THE ENUMERATION OF PLANE PARTITIONS VIA THE BURGE CORRESPONDENCE

BY

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1. Introduction

The task of enumerating plane partitions was first undertaken by MacMahon at the turn of this century. With a remarkable instinct for form and underlying structure, he was able to compute the generating function for plane partitions of a fixed shape, among many other results. These are compiled in the second volume of his *Combinatory analysis* [18].

Recently, new tools have been forged for working with plane partitions. One of these is the Schensted correspondence between matrices and plane partitions, as extended by Knuth [15], which was used by Bender and Knuth [3] to obtain new results and to elegantly prove some of MacMahon's old ones. This work led to more results and more correspondences. In particular, a correspondence of Burge, one of several presented in [4], is the basis for the work in this paper. Using it, we can obtain, in fairly simple form, a variety of new or extended generating functions.

Following a presentation of the basic definitions, the Burge correspondence is given in a form that best serves our purpose. The characterization is nonalgorithmic, at least in comparison with the more usual and very constructive "bumping" definition. However, in this form, the shape, symmetry and other properties of a plane partition can be more easily determined from its associated matrix.

Coming naturally from the Burge correspondence, we have the ability to generalize the usual generating functions for the sum of the parts (the norm generating function) or for the trace of a plane partition, and obtain ones, called trace generating functions, that take into account the sum of the parts of every diagonal of a plane partition. Such extensions have also arisen naturally in [8] and [10], and it might be worthwhile to consider for what other generating functions for plane partitions can such extensions be made.

Rectangular plane partitions are dealt with in the above fashion in Section 4. In Section 5, Burge's correspondence is used to derive the trace generating function for plane partitions of nonrectangular shape and, as a corollary, a new form for the norm generating function. This is compared to MacMahon's norm

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generating function in Section 6. It is shown that, in certain cases, our form can be converted to MacMahon's form. This leads to a conjecture of the existence of a simple method for obtaining MacMahon's generating function.

The correspondence of Burge also preserves symmetry. This allows us to deal easily with symmetric plane partitions, both of square and nonsquare shapes, in Section 7. Finally, in Section 8, we note the applicability of these techniques to the enumeration of shifted plane partitions.

2. Definitions and notation

The study of plane partitions is a natural outgrowth of the study of the number-theoretic partitions of an integer and, as such, many of our definitions and concepts will be derived from the latter theory.

A *shape* λ is a finite nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ of positive integers. We let $\rho(\lambda) = k$, the length of the sequence, and

$$\sigma(\lambda) = \lambda_1 + \cdots + \lambda_k.$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the *parts* of λ and $\sigma(\lambda)$ in the *norm* of λ. With each shape λ, we can associate the set $\Gamma(\lambda)$ of all pairs (i, j) of integers such that $1 \leq i \leq \rho(\lambda)$ and $1 \leq j \leq \lambda_i$. A shape λ is nothing more than a partition of the integer $\sigma(\lambda)$, and the set $\Gamma(\lambda)$ is the Ferrers graph associated with λ.

Another notion we wish to borrow from the theory of partitions is that of conjugacy. If λ is a shape, let Γ^0 be the set of all pairs of integers such that (i, j) is in Γ^0 if and only if (j, i) is in $\Gamma(\lambda)$. Let λ' be the unique shape with $\Gamma(\lambda') = \Gamma^0$. We call λ' the *conjugate* of λ. If $\lambda = \lambda'$, we say that λ is *symmetric*.

Given a shape λ, suppose that λ has m_i parts equal to i. We can then represent λ by the alternate notation $(1^{m_1} 2^{m_2} \cdots)$. In particular, if λ is a constant sequence with $\rho(\lambda) = r$ and $\lambda_1 = \cdots = \lambda_r = c$, we denote λ by (c^r). In this case, λ is said to be *rectangular*.

Let λ be some shape. A *plane partition of shape* λ is an array $P = (p_{ij})$ of nonnegative integers indexed by $\Gamma(\lambda)$ such that $p_{ij} \geq p_{ij+1}$ and $p_{ij} \geq p_{i+1j}$. The integers p_{ij}, including multiplicities, are called the *parts* of P, and the *norm* $\sigma(P)$ of P is the sum of all the parts in P. Below, we exhibit a plane partition of shape $(5, 3, 3, 2, 1)$ and norm 31:

$$
\begin{array}{ccccc}
6 & 4 & 3 & 2 & 1 \\
3 & 3 & 3 & & \\
2 & 1 & 0 & & \\
2 & 0 & & & \\
1. & & & &
\end{array}
$$

If l is an integer between $1 - \rho(\lambda)$ and $\lambda_1 - 1$, we define the *l-diagonal* of P to be the sequence of parts p_{ij} with $j - i = l$; the *l-trace* of P, denoted by $t_l(P)$, is

the sum of the parts in the l-diagonal of P. The 1-trace and (-1)-trace for the plane partition given above are 7 and 4, respectively. The function t_l extends naturally to shapes. Let P be the plane partition of shape λ all of whose parts equal 1. We then define $t_l(\lambda) = t_l(P)$.

Let $P = (p_{ij})$ be a plane partition of shape λ. We let $TP = (q_{ij})$ be the plane partition of shape λ' where $q_{ij} = p_{ji}$. If $P = TP$, we say that P is *symmetric*.

It will be useful to extend the use of T and σ to matrices in the obvious way: if M is a matrix, TM represents the transpose of M and $\sigma(M)$ represents the sum of all the entries in M. In addition, if $M = (m_{ij})$, the l-diagonal of M is the sequence of all m_{ij} with $j - i = l$.

3. The Burge correspondence

In order to give combinatorial proofs for a collection of Schur function identities due to Littlewood [16, p. 238], Burge [4] devised several variations on the well-known Schensted-Knuth correspondence [15, 19] between nonnegative integer matrices and pairs of plane partitions of the same shape whose parts strictly decrease down the columns. (More precisely, the correspondence of Burge that is of interest to us is a variation on the "dual" correspondence of Knuth.) By extending a construction of Frobenius on shapes to one which merges two column-strict plane partitions into a single plane partition, we extend Burge's correspondence to a map from matrices to plane partitions.

The details for the construction of this map can be found in [9], where it also shown that Burge's correspondence is related to a correspondence of Hillman and Grassl [13]. This relationship provides another, and simpler, method for defining the correspondence. For our purposes, we will only need the characterization of the correspondence given in Theorem 3.1. Before we present that theorem, we must introduce some additional notation.

Let M be an $r \times c$ matrix, and let l be an integer, $1 - r \le l \le c - 1$. We let M_l be the largest submatrix of M with the l-diagonal of M as its 0-diagonal.

Given an $r \times c$ matrix $M = (m_{ij})$, we define a *chain* in M to be a sequence

$$((i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k))$$

of pairs of integers such that

$$r \ge i_1 \ge i_2 \ge \cdots \ge i_k \ge 1 \quad \text{and} \quad 1 \le j_1 \le j_2 \le \cdots \le j_k \le c,$$

with the additional property that the pair (i, j) can appear at most m_{ij} times in the chain. A *cross chain* in M is defined similarly, except we require instead

$$1 \le i_1 \le i_1 \le i_2 \le \cdots \le i_k \le r.$$

For $k \ge 1$, define $a_k(M)$ $(d_k(M))$ to equal the maximum of

$$|C_1| + |C_2| + \cdots + |C_k|,$$

where the maximum is taken over all collections of chains (cross chains) C_1, $C_2, \ldots, C_k$ such that the number of times the pair (i, j) appears in all the C_i's

combined is at most m_{ij}. ($|C_l|$ is the number of terms in the sequence C_l.) We allow empty chains and cross chains. Let $a_0(M) = d_0(M) = 0$. If $C_1, C_2, \ldots, C_k$ have the property that (i, j) appears exactly m_{ij} times for all pairs (i, j), we say that $C_1, \ldots, C_k$ *cover* M. If M is an $r \times c$ matrix and $k \geq \min\{r, c\}$, it is clear that M can always be covered with k chains or k cross chains; thus $a_k(M) = d_k(M) = \sigma(M)$.

As an example of these definitions, consider the matrix

$$M = \begin{pmatrix} 0 & 2 & 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 2 & 0 & 1 \end{pmatrix}.$$

Then, M_2 is the matrix

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

The longest chain in M_2 is

$$((4, 2), (4, 2), (3, 3), (3, 3), (2, 3), (1, 3), (1, 3), (1,3))$$

and the longest cross chain is

$$((1, 1), (1, 3), (1, 3), (1, 3), (2, 3), (3, 3), (3, 3), (3, 4), (4, 4)).$$

Thus, $a_1(M_2) = 8$ and $d_1(M_2) = 9$. Similarly, $a_2 = 11$, $a_3 = 12$, $d_2 = 12$, $d_3 = 13$ and $a_k = d_k = 13$ for all $k \geq 4$.

We can now describe Burge's correspondence in the form that will be the most useful for our purposes. To help clarify, in part, the relation between this correspondence and the Schensted-Knuth correspondence, and for later use, we also give the analogous description for the latter correspondence.

THEOREM 3.1. *Let M be an $r \times c$ nonnegative integer matrix. Define two $r \times c$ matrices $B(M)$ and $K(M)$ as follows. For all l between $1 - r$ and $c - 1$, let the l-diagonal of $B(M)$ (resp. $K(M)$) be given by*

$$a_i(M_l) - a_{i-1}(M_l) \text{ (resp. } d_i(M_l) - d_{i-1}(M_l)), \quad i \geq 1.$$

Then $B(M)$ and $K(M)$ are plane partitions, and B and K are bijections from the set of $r \times c$ nonnegative integer matrices onto the set of plane partitions of shape (c^r).

Proofs for this theorem can be found in [9] and [7, Theorem 2.10]. Also, a proof for B is essentially given in [14]. This characterization of these correspondences makes manifest certain properties that are not at all obvious from their original constructions. We list several of these properties below as corollaries in

terms of B. The same results hold if B is replaced by K. Their proofs follow easily from the theorem and are omitted.

COROLLARY 3.2. *Let M be an $r \times c$ nonnegative integer matrix, and let l be an integer, $1 - r \leq l \leq c - 1$. Then $t_l(B(M)) = \sigma(M_l)$.*

COROLLARY 3.3. *If M is a nonnegative integer matrix, $TB(M) = B(TM)$.*

COROLLARY 3.4. *B is a bijection from the set of all symmetric $r \times r$ nonnegative integer matrices onto the set of all symmetric plane partitions of shape (r^r).*

A variety of other properties of B and K can be found in the sources cited above.

4. Plane partitions of rectangular shape

We now turn to the task of enumerating plane partitions, in the sense of finding the generating function for the number of plane partitions that fulfill certain conditions. The Burge correspondence gives us a potent tool for deriving fairly general generating functions. To begin, we have need of some new notation.

For any pair (i, j) of positive integers, define a weight

$$w(i, j) = \prod_{l = 1 - i}^{j - 1} x_l.$$

With every plane partition $P = (p_{ij})$ of shape λ, we associate its *trace weight*

$$W(P) = \prod x_{j-i}^{p_{ij}}$$

where the product is taken over all (i, j) in $\Gamma(\lambda)$. Thus, the exponent of x_l in $W(P)$ is $t_l(P)$.

Given an $r \times c$ matrix M, it is simple, using Corollary 3.2 and the observation that, if $M = (m_{ij})$,

$$\prod_{l = 1 - r}^{c - l} x_l^{\sigma(M_l)} = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq c}} w(i, j)^{m_{ij}}.$$

to evaluate the trace weight of $B(M)$ in terms of the $w(i, j)$.

LEMMA 4.1. *Let $M = (m_{ij})$ be an $r \times c$ matrix. Then*

$$W(B(M)) = \prod_{\substack{1 \leq i \leq r \\ 1 \leq i \leq c}} w(i, j)^{m_{ij}}.$$

If $\mathscr{C}$ is some class of plane partitions, the *trace generating function for $\mathscr{C}$* is defined as

$$G(\mathscr{C}) = \sum_{P \in \mathscr{C}} W(P)$$

and the *norm generating function for* $\mathscr{C}$ is given by

$$g(\mathscr{C}) = \sum_{P \in \mathscr{C}} x^{\sigma(P)}.$$

Clearly, $g(\mathscr{C})$ can be obtained from $G(\mathscr{C})$ by replacing all x_i by x. For a given shape λ, let $\mathscr{P}(\lambda)$ be the set of all plane partitions of shape λ, and define $G(\lambda) = G(\mathscr{P}(\lambda))$ and $g(\lambda) = g(\mathscr{P}(\lambda))$.

The easiest plane partitions to enumerate are those of rectangular shape, in which we only restrict the number of rows and columns.

THEOREM 4.2.   *Let r and c be two positive integers. Then*

$$G(c^r) = \prod_{\substack{1 \le i \le r \\ 1 \le j \le c}} (1 - w(i, j))^{-1}.$$

Proof.   We employ the Burge correspondence and Lemma 4.1. Let $\mathscr{M}$ be the set of all $r \times c$ nonnegative integer matrices, and let $\lambda = (c^r)$. Then

$$G(c^r) = \sum_{P \in \mathscr{P}(\lambda)} W(P)$$

$$= \sum_{\substack{M \in \mathscr{M} \\ M = (m_{ij})}} \prod_{\substack{1 \le i \le r \\ 1 \le j \le c}} w(i, j)^{m_{ij}}$$

$$= \prod_{\substack{1 \le i \le r \\ 1 \le j \le c}} \sum_{m_{ij}=0}^{\infty} w(i, j)^{m_{ij}}$$

$$= \prod_{\substack{1 \le i \le r \\ 1 \le j \le c}} (1 - w(i, j))^{-1}.$$

To concentrate our attention on the norm and 0-trace, we can replace x_0 by xy and x_l, $l \ne 0$, by x in $G(c^r)$. Thus, $w(i, j)$ becomes $x^{i+j-1}y$.

COROLLARY 4.3.   *Let $\lambda = (c^r)$. Then*

$$\sum_{P \in \mathscr{P}(\lambda)} x^{\sigma(P)} y^{t_0(P)} = \prod_{\substack{1 \le i \le r \\ 1 \le j \le c}} (1 - x^{i+j-1} y)^{-1}.$$

In particular,

$$g(c^r) = \prod_{\substack{1 \le i \le r \\ 1 \le j \le c}} (1 - x^{i+j-1})^{-1}.$$

The first part of the corollary is due to Stanley [22], though he presents the result in a slightly different form. The second part of the corollary was first given by MacMahon [18, p. 234]. Bender and Knuth [3] are responsible for the matrix correspondence proof of this result.

To remove the restrictions on the number of rows, the number of columns, or both, we simply let $r \to \infty$ or $c \to \infty$ in Theorem 4.2 or Corollary 4.3. In

particular, we have the following generating functions due to MacMahon [18, p. 234].

COROLLARY 4.4. *The norm generating function for plane partitions with at most r rows is*

$$\prod_{k=1}^{\infty} (1 - x^k)^{-\min\{k, r\}}.$$

The norm generating function for all plane partitions is

$$\prod_{k=1}^{\infty} (1 - x^k)^{-k}.$$

It should be mentioned that a limiting argument is not necessary to obtain the generating functions in Corollary 4.4. A closer look at Theorem 3.1 reveals that B is a bijection from the set of all semi-infinite arrays $M = (m_{ij})$, $1 \le i < \infty$, $1 \le j < \infty$, with a finite number of nonzero entries onto the set of all plane partitions (ignoring zero parts). Since a 1 in position (i, j) of M contributes $(i + j - 1)$ to the norm of $B(M)$, the proof of Corollary 4.4 follows immediately. Thus, the Burge correspondence provides a straightforward, constructive reason for the simplicity of these functions.

By a more careful use of Corollary 3.2, we can obtain a slight generalization of the norm generating function $g(c^r)$. More generality can be obtained, at the cost of much more work and a much uglier result. Perhaps the nicest extension to be hoped for in this direction can be found in [5].

Let p and q be two nonnegative integers, and let r and c be two positive integers. By Corollary 3.2, Burge's correspondence induces a bijection between $r \times c$ matrices $M = (m_{ij})$ such that

$$\sum_{j=1}^{c} m_{rj} = p \quad \text{and} \quad \sum_{i=1}^{r} m_{ic} = q$$

and plane partitions $P = (p_{ij})$ of shape (c^r) such that $p_{r1} = p$ and $p_{1c} = q$. We also know that m_{ij} contributes $m_{ij}(i + j - 1)$ to $\sigma(B(M))$.

Thus, to enumerate the plane partitions described above, we need to consider the set of corresponding matrices $M = (m_{ij})$. We can choose m_{ij}, $1 \le i < r$, $1 \le j < c$, arbitrarily. This will contribute a factor of

$$\prod_{\substack{1 \le i < r \\ 1 \le j < c}} (1 - x^{i+j-1})^{-1}$$

to the norm generating function.

Secondly, we can choose m_{rc} to be any nonnegative integer less than or equal to min $\{p, q\}$. Let $m_{rc} = k$. Finally, we must spread $p - k$ and $q - k$ over the rest of the positions in the rth row and cth column, respectively. Weighted by

$r + j - 1$, the possible choices for m_{rj}, $1 \leq j < c$, yield a term in the generating function equal to

$$\sum_{b_1 + \cdots + b_{c-1} = p-k} x^{(rb_1 + \cdots + (r+c-2)b_{c-1})} = x^{r(p-k)} \sum_{b_1 + \cdots + b_{c-1} = p-k} x^{((1)b_2 + \cdots + (c-2)b_{c-1})}$$

$$= x^{r(p-k)} \prod_{l=1}^{p-k} (1 - x^{c+l-2})(1 - x^l)^{-1}.$$

The last equality holds by the well-known expression for the norm generating function for shapes with at most $p - k$ parts and whose parts are bounded by $c - 2$ ([18, p. 5]).

Similarly, the choices for m_{ic}, $1 \leq i < r$, yield the term

$$x^{c(q-k)} \prod_{l=1}^{q-k} (1 - x^{r+l-2})(1 - x^l)^{-1}.$$

Putting these remarks, and terms, together, we have proved the following result.

THEOREM 4.5. *The norm generating function for plane partitions $P = (p_{ij})$ of shape (c^r) with $p_{r1} = p$ and $p_{1c} = q$ is*

$$\left(\sum_{k=0}^{\min\{p, q\}} x^{cq + rp - k} \prod_{l=1}^{q-k} (1 - x^{r+l-2})(1 - x^l)^{-1} \prod_{l=1}^{p-k} (1 - x^{c+l-2})(1 - x^l)^{-1} \right)$$

$$\times \left(\prod_{\substack{1 \leq i < r \\ 1 \leq j < c}} (1 - x^{i+j-1})^{-1} \right),$$

with the convention that $\prod_{l=1}^{0} a_l = 1$.

Summing this generating function over all q from 0 to ∞ or using the same analysis as in the proof of Theorem 4.4, but with no restriction on the sum of the entries in column c of M, we obtain a theorem of Bender and Knuth [3], given by them in a slightly different form.

COROLLARY 4.6. *The norm generating function for plane partitions $P = (p_{ij})$ of shape (c^r) such that $p_{r1} = p$ is*

$$\left[\prod_{\substack{1 \leq i \leq r-1 \\ 1 \leq j \leq c}} (1 - x^{i+j-1})^{-1} \right] \left[x^{rp} \prod_{l=1}^{p} (1 - x^{c+l-1})(1 - x^l)^{-1} \right].$$

Before leaving plane partitions of rectangular shape, it should be recalled that Theorem 3.1 implies that all of the properties of B that we have used in this section also hold for the Schensted-Knuth correspondence. Thus, all the results that we have displayed could equally well have been derived using K. This will not be the case for plane partitions of nonrectangular shape.

5. Plane partitions of arbitrary shape

To get a plane partition of rectangular shape, we can pick any nonnegative integer matrix of the same shape and apply the Burge correspondence. For nonrectangular shapes, we no longer have this freedom. However, we can show that the set of matrices that are associated by the Burge correspondence with plane partitions of shape λ can be partitioned into subsets of matrices such that the matrices in each subset are freely generated. It is then an easy task to derive the trace generating function for the plane partitions that correspond to the matrices in a given subset. Adding these terms together, we obtain the trace generating function for all plane partitions of the given shape.

Let λ be a shape, and let $r = \rho(\lambda)$ and $c = \lambda_1$. Theorem 3.1 tells us that every plane partition of shape (c^r) is the image under Burge's correspondence B of a unique $r \times c$ matrix. Hence, the same holds for every plane partition of shape λ. Let $\mathcal{M}$ be the set of all $r \times c$ matrices M such that $B(M)$ corresponds to a plane partition of shape λ, i.e., if $B(M) = (p_{ij})$, then $p_{ij} > 0$ only if $(i, j) \in \Gamma(\lambda)$.

It will be useful to place the natural partial order on $\Gamma(c^r)$, setting $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$. Also, as we have already taken care of plane partitions of rectangular shape in the preceding section, we may assume that $\lambda \neq (c^r)$. Thus, letting $\Gamma = \Gamma(c^r)$ and $\Gamma' = \Gamma - \Gamma(\lambda)$, we have $\Gamma' \neq \phi$.

We want to consider the finite set $\mathcal{F}$ of one-to-one functions $f : \Gamma' \to \Gamma$ that satisfy the following two properties:

(i)   If $f(s, t) = (i, j)$, then $j - i = t - s$;
(ii) If $(s, t) \leq (s', t')$, then $f(s, t) \leq f(s', t')$.

Let b be the cardinality of $\mathcal{F}$. We then denote the functions in $\mathcal{F}$ by f_1, $f_2, \ldots, f_b$, where we take f_1 to be the identity function.

For a fixed k, $1 \leq k \leq b$, we can partition the positions in Γ into three classes. The position (i, j) in Γ is of *type I* if $(i, j) \in f_k(\Gamma')$. It is of *type II* if it possesses the following properties:

(i) $(i, j) \notin f_k(\Gamma')$;
(ii)   There exists $(i', j') < (i, j)$ such that $j - i = j' - i'$ and $(i', j') \in f_k(\Gamma')$;
(iii) If (i_0, j_0) is the pair whose existence is guaranteed by (ii) with the largest value of j' (or i'), say $f_k(x, y) = (i_0, j_0)$, then for all $(u, v) > (x, y)$, $f_k(u, v) > (i, j)$.

A position is of *type III* if it is not of types I or II. Note that these types depend upon a specific choice of f_k.

It is easy to see that (i, j) is of type II if and only if there exist (x, y) in Γ' and $l \neq k$ such that

$$f_k(s, t) = f_l(s, t) \quad \text{for all } (s, t) \neq (x, y) \quad \text{and} \quad f_k(x, y) < f_l(x, y) = (i, j).$$

If we think of the functions in $\mathcal{F}$ as specifying the placement of zeros on an $r \times c$ chessboard with certain restrictions, a position (i, j) is of type I if a zero

occurs there and of type II if no zero occurs there but a zero occurs at some position (k, l), with $l - k = j - i$ and $(k, l) < (i, j)$, and that zero can be slid to position (i, j), disturbing no other zeros, and still leave an allowed placement of zeros.

The functions in $\mathscr{F}$ can be used to define b classes of matrices. For $1 \leq k \leq b$, we let $\mathscr{M}_k$ be the set of all $r \times c$ nonnegative integer matrices $M = (m_{ij})$ such that

$$m_{ij} \begin{cases} = 0 & \text{if } (i, j) \text{ is of type I} \\ \geq 1 & \text{if } (i, j) \text{ is of type II} \\ \geq 0 & \text{if } (i, j) \text{ is of type III.} \end{cases}$$

Although these definitions may seem complicated, they have an obvious geometric significance, as suggested above, and the classes $\mathscr{M}_k$ are simple to represent. We illustrate this in Figure 1, using $\lambda = (4, 2, 1)$ with its nine associated matrix classes.

$$\begin{pmatrix} \geq 0 \geq 0 \geq 0 \geq 0 \\ \geq 0 \geq 0 = 0 = 0 \\ \geq 0 = 0 = 0 = 0 \end{pmatrix} \quad \begin{pmatrix} \geq 0 = 0 \geq 0 \geq 0 \\ \geq 0 \geq 0 \geq 1 = 0 \\ \geq 0 = 0 = 0 = 0 \end{pmatrix} \quad \begin{pmatrix} \geq 0 = 0 = 0 \geq 0 \\ \geq 0 \geq 0 \geq 0 \geq 1 \\ \geq 0 = 0 = 0 = 0 \end{pmatrix}$$

$$\begin{pmatrix} \geq 0 \geq 0 \geq 0 \geq 0 \\ = 0 \geq 0 = 0 = 0 \\ \geq 0 \geq 1 = 0 = 0 \end{pmatrix} \quad \begin{pmatrix} \geq 0 = 0 \geq 0 \geq 0 \\ = 0 \geq 0 \geq 1 = 0 \\ \geq 0 \geq 1 = 0 = 0 \end{pmatrix} \quad \begin{pmatrix} \geq 0 = 0 \geq 0 \geq 0 \\ = 0 = 0 \geq 0 = 0 \\ \geq 0 \geq 0 \geq 1 = 0 \end{pmatrix}$$

$$\begin{pmatrix} \geq 0 = 0 = 0 \geq 0 \\ = 0 \geq 0 \geq 0 \geq 1 \\ \geq 0 \geq 1 = 0 = 0 \end{pmatrix} \quad \begin{pmatrix} \geq 0 = 0 = 0 \geq 0 \\ = 0 = 0 \geq 0 \geq 1 \\ \geq 0 \geq 0 \geq 1 = 0 \end{pmatrix} \quad \begin{pmatrix} \geq 0 = 0 = 0 \geq 0 \\ = 0 = 0 = 0 \geq 0 \\ \geq 0 \geq 0 \geq 0 \geq 1 \end{pmatrix}$$
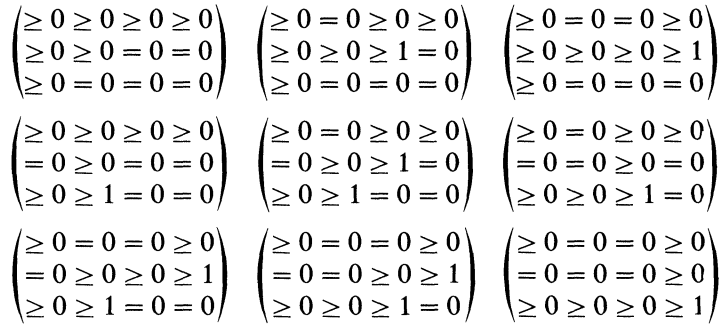
FIG. 1. The nine matrix classes associated with $\lambda = (4, 2, 1)$

Our main objective now is to show that $\mathscr{M}$ is the disjoint union of $\mathscr{M}_1$, $\mathscr{M}_2, \ldots, \mathscr{M}_b$. That the $\mathscr{M}_k$ are disjoint is easy to demonstrate, for each matrix in $\mathscr{M}_k$ uniquely determines f_k. If $M = (m_{ij}) \in \mathscr{M}_k$, then

$$f_k(r, c) = (l, c - r + l) \quad \text{where} \quad l = \max \{p: m_{p\, c - r + p} = 0\}.$$

Assuming that we have uniquely determined $f_k(s', t')$ for all $(s', t') > (s, t)$, we then note that $f_k(s, t) = (l, t - s + l)$, where

$$l = \max \{p: m_{p,\, t - s + p} = 0 \quad \text{and}$$

$$(p, t - s + p) < f_k(s', t') \quad \text{for all } (s', t') > (s, t)\}.$$

The next step is supplied by the following lemma.

LEMMA 5.1. $\mathscr{M}_k \subseteq \mathscr{M}$ for all k, $1 \leq k \leq b$.

Proof. Suppose we can cover $\Gamma - f_k(\Gamma')$ with $s = t_0(\lambda)$ nonintersecting paths, each of which starts in the first column, moves only to the right or up, and ends

in the first row. In addition, suppose that these paths begin in rows $r_1 < r_2 < \cdots < r_s$ and end in columns $c_1 < c_2 < \cdots < c_s$, where

$$c_j = \left| \{l: l \geq 0, t_l(\lambda) \geq s - j + 1\} \right|,$$
$$r_j = \left| \{l: l \leq 0, t_l(\lambda) \geq s - j + 1\} \right|,$$

for $j \geq 1$.

Let $M = (m_{ij})$ be an $r \times c$ matrix such that $m_{ij} = 0$ if $(i, j) \in f_k(\Gamma')$. It then follows from Theorem 3.1 that the number of nonzero parts in the l-diagonal of $B(M)$ is at most

$$\left| \{j: c_j \geq l + 1\} \right| \quad \text{for} \quad l \geq 0$$

and

$$\left| \{i: r_i \geq 1 - l\} \right| \quad \text{for} \quad l \leq 0.$$

Combining the suppositions and remarks from the previous two paragraphs, we see that the number of nonzero parts in the l-diagonal of $B(M)$, $l \geq 0$, is at most

$$\left| \{j: 1 \leq j \leq s \quad \text{and} \quad \left| \{p: p \geq 0 \quad \text{and} \quad t_p(\lambda) \geq s - j + 1\} \right| \geq l + 1\} \right|.$$

A closer look at this expression reveals that it is the $(l + 1)$th term in the conjugate of the conjugate of the shape $(t_0(\lambda), t_1(\lambda), t_2(\lambda), \ldots)$, and therefore equals $t_l(\lambda)$. Similarly, we find that the number of nonzero parts in the l-diagonal of $B(M)$, $l \leq 0$, is at most $t_l(\lambda)$. Hence $B(M)$ can be considered as a plane partition of shape λ.

To complete the proof of this lemma, we need to show that such a covering by paths of $\Gamma - f_k(\Gamma')$ exists for all k. It is easy to verify that $\Gamma - f_1(\Gamma') = \Gamma(\lambda)$ can be covered (in a unique way) by $t_0(\lambda)$ such paths and that c_j and r_j satisfy the corresponding equalities for all j. These paths can be altered in such a way as to yield $t_0(\lambda)$ paths with the same end points that cover $\Gamma - f_k(\Gamma')$ for any $k \neq 1$.

This alteration can be done recursively by noting that any $\Gamma - f_k(\Gamma')$ can be derived from $\Gamma(\lambda)$ through a sequence of stages

$$\Gamma(\lambda), \; \Gamma - f_x(\Gamma'), \; \Gamma - f_y(\Gamma'), \; \ldots, \; \Gamma - f_k(\Gamma'),$$

where we pass from one stage to the next by deleting a position (i, j) and adding on a position $(i + 1, j + 1)$. In other words, we pass from $\Gamma(\lambda)$ to $\Gamma - f_k(\Gamma')$ by moving zeros one space at a time down the diagonals such that the placement of the zeros always corresponds to the positions in $f_z(\Gamma')$ for some f_z in $\mathcal{F}$.

Suppose we have managed to cover $\Gamma - f_x(\Gamma')$ with paths as desired, and that we can obtain $\Gamma - f_y(\Gamma')$ by deleting (i, j) from $\Gamma - f_x(\Gamma')$ and adding $(i + 1, j + 1)$. Then $(i + 1, j)$ and $(i, j + 1)$ must belong to $\Gamma - f_x(\Gamma')$. Otherwise, we

would have, say, $(i + 1, j)$ in $f_x(\Gamma')$ as well as $(i + 1, j + 1)$. This would imply that $f_x^{-1}(i + 1, j) < f_x^{-1}(i + 1, j + 1)$, so that

$$(i, j) < (i + 1, j) = f_y f_x^{-1}(i + 1, j) < f_y f_x^{-1}(i + 1, j + 1) = (i, j),$$

a contradiction.

