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AN EXACT FORMULA FOR AN AVERAGE OF *L*-SERIES

BY
HERBERT WALUM

Let χ be a character of a prime p. As usual,

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \tau(\chi) = \sum_{r=1}^{p} \chi(r) e\left(\frac{r}{p}\right)$$

where s is a complex variable and $e(z) = e^{2\pi i z}$.

The average in question is

$$(1) \qquad \sum_{\chi(-1)=-1} |L(1, \chi)|^2 = \frac{(p-1)^2(p-2)}{12p^2} \pi^2.$$

Chowla [1] gave the asymptotic value $(\pi^2 p)/12$ for the left hand side of (1) and noted that the asymptotic form of (1) was implicit in the work of Paley [2] and Selberg [3].

If $F(t)$ is the fractional part of t minus 1/2 and $e(t) = e^{2\pi i t}$, then

$$(2) \qquad F(t) = -\frac{1}{2\pi i} \sum_{n} \frac{e(nt)}{n}$$

where n runs over all non zero integers. Let ψ be a non-principal character mod p. From (2),

$$\sum_{r=1}^{p-1} \bar{\psi}(r) F\left(\frac{r}{p}\right) = -\frac{1}{2\pi i} \sum_{n} \frac{1}{n} \sum_{r=1}^{p-1} \bar{\psi}(r) e\left(\frac{rn}{p}\right).$$

Since

$$\sum_{r=1}^{p-1} \bar{\psi}(r) e\left(\frac{rn}{p}\right) = \tau(\bar{\psi})\psi(n)$$

we have

$$(3) \qquad \sum_{r=1}^{p-1} \bar{\psi}(r) F\left(\frac{r}{p}\right) = -\frac{\tau(\bar{\psi})}{2\pi i} \begin{cases} 2L(1, \psi) & \text{if } \psi(-1) = -1 \\ 0 & \text{if } \psi(-1) = 1 \end{cases}$$

even when ψ is principal, as an easy computation shows. Thus,

$$\sum_{r,t=1}^{p-1} F\left(\frac{r}{p}\right) F\left(\frac{t}{p}\right) \bar{\psi}(r)\psi(t) = \begin{cases} \dfrac{p}{\pi^2} |L(1, \psi)|^2 & \text{if } \psi(-1) = -1 \\ 0 & \text{if } \psi(-1) = 1 \end{cases}$$

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Summing over all characters, ψ,

$$(p-1)\sum_{r=1}^{p-1}\left(F\left(\frac{r}{p}\right)\right)^2 = \sum_{r,t=1}^{p-1} F\left(\frac{r}{p}\right)F\left(\frac{t}{p}\right)\sum_{\psi}\bar{\psi}(r)\psi(r) = \frac{p}{\pi^2}\sum_{\psi(-1)=-1}|L(1,\psi)|^2.$$

This last equation is true since $\sum_{\psi}\bar{\psi}(r)\psi(t)$ is 0 for $r \not\equiv t$ or $r \equiv t \equiv 0$ and $p-1$ for $r \equiv t \not\equiv 0$. Thus

$$\sum_{\psi(-1)=-1}|L(1,\psi)|^2 = \pi^2\frac{(p-1)}{p}\sum_{r=1}^{p-1}\left(F\left(\frac{r}{p}\right)\right)^2.$$

Now

$$\sum_{r=1}^{p-1}\left(F\left(\frac{r}{p}\right)\right)^2 = \sum_{r=1}^{p-1}\left(\frac{r}{p}-\frac{1}{2}\right)^2 = \frac{(p-1)(p-2)}{12p}$$

and from this we obtain our average.

By an extension of the above argument, it is possible to prove that

$$\sum_{\psi(-1)=-1}|L(1,\psi)|^{2k} \tag{4}$$

is a rational multiple of π^{2k}.

However, the only value of k for which the sum (4) seems to have a simple formula is $k = 1$ in (1). In fact for values of $k > 1$ it is not even clear that (4) has an asymptotic formula. For example, if $k = 2$, formula (3) gives

$$\begin{aligned}\frac{p^2}{\pi^4}\sum_{\psi(-1)=-1}|L(1,\psi)|^4 &= \sum_{\psi}\sum_{r,s,t,u=1}^{p-1} F\left(\frac{r}{p}\right)F\left(\frac{s}{p}\right)F\left(\frac{t}{p}\right)F\left(\frac{u}{p}\right)\psi(rt)\bar{\psi}(su)\\ &= (p-1)\sum_{\substack{r,s,t,u \bmod p,\\ rt\equiv su}} F\left(\frac{r}{p}\right)F\left(\frac{s}{p}\right)F\left(\frac{t}{p}\right)F\left(\frac{u}{p}\right)\end{aligned}$$

In this last sum replace t by ts (this can be done by a suitable rearrangement of the sum: make t be the last variable of summation in an iterated form of the sum) to obtain

$$\begin{aligned}\sum_{\psi(-1)=-1}|L(1,\psi)|^4 &= \frac{\pi^4(p-1)}{p^2}\sum_{\substack{r,s,t,u \bmod p,\\ rts=su}} F\left(\frac{r}{p}\right)F\left(\frac{s}{p}\right)F\left(\frac{ts}{p}\right)F\left(\frac{u}{p}\right)\\ &= \frac{\pi^4(p-1)}{p^2}\sum_{r,s,t \bmod p} F\left(\frac{r}{p}\right)F\left(\frac{s}{p}\right)F\left(\frac{tr}{p}\right)F\left(\frac{tr}{p}\right)\\ &= \frac{\pi^4(p-1)}{p^2}\sum_{t \bmod p}\left(\sum_{r \bmod p} F\left(\frac{r}{p}\right)F\left(\frac{rt}{p}\right)\right)\left(\sum_{s \bmod p} F\left(\frac{s}{p}\right)F\left(\frac{ts}{p}\right)\right)\\ &= \frac{\pi\psi(p-1)}{p^2}\sum_{t \bmod p}(s(t,p))^2\end{aligned}$$

where $s(t, p)$ is the Dedekind sum

$$s(t, p) = \sum_{n=1}^{p-1} F\left(\frac{n}{p}\right)F\left(\frac{nt}{p}\right).$$

The average

$$\sum_{t \bmod p} (s(t, p))^2 \tag{5}$$

when computed for primes below 400 seems to have an oscillatory sort of behavior. Thus it appears that the average (4) may not have an exact formula of the same sort as in (1), and perhaps not even an asymptotic value. Further information about the growth of (4) appears to be hard to obtain.

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THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO