre}^*\ A\colon A \in \mathscr{K}\} \subseteq \mathscr{N}_j.$$

As a final remark, the $W(\mathscr{H})$'s can easily be chosen so that Int $C = \phi$ (C is defined in the proof of Proposition A.1); since T is an n-manifold, it follows that dim $C \leq n - 1$. This leads to the conclusion that dim $T/G \leq$ dim T.

Appendix 2

Frequently, upper semi-continuous decompositions of PL manifolds are described by defining sequences $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \ldots\}$ where each $\mathscr{M}_i$ consists of PL manifolds; in fact, every upper semi-continuous decomposition of a PL manifold has such a defining sequence.

THEOREM A.2. *If G is an upper semi-continuous decomposition of a PL manifold Q^n, then G has a defining sequence $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \ldots\}$ where each $\mathscr{M}_i$ consists of PL manifolds.*

Proof. Let $\pi\colon Q \to Q/G$ be the induced map. Let $\mathscr{T} = \{\mathscr{P}_1, \mathscr{P}_2, \ldots\}$ be a defining sequence in Q/G for the trivial decomposition consisting of points with each $\mathscr{P}_k$ a cover of Q/G and with

$$\mathrm{Int}\,(\pi^{-1}(\mathrm{Fr}\,(A)) = \phi \quad \text{for each } A \in \mathscr{P}_k.$$

The $\mathscr{P}_k$'s should satisfy two additional properties: first,

$$st^6(x, \mathscr{P}_{k+1}) \subseteq \mathrm{Int}\,(st(x, \mathscr{P}_k))$$

and, second,

$$(A \cap A') \cap (\mathrm{Int}\,(A \cup A')) \neq \phi$$

whenever $A, A' \in \mathscr{P}_k$ with $A \cap A' \neq \phi$. For example, the brick partitions in [2] can be used to produce the defining sequence $\mathscr{T}$ (a detailed discussion of this can be found in [17, Section 2]). Let $\mathscr{R} = \{\mathscr{N}_1, \mathscr{N}_2, \ldots\}$ be the defining sequence for G determined by letting

$$\mathscr{N}_k = \{\pi^{-1}(A)\colon A \in \mathscr{P}_k\}.$$

For each $x \in Q$ and each k, let $U(k, x)$ be an open set with

$$st^6(x, \mathscr{N}_{k+1}) \subseteq U(k, x) \subseteq \mathrm{Cl}\,(U(k, x)) \subseteq \mathrm{Int}\,(st(x, \mathscr{N}_k))$$

and with $U(k, x) = U(k, x')$ whenever $st^6(x, \mathcal{N}_{k+1}) = st^6(x', \mathcal{N}_{k+1})$. Let $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$ be a sequence of handlebody decompositions of Q satisfying:

(1) For $A, A' \in \mathcal{N}_k$ with $A \cap A' \neq \phi$, there are elements $h, h', h'' \in \mathcal{H}_k$ with $h \subseteq \text{Int}\,(A)$, $h' \subseteq \text{Int}\,(A')$ and $h'' \subseteq \text{Int}\,(A \cup A')$ and with $h \cap h'' \neq \phi$ and $h' \cap h'' \neq \phi$.

(2) For each $A \in \mathcal{N}_k$, $st^3(A, \mathcal{H}_k) \subseteq \text{Int}\,(st(A, \mathcal{N}_k))$.

(3) If $h \in \mathcal{H}_{k+1}$ and $h \cap st^6(x, \mathcal{N}_{k+1}) \neq \phi$, then $h \subseteq U(k, x)$.

(4) If $h \in \mathcal{H}_k$ and $h \cap (\text{Cl}\,(Q - st(x, \mathcal{N}_k)) \neq \phi$, then $h \cap \text{Cl}\,(U(k, x)) = \phi$.

The second special property of the $\mathcal{P}_k$'s listed in the preceding paragraph is needed to achieve condition 1, and the local finiteness of $\{U(k, x)\colon x \in Q\}$ is needed to achieve conditions (3) and (4).

For each k, let $\omega_k\colon \mathcal{H}_k \to \mathcal{N}_k$ be a function which satisfies $\omega_k(h) \cap h \neq \phi$ for each $h \in \mathcal{H}_k$; condition (1) forces ω_k to be onto. For each $A \in \mathcal{N}_k$, let

$$\Omega_k(A) = \bigcup \{h \in \mathcal{H}_k\colon \omega_k(h) = A\}$$

and let $\mathcal{M}_k = \{\Omega_k(A)\colon A \in \mathcal{N}_k\}$. The function Ω_k induces a bijection between $\mathcal{N}_k$ and $\mathcal{M}_k$ and conditions (1) and (2) insure that $\Omega_k(A) \cap \Omega_k(A') \neq \phi$ if and only if $A \cap A' \neq \phi$ and that $\Omega_k(A) \subseteq \text{Int}\,st(A, \mathcal{N}_k)$. It should be clear that the elements of $\mathcal{M}_k$ are PL manifolds and that $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ satisfies Axiom 1. It remains to show that Axiom 2 holds for $\mathcal{S}$ and that G is the associated decomposition.

The property that $\Omega_k(A) \cap \Omega_k(A') \neq \phi$ if and only if $A \cap A' \neq \phi$ for $A, A' \in \mathcal{N}_k$ combined with the containment $\Omega_k(A) \subseteq st(A, \mathcal{N}_k)$ implies that

$$st^j(x, \mathcal{M}_k) \subseteq st^{j+3}(x, \mathcal{N}_k)$$

for each j and each $x \in Q$. This same property combined with conditions 3 and 4 implies that

$$st^6(x, \mathcal{N}_{k+1}) \subseteq \text{Int}\,(st^2(x, \mathcal{M}_k)).$$

Axiom 2 is an immediate consequence of the containments

$$st^3(x, \mathcal{M}_{k+1}) \subseteq st^6(x, \mathcal{N}_{k+1}) \subseteq \text{Int}\,(st^2(x, \mathcal{M}_k)).$$

Since $G(x) = \bigcap_{k=1}^{\infty} st^6(x, \mathcal{N}_k)$, these containments also show that G is the decomposition associated with $\mathcal{S}$.

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THE UNIVERSITY OF TENNESSEE
KNOXVILLE, TENNESSEE