It is then an easy observation that one of the paths in the given covering of $\Gamma - f_x(\Gamma')$ contains the positions $(i + 1, j)$, (i, j) and $(i, j + 1)$. We get an appropriate covering of $\Gamma - f_y(\Gamma')$ by using the same paths employed to cover $\Gamma - f_x(\Gamma')$, except in the path containing (i, j), we use $(i + 1, j + 1)$ instead. This finishes the proof of the lemma.

All that we have remaining to show is that $\mathcal{M}$ is contained in the union of the $\mathcal{M}_k$. We do this by using a given $M = (m_{ij})$ in $\mathcal{M}$ to define a function $f: \Gamma \to \Gamma$ such that $f = f_k$ for some k and $M \in \mathcal{M}_k$. If there are no zeros on the $(c - r)$-diagonal of M, we need min $\{r, c\}$ chains to cover M. This implies, by Theorem 3.1, that $t_{c-r}(\lambda) = \min \{r, c\}$ and thus, $\lambda = (c^r)$, contrary to assumption. So, some entry on the $(c - r)$-diagonal is zero. Let

$$f(r, c) = (i, c - r + i) \quad \text{where } i = \max \{l: m_{l, c-r+l} = 0\}.$$

Let $(s, t) \in \Gamma'$, $(s, t) < (r, c)$, and assume we have defined f on all (s', t') in Γ', $(s', t') > (s, t)$. We then let $f(s, t) = (i, t - s + i)$, where

$$i = \max \{l: m_{l, t-s+l} = 0 \quad \text{and} \quad (l, t - s + l) < f(s', t') \quad \text{for all } (s', t') > (s, t)\}.$$

For f to be defined at (s, t), we need to know that the set over which we maximize to find i is nonempty. This want is supplied by the next result.

LEMMA 5.2. *Let $(s, t) \in \Gamma'$, and suppose we have defined f at (s, t): $f(s, t) = (i, j)$. Let $M' = (m'_{kl})$ be the $r \times c$ matrix obtained from M by letting*

$$m'_{kl} = \begin{cases} m_{kl} & \text{if } i < k \leq r \quad \text{and} \quad j < l \leq c \\ 0 & \text{otherwise} \end{cases}$$

Then we need at least $s - i(= t - j)$ chains to cover M'.

Proof. We first note that, since $f(s, t)$ is defined, $f(s', t')$ is also defined for all $(s', t') > (s, t)$. If $f(s, t) = (s, t)$, then $f(s', t') = (s', t')$ for all $(s', t') \geq (s, t)$ and $M' = 0$. Hence, the lemma holds.

So, we assume that $f(s, t) \neq (s, t)$. We will show by induction that we can find a set

$$\{(x_t, y_t): 1 \leq t \leq s - i; \quad i < x_1 < x_2 < \cdots < x_{s-1} \leq r,$$

$$j < y_1 < y_2 < \cdots < y_{s-i} \leq c; \quad m_{x_t y_t} \neq 0\},$$

the existence of which implies the lemma. This clearly holds if $(s, t) = (r, c)$, for

$$\{(i + 1, j + 1), (i + 2, j + 2), \ldots, (r, c)\}$$

is such a set.

We can now assume that $(s, t) < (r, c)$. If $t = c$, let $(u, v) = f(s + 1, t)$. Then, either $(u, v) = (s + 1, t)$ or, by induction, we have x's and y's such that

$$u < x_1 < x_2 < \cdots < x_{s+1-u} \leq r, \quad v < y_1 < y_2 < \cdots < y_{s+1-u} \leq c$$

and

$$m_{x_t y_t} \neq 0 \quad \text{for } 1 \leq t \leq s + 1 - u.$$

The definition of $f(s, t)$ tell us that $f(s, t) \leq (u - 1, v)$ and $m_{kl} \neq 0$ for $i + 1 \leq k \leq u - 1$ and $l = t - s + k$. We thus have

$$i < i + 1 < \cdots < u - 1 < x_1 < \cdots < x_{s+1-u} \leq r$$

and

$$j < j + 1 < \cdots < t - s + u - 1 = v < y_1 < \cdots < y_{s+1-u} \leq c,$$

where the subsequences of the x's and y's are empty if $u = s + 1$. Hence, we have the required set of cardinality $(u - 1 - i) + (s + 1 - u) = s - i$.

If, instead, we have $s = r$, the proof is similar, using $f(s, t + 1)$ instead of $f(s + 1, t)$. (Or, transpose M, apply the previous case, and transpose back.) In the case where $s < r$ and $t < c$, let

$$(u, v) = f(s + 1, t) \quad \text{and} \quad (u', v') = f(s, t + 1).$$

If $v' > v$, then we must have $f(s, t) \leq (u - 1, v)$ and the same proof used in the preceding paragraph holds, without the assumption that $t = c$. If $v' \leq v$, we can use the same proof suggested at the beginning of this paragraph, without the assumption that $s = r$. In either case, the proof of this lemma is complete.

To now see that f can be defined on all of Γ', assume we have $(s, t) \in \Gamma'$, $(s, t) < (r, c)$ and suppose we have defined $f(s', t')$ for all $(s', t') > (s, t)$. Then $(s + 1, t)$ or $(s, t + 1)$ or both belong to Γ' and the set

$$\{l: (l, t - s + l) < f(s', t') \quad \text{for all } (s', t') > (s, t)\}$$

equals either

$$\{l: (l, t - s + l) < f(s + 1, t)\} \quad \text{or} \quad \{l: (l, t - s + l) < f(s, t + 1)\}.$$

We can assume that it equals the former set, the proof for the latter case being similar. Let $(u, v) = f(s + 1, t)$. By Lemma 5.2, the submatrix of M consisting of all m_{kl} with $k > u$ and $l > v$ requires at least $(s + 1 - u)$ chains to cover it.

Suppose that $t \geq s$. Then, if we cannot define $f(s, t)$, we must have $m_{k, t-s+k}$, $1 \leq k \leq u - 1$, all nonzero. Therefore, the submatrix of M consisting of all m_{kl} with $1 \leq k < u$ and $t - s + 1 \leq l \leq v$ requires at least $u - 1$ chains to cover it. Hence, M_{t-s} (as defined in Section 3) requires at least $(s + 1 - u) + (u - 1) = s$ chains to cover it. By Theorem 3.1, this implies that the $(t - s)$-trace of λ is at least s since $M \in \mathcal{M}$, and hence, $(s, t) \in \Gamma(\lambda)$, contradicting the fact that $(s, t) \in \Gamma = \Gamma - \Gamma(\lambda)$. A similar analysis holds if $t \leq s$.

We have thus shown that f can be defined on all of Γ'. It is then clear from the

construction of the function that $f = f_k$ for some k and that $M \in \mathcal{M}_k$. We have now reached our objective.

LEMMA 5.3. $\mathcal{M}$ is the disjoint union of $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_b$.

As suggested at the beginning of this section, this partition allows us to compute $G(\lambda)$.

THEOREM 5.4.   Let λ be a nonrectangular shape and let $\mathcal{F}$ be the set of functions associated with λ as described above. Then

$$G(\lambda) = \sum_{f \in \mathcal{F}} \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II}}} w(i,\,j) \right) \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II or III}}} (1 - w(i,\,j))^{-1} \right).$$

Proof.   Let $r = \rho(\lambda)$ and $c = \lambda_1$. Then, by Lemma 4.1, we have

$$\sum_{P \in B(\mathcal{M}_k)} W(P) = \sum_{\substack{M \in \mathcal{M}_k \\ M = (m_{ij})}} \prod_{\substack{1 \le i \le r \\ 1 \le j \le c}} w(i,\,j)^{m_{ij}}$$

$$= \prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II}}} \left(\sum_{m_{ij}=1}^{\infty} w(i,\,j)^{m_{ij}} \right) \prod_{\substack{(i,\,j)\,\text{of} \\ \text{type III}}} \left(\sum_{m_{ij}=0}^{\infty} w(i,\,j)^{m_{ij}} \right)$$

$$= \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II}}} w(i,\,j) \right) \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II or III}}} (1 - w(i,\,j))^{-1} \right).$$

From Lemma 5.3, we know that $G(\lambda) = \sum_{f_k \in \mathcal{F}} \sum_{P \in B(\mathcal{M}_k)} W(P)$. This fact combined with the previous computations completes the proof.

As before, to focus our attention on the 0-trace and norm functions, we let $x_0 = xy$ and $x_l = x$, $l \neq 0$, in $G(\lambda)$.

COROLLARY 5.5.   *Using the notation of Theorem 5.4, we have*

$$\sum_{P \in \mathscr{P}(\lambda)} x^{\sigma(P)} y^{t_0(P)} = \sum_{f \in \mathscr{F}} \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II}}} yx^{i+j-1} \right) \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II or III}}} (1 - yx^{i+j-1})^{-1} \right).$$

In particular,

$$g(\lambda) = \sum_{f \in \mathscr{F}} \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II}}} x^{i+j-1} \right) \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II or III}}} (1 - x^{i+j-1})^{-1} \right).$$

and

$$\sum_{P \in \mathscr{P}(\lambda)} y^{t_0(P)} = \left(\sum_{f \in \mathscr{F}} \left(\prod_{\substack{(i,\,j)\,\text{of} \\ \text{type II}}} y \right) \right) (1 - y)^{-\sigma(\lambda)}.$$

Example. Referring to Figure 1, we can use Corollary 5.5 to give us the following generating functions when $\lambda = (4, 2, 1)$. The norm generating function is

$$g(\lambda) = [(1)(3)(4)]^{-1}\{[(2)^2(3)^2]^{-1} + 2x^4[(2)(3)^2(4)]^{-1}$$
$$+ x^5[(2)(3)(4)(5)]^{-1} + x^8[(3)^2(4)^2]^{-1} + (x^5 + x^9)[(3)(4)^2(5)]^{-1}$$
$$+ x^6[(4)(5)^2(6)]^{-1} + x^{10}[(4)^2(5)^2]^{-1}\}$$

where we have used the notation (n) for $(1 - x^n)$. The 0-trace generating function is given by

$$\sum_{P \in \mathscr{P}(\lambda)} y^{t0(P)} = (1 + 5y + 3y^2)/(1 - y)^7.$$

Corollaries 4.3 and 5.5 imply that the number of plane partitions of a fixed shape and 0-trace has an interesting property. If λ is a given shape, we find that the 0-trace generating function $\sum_{P \in \mathscr{P}(\lambda)} y^{t0(P)}$ has the form $r(y)(1 - y)^{-\sigma(\lambda)}$, where $r(y)$ is a polynomial. In the rectangular case, $r(y) = 1$. If λ is not rectangular, the degree of $r(y)$ is the maximum number of positions of type II taken over all f in $\mathscr{F}$, the set of functions associated with λ. It is easy to see that this degree is less than $\sigma(\lambda)$. In addition, $r(1) = b$, where $b = 1$ if λ is rectangular and $b = |\mathscr{F}|$ if λ is not rectangular. It is well-known (e.g., [12, pp. 20–22]) that these facts imply that the number of plane partitions of shape λ with 0-trace equal to n is a polynomial in n of degree $\sigma(\lambda) - 1$ with leading coefficient b.

6. Variations and MacMahon's Theorem

The analysis carried out in Section 5 depended greatly upon the relation between chains, shapes and Burge's correspondence. If $\mathscr{M}^*$ is the set of all $\rho(\lambda) \times \lambda_1$ matrices whose image under K is a plane partition of shape λ, we do not know of a partition of $\mathscr{M}^*$ analogous to the one given in Lemma 5.3 for $\mathscr{M}$. However, for a certain class of shapes, such partitions do exist.

LEMMA 6.1. *Let λ be a shape of the form $(q^t p^s)$, where $p > q \geq s > 0$ and $t > 0$. Let $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_b$ be as in Lemma 5.3 for λ and let $\mathscr{M}^*$ be as defined above. For $1 \leq k \leq b$, let $\mathscr{M}_k^* = \{RM: M \in \mathscr{M}_k\}$, where R is the operator which reflects a matrix about a horizontal axis. Then $\mathscr{M}^*$ is the disjoint union of $\mathscr{M}_1^*$, $\mathscr{M}_2^*, \ldots, \mathscr{M}_b^*$.*

Proof. In light of Lemma 5.3, it suffices to show that $\mathscr{M}^* = \{RM: M \in \mathscr{M}\}$. Let $M \in \mathscr{M}$. Then, by Theorem 3.1, M_{q-s} can be covered by at most s chains. These chains, when reflected about a horizontal axis, yield s cross chains that cover $(RM)_{q-s}$. Hence, again using Theorem 3.1, we see that the number of nonzero parts on the $(q - s)$-diagonal of $K(RM)$ is at most s, implying that $RM \in \mathscr{M}^*$.

This argument can be reversed, telling us that, if $M' \in \mathcal{M}^*$, $RM' \in \mathcal{M}$ and hence, $M' \in \{RM: M \in \mathcal{M}\}$. This completes the proof.

THEOREM 6.2   *Let λ be a shape of type specified in Lemma* 6.1, *and let* $r = s + t$. *Let $\mathcal{F}$ be the set of associated functions for λ as defined in Section* 5. *Then*

$$G(\lambda) = \sum_{f \in \mathcal{F}} \left(\prod_{\substack{(i,\,j) \text{ of} \\ \text{type II}}} w(r - i + 1, j) \right) \left(\prod_{\substack{(i,\,j) \text{ of} \\ \text{type II or III}}} (1 - w(r - i + 1, j))^{-1} \right).$$

The proof can be obtained by an obvious modification of the proof of Theorem 5.4, taking into account the reflection of the matrices. With this approach, the norm generating function takes on a particularly nice form.

COROLLARY 6.3.   *Under the same hypotheses as in Theorem* 6.2, *we have*

$$g(\lambda) = h(x) \prod_{(i,\,j) \in \Gamma(\lambda)} (1 - x^{r+j-i}) \quad \text{where } h(x) = \sum_{f \in \mathcal{F}} \left(\prod_{\substack{(i,\,j) \text{ of} \\ \text{type II}}} x^{r+j-i} \right).$$

Proof.   As usual, we set every $x_l = x$ in $G(\lambda)$. This changes $w(r - i + 1, j)$ to x^{r+j-i}. To finish his proof, it suffices to note that regardless of the choice of f in $\mathcal{F}$, there will always be the same number, $t_l(\lambda)$, of positions (i, j) of type II or III with $j - i = l$, each yielding the same factor $(1 - x^{r+l})^{-1}$. Thus, for all f in $\mathcal{F}$,

$$\prod_{\substack{(i,\,j) \text{ of} \\ \text{type II or III}}} (1 - w(r - i + 1, j))^{-1}$$

becomes

$$\prod_{l=1-r}^{p-1} (1 - x^{r+l})^{-t_l(\lambda)} = \prod_{(i,\,j) \in \Gamma(\lambda)} (1 - x^{r+j-i})^{-1}.$$

The form of the norm generating function given in Corollary 6.3 leads us to a theorem of MacMahon. Through a magnificent analysis of the problem [18, pp. 213–234], he arrived at the following norm generating function for the plane partitions of shape λ.

THEOREM 6.4.   *Let λ be a shape, and let $r = \rho(\lambda)$. Then*

$$g(\lambda) = h(x) \prod_{(i,\,j) \in \Gamma(\lambda)} (1 - x^{r+j-i})^{-1},$$

where $h(x)$ is a polynomial with integer coefficients. Specifically, $h(x)$ is the determinant of the $r \times r$ matrix whose (i, j)-entry is the polynomial

$$x^{e_{ij}} \prod_{l=1}^{r-i} ((1 - x^{\lambda_j + i - j + l})/(1 - x^l))$$

where

$$e_{ij} = \binom{i - j + 1}{2} \quad if \ i \geq j$$

and

$$e_{ij} = \binom{j - i}{2} \quad if \ i < j.$$

MacMahon's theorem gives the norm generating function in a very elegant form as the quotient of two polynomials. However, the actual determination of the coefficients of the numerator polynomial $h(x)$ via the determinant can be extremely tedious. In addition, huge amounts of cancellation take place.

We also note that, in the case covered by Corollary 6.3, $h(x)$ has only non-negative integer coefficients. The same can be shown to be true for a variety of other shapes, and we conjecture that it is true for all shapes. In fact, we believe a stronger result holds.

Conjecture. The set of all plane partitions of shape λ can be partitioned into a finite number of subsets such that the norm generating function for the plane partitions in each subset has the form

$$x^c \prod_{(i, j) \in \Gamma(\lambda)} \left(1 - x^{\rho(\lambda) + j - i}\right)^{-1}$$

for some nonnegative integer c depending on the subset.

The validity of this conjecture would imply that $h(x)$ has only nonnegative coefficients. A solution of the conjecture in the spirit of Lemma 5.3 would also provide a very efficient means for computing $h(x)$.

In proving the result given in Theorem 6.4, MacMahon was able to determine $h(1)$, the sum of the coefficients in $h(x)$. Indeed, it was the expression for this value that led him to the form for his solution. MacMahon [18, p. 217] discovered that the norm generating function for plane partitions of shape λ has the form $q(x) \prod_{l=1}^{\sigma(\lambda)} (1 - x^l)^{-1}$, where $q(x)$ is a polynomial with integer coefficients such that $q(1)$ equals the number of plane partitions of shape λ whose parts consist of all integers from 1 to $\sigma(\lambda)$. (This result has been greatly extended by Stanley [21].) Combining this with Theorem 6.4, we have

$$h(x) = q(x) \prod_{(i, j) \in \Gamma(\lambda)} \left(1 - x^{\rho(\lambda) + j - i}\right) \prod_{l=1}^{\sigma(\lambda)} (1 - x^l)^{-1}.$$

Thus, letting $x \to 1$,

$$h(1) = \left(q(1)/\sigma(\lambda)!\right) \prod_{(i, j) \in \Gamma(\lambda)} \left(\rho(\lambda) + j - i\right).$$

MacMahon has evaluated $q(1)$, but we invoke a more recent result of Frame, Robinson and Thrall [6], who showed that

$$q(1) = \sigma(\lambda)! \prod_{(i, j) \in \Gamma(\lambda)} (\lambda_i - j + \lambda_j' - i + 1)^{-1}.$$

Combining these last two equations, we arrive at the desired value.

COROLLARY 6.5.   *Using the notation of Theorem* 6.4, *we have*

$$h(1) = \prod_{(i, j) \in \Gamma(\lambda)} (\rho(\lambda) + j - i)(\lambda_i - j + \lambda_j' - i + 1)^{-1}.$$

The terms $(\lambda_i - j + \lambda_j' - i + 1)$ used above are the hook lengths of λ, which appear frequently in enumerative problems concerning plane partitions (cf. [13], [20]). It is known [20, p. 263] that the value for $h(1)$ given in Corollary 6.5 is the number of plane partitions of shape λ whose parts strictly decrease down the columns and are bounded by $\rho(\lambda) - 1$. It is therefore conceivable that a solution to the conjecture given above could involve these special plane partitions in some manner.

7. Symmetric plane partitions

The technique that worked so well in the previous sections for plane partitions can also be applied to symmetric plane partitions, thanks to Corollary 3.4. Once we have defined the appropriate weight function for the symmetric case, we will see that almost all of the earlier results have symmetric analogues. In addition, the proofs involved essentially mimic the earlier proofs, changing only in the details.

Let λ be a symmetric shape. We define a new weight function w^s on $\Gamma(\lambda)$, letting

$$w^s(i, j) = \begin{cases} w(i, j) & \text{if } i = j \\ w(i, j)w(j, i) & \text{if } i \neq j \end{cases}.$$

Applying this definition and Corollary 3.4 to Lemma 4.1, we obtain the following result.

LEMMA 7.1.   *Let* $M = (m_{ij})$ *be a symmetric* $r \times r$ *matrix. Then*

$$W(B(M)) = \prod_{1 \leq i \leq j \leq r} w^s(i, j).$$

Let $\mathscr{P}^s(\lambda)$ denote the collection of all symmetric plane partitions of shape λ; then the trace generating function for $P^s(\lambda)$ is

$$G^s(\lambda) = G(\mathscr{P}^s(\lambda))$$

and the norm generating function is given by

$$g^s(\lambda) = g(\mathscr{P}^s(\lambda)).$$

The rectangular case is again the easiest to consider.

THEOREM 7.2.   *Let r be a positive integer. Then*

$$G^s(r^r) = \prod_{1 \le i \le j \le r} (1 - w^s(i, j))^{-1}.$$

The proof is the same as the one given for Theorem 4.2, using, in addition, the fact that the matrices involved are symmetric (Corollary 3.4) and Lemma 7.1. If we replace x_0 by xy and x_l, $l \neq 0$, by x in $w^s(i, j)$, we find that

$$w^s(i, i) = yx^{2i-1} \quad \text{and} \quad w^s(i, j) = y^2 x^{2(i+j-1)}, \quad i \neq j.$$

Applying this to $G^s(r^r)$, we obtain the symmetric version of Corollary 4.3.

COROLLARY 7.3.   *Let $\lambda = (r^r)$, where r is a positive integer. Then*

$$\sum_{P \in \mathscr{P}^s(\lambda)} x^{\sigma(P)} y^{t_0(P)} = \prod_{1 \le i \le r} (1 - yx^{2i-1})^{-1} \prod_{1 \le i < j \le r} (1 - y^2 x^{2(i+j-1)})^{-1}.$$

In particular, the norm generating function for $\mathscr{P}^s(\lambda)$ is

$$g^s(r^r) = \prod_{1 \le i \le r} (1 - x^{2i-1})^{-1} \prod_{1 \le i < j \le r} (1 - x^{2(i+j-1)})^{-1}.$$

The latter part of this corollary is also an easy consequence of a conjecture of MacMahon that has recently been proved by Andrews [2] and I. MacDonald [17, pp. 50–52]. Allowing $r \to \infty$ in the above result, we obtain the norm generating function for all symmetric plane partitions, originally discovered by Gordon [11].

COROLLARY 7.4.   *The norm generating function for symmetric plane partitions equals*

$$\prod_{1 \le i < \infty} (1 - x^{2i-1})^{-1} \prod_{1 \le i < j < \infty} (1 - x^{2(i+j-1)})^{-1}.$$

To sharpen these results a bit, we can imitate the analysis used to obtain Theorem 4.5, but restrict ourselves to symmetric matrices. In this manner, we find that the norm generating function for symmetric plane partitions $P = (p_{ij})$ of shape (r^r) such that $p_{r1} = p_{1r} = p$ is given by

$$\left[\prod_{1 \le i \le r-1} (1 - x^{2i-1})^{-1} \right] \left[\prod_{1 \le i < j \le r-1} (1 - x^{2(i+j-1)})^{-1} \right]$$

$$\times \left[\sum_{b_1 + \cdots + b_r = p} x^{((2r-1)b_r) + (2r)b_1 + (2r+2)b_2 + \cdots + (4r-4)b_{r-1}} \right].$$

The third factor above can be rewritten as

$$x^{(2r-1)p} \sum_{b_1 + \cdots + b_r = p} x^{(b_1 + 3b_2 + \cdots + (2r-3)b_{r-1})}$$

when $r \geq 2$, or as x^p when $r = 1$. This summation is the norm generating function for shapes λ such that $\rho(\lambda) \leq p$, the λ_i are odd and $\lambda_i \leq 2r - 3$ for all i. This is known to be

$$\sum_{i=0}^{p} x^i \prod_{l=1}^{i} (1 - x^{2(r+l-2)})(1 - x^{2l})^{-1} \qquad [18, \text{p. } 11]$$

Hence, we have the following strengthening of Corollary 7.3.

THEOREM 7.5. *The norm generating function for symmetric plane partitions* $P = (p_{ij})$ *with at most r rows and columns such that $p_{1r} = p_{r1} = p$ equals*

$$\left[\prod_{1 \leq i \leq r-1} (1 - x^{2i-1})^{-1} \right] \left[\prod_{1 \leq i < j \leq r-1} (1 - x^{2(i+j-1)})^{-1} \right]$$

$$\times \left[x^{(2r-1)p} \sum_{i=0}^{p} x^i \prod_{l=1}^{i} (1 - x^{2(r+l-2)})(1 - x^{2l})^{-1} \right],$$

with the convention that $\prod_{l=1}^{0} a_l = 1$.

Moving on to symmetric plane partitions with fixed, nonrectangular shape, it turns out that the machinery developed in Section 5 can be applied here as well, with a little modification. Given a nonrectangular symmetric shape λ, we let $\mathscr{F}$ be the associated set of functions as defined in Section 5. Let $\mathscr{F}^s$ be the collection of all f in $\mathscr{F}$ which are *symmetric*, i.e., if $(i, j) \in f(\Gamma')$, then $(j, i) \in f(\Gamma')$. It is easy to see that f is symmetric if and only if $f(s, t) = (i, j)$ whenever $f(t, s) = (j, i)$. Letting $b = |\mathscr{F}|$ and $a = |\mathscr{F}^s|$, we can assume that $\mathscr{F}^s = \{f_1, f_2, \ldots, f_a\}$.

Let $\mathscr{M}, \mathscr{M}_1, \ldots, \mathscr{M}_b$ be defined as in Section 5, and let $\mathscr{M}^s, \mathscr{M}_1^s, \ldots, \mathscr{M}_b^s$ be their respective subsets of symmetric matrices. It is clear from the definition of $\mathscr{M}_k$ that if f_k is symmetric, $\mathscr{M}_k^s$ is nonempty.

Conversely, if f_k is not symmetric, we can find a maximal (s, t) such that $f_k(s, t) = (i, j)$ but $f_k(t, s) < (j, i)$. Since (s, t) is maximal with this property and $f_k(s', t') > (i, j)$ for all $(s', t') > (s, t)$, we have, for all $(t', s') > (t, s)$, $f_k(t', s') > (j, i)$. Hence, (j, i) is of type II while (i, j) is of type I. Then, if $M = (m_{ln}) \in \mathscr{M}_k$, we must have $m_{ij} = 0$ while $m_{ji} \geq 1$, so M cannot be symmetric. Combining these remarks with Lemma 5.3 gives us the symmetric version of that result.

LEMMA 7.6. $\mathscr{M}^s$ *is the disjoint union of the nonempty sets* $\mathscr{M}_1^s, \mathscr{M}_2^s, \ldots, \mathscr{M}_a^s$.

We can now compute the trace generating function for symmetric plane partitions. The derivation is essentially the same as the one given for Theorem 5.4, using Lemma 7.1, and is omitted.

THEOREM 7.7. *Let λ be a symmetric shape, and let $\mathscr{F}^s$ be defined as above. Then*

$$G^s(\lambda) = \sum_{f \in \mathscr{F}^s} \left(\prod_{\substack{(i, j) \text{ of type} \\ \text{II}, \, i \leq j}} w^s(i, j) \right) \left(\prod_{\substack{(i, j) \text{ of type} \\ \text{II or III}, \, i \leq j}} (1 - w^s(i, j))^{-1} \right).$$

The symmetric plane partition version of Corollary 5.5 can be obtained in the usual manner.

8. Shifted plane partitions

Another type of plane partition that appears fairly frequently is the shifted plane partition. Let λ be a symmetric shape. A *shifted plane partition* $P = (p_{ij})$ *of shape* λ is an array of nonnegative integers indexed by (i, j) in $\Gamma(\lambda)$ with $i \leq j$ such that $p_{ij} \geq p_{i+1j}$ and $p_{ij} \geq p_{ij+1}$. The norm, *l*-trace and trace weight of a shifted plane partition are defined in the obvious way.

Having dealt with symmetric plane partitions, shifted plane partitions are easily enumerated. Given a symmetric plane partition of shape λ, if we ignore everything below the 0-diagonal, we get a shifted plane partition of shape λ. This defines a bijection between these sets of plane partitions. Thus, to obtain the trace generating function for shifted plane partitions of shape λ, we only need to replace x_l and x_{-l}, $l \neq 0$, by $x_l^{1/2}$ in $G^s(\lambda)$. Once we have the trace generating function, we can restrict our attention to the 0-trace or norm in the usual manner.

For example, if r is a positive integer, we find, using the expression for $G^s(r^r)$ given in Theorem 7.2, that the trace generating function for shifted plane partitions of shape (r^r) equals

$$\prod_{1 \leq i \leq r} \left(1 - \prod_{l=0}^{i-1} x_l\right)^{-1} \prod_{1 \leq i < j \leq r} \left(1 - \left(\prod_{l=0}^{i-1} x_l^2\right)\left(\prod_{l=i}^{j-1} x_l\right)\right)^{-1}.$$

The corresponding norm generating function is

$$\prod_{1 \leq i \leq r} (1 - x^i)^{-1} \prod_{1 \leq i < j \leq r} (1 - x^{i+j})^{-1}.$$

The latter result is a special case (and in a different form) of a conjecture of Bender and Knuth [3] that has been verified by Gordon (unpublished) and Andrews [1].

REFERENCES

1. G. E. ANDREWS, *Plane partitions (II): the equivalence of the Bender-Knuth and MacMahon conjectures*, Pacific J. Math., vol. 72 (1977), pp. 282–291.
2. ———, "Plane partitions (I): the MacMahon conjecture" in *Studies in foundations and combinatorics*, G.-C. Rota, editor, Academic Press, N.Y., 1978.
3. E. A. BENDER and D. E. KNUTH, *Enumeration of plane partitions*, J. Combinatorial Theory Ser. A, vol. 13 (1972), pp. 40–54.
4. W. H. BURGE, *Four correspondences between graphs and generalized Young tableaux*, J. Combinatorial Theory Ser. A, vol. 17 (1974), pp. 12–30.
5. L. CARLITZ, *Rectangular arrays and plane partitions*, Acta Arith., vol. 13 (1967), pp. 29–47.
6. J. S. FRAME, G. DE B. ROBINSON, and R. M. THRALL, *The hook graphs of the symmetric group*, Canadian J. Math., vol. 6 (1954), pp. 316–325.
7. E. R. GANSNER, *Matrix correspondences and the enumeration of plane partitions*, Ph.D. Dissertation, M.I.T., February 1978.
8. ———, *The Hillman-Grassl correspondence and the enumeration of reverse plane partitions*, J. Combinatorial Theory Ser. A, to appear.
9. ———, *Matrix correspondences of plane partitions*, Pacific J. Math., to appear.
10. I. M. GESSEL, *Determinants and plane partitions*, preprint.
11. B. GORDON, *Notes on plane partitions V*, J. Combinatorial Theory Ser. B, vol. 11 (1971), pp. 157–168.

12. M. HALL, *Combinatorial theory*, Blaisdell, Waltham, Mass., 1967.
13. A. P. HILLMAN and R. M. GRASSL, *Reverse plane partitions and tableau hook numbers*, J. Combinatorial Theory Ser. A, vol. 21 (1976), pp. 216–221.
14. ———, Functions on tableau frames, Discrete Math., vol. 25 (1979), pp. 245–255.
15. D. E. KNUTH, *Permutations, matrices and generalized Young tableaux*, Pacific J. Math., vol. 34 (1970), pp. 709–727.
16. D. E. LITTLEWOOD, *The theory of group characters*, 2nd ed., Oxford, 1950.
17. I. G. MACDONALD, *Symmetric functions and Hall polynomials*, Oxford, 1979.
18. P. A. MACMAHON, *Combinatory analysis*, vol. 2, Cambridge, 1916; reprinted by Chelsea, New York, 1960.
19. C. SCHENSTED, *Longest increasing and decreasing subsequences*, Canadian J. Math., vol. 13 (1961), pp. 179–191.
20. R. P. STANLEY, *Theory and applications of plane partitions II*, Studies in Appl. Math., vol. 50 (1971), pp. 259–279.
21. ———, *Ordered structures and partitions*, Mem. Amer. Math. Soc., no. 119.
22. ———, *The conjugate trace and trace of a plane partition*, J. Combinatorial Theory Ser. A, vol. 14 (1973), pp. 53–65.

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