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TRANSFORMATION GROUP C^*-ALGEBRAS WITH HAUSDORFF SPECTRUM

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The results in this note grew out of an attempt to investigate the topology of the primitive ideal space of transformation group C^*-algebras. In [8], some progress was made when the group concerned was abelian. In particular, the primitive ideal space was shown to be homeomorphic to a quotient topological space where the topology is easy to compute [8, Theorem 5.5 and Corollary 5.6].

With this description of the topology at hand for abelian groups, it seems appropriate to ask which transformation group C^*-algebras have Hausdorff primitive ideal space, or even Hausdorff spectrum. The characterization of such algebras given here depends on the fact that the space of closed subgroups of an abelian group and the space of closed subgroups of its dual group are homeomorphic in a natural way. The space of subgroups is given the compact, Hausdorff topology introduced by J. M. G. Fell in [1].

The existence of the homeomorphism mentioned above may be of some interest in itself. Finding an explicit description of the topology on the subgroups, even for abelian groups, can be a formidable task, and one is encouraged to produce alternate, if not necessarily easier, means of calculating it.

Let G be a locally compact abelian group and let $\hat{G}$ be the dual group. Denote the corresponding spaces of closed subgroups by Σ and $\hat{\Sigma}$, respectively. Recall that a base for the topology on Σ is indexed by finite collections of open subsets of G, $\mathscr{F} = \{O_1, O_2, \ldots, O_n\}$, and compact subsets of G, K, and that a typical bases element is

$$\mathscr{U}(\mathscr{F}, K) = \{H \in \Sigma\colon H \cap O_i \neq \phi \quad i = 1, 2, \ldots, n \text{ and } H \cap K = \phi\}.$$

If $H \in \Sigma$, then let $H^\perp = \{\sigma \in \hat{G}\colon \sigma(h) = 1, \text{ for all } h \in H\}$. It follows from the Pontryagin Duality Theorem and lemma 2.13 of [7] that the map defined by $H \to H^\perp$ is a bijection of Σ onto $\hat{\Sigma}$.

THEOREM. *The map of Σ onto $\hat{\Sigma}$ defined by $H \to H^\perp$ is a homeomorphism.*

Proof. Since Σ is compact and $\hat{\Sigma}$ is Hausdorff, it will suffice to show only continuity. Let $\{H_\alpha\}_{\alpha \in \Lambda}$ be a net in Σ converging to H. It will be enough to show that every subnet of $\{H_\alpha^\perp\}_{\alpha \in \Lambda}$ has a subnet which converges to $H^\perp$.

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Let $\{H_\beta^\perp\}_{\beta \in \Lambda'}$ be a subnet. Since $\hat{\Sigma}$ is compact there is a sub-subnet, $\{H_\gamma^\perp\}_{\gamma \in \Lambda''}$, which must converge to something. Thus, there is a $C \in \Sigma$ such that $H_\gamma^\perp \to C^\perp$.

Let $\sigma \in C^\perp$ and $h \in H$. By passing to a still further subnet if necessary, we may assume there exist $h_\gamma \in H_\gamma$ and $\sigma_\gamma \in H_\gamma^\perp$ such that the h_γ converge to h in G and the σ_γ converge to σ in $\hat{G}$. Since the h_γ eventually lie in a compact subset of G, the $\sigma_\gamma(h_\gamma)$ converge to $\sigma(h)$. In particular, $\sigma(h) = 1$ and $C^\perp \subseteq H^\perp$.

To complete the proof we need to show the reverse inclusion. Somewhat surprisingly, this fact is not immediate. The argument given here uses the continuity of inducing representations, so we postpone the proof of the theorem to give two lemmas and introduce some notation.

Let f_0 be a non-negative, real-valued function in $C_c(G)$ which does not vanish at the identity. For each $H \in \Sigma$, define α_H to be the left Haar measure on H with the property that

$$\int_H f_0(t)\, d\alpha_H(t) = 1.$$

The $\{\alpha_H\}$ are called a continuous choice of Haar measures and have the property that, for every $f \in C_c(G)$,

$$H \mapsto \int_H f(t)\, d\alpha_H(t)$$

is continuous [3, p. 908].

Also, let $\mathrm{Ind}_H^G(1)$ denote the representation of G induced from the trivial representation on H. Recall from the definitions in [6] that $\mathrm{Ind}_H^G(1)$ acts on the completion of $C_c(G)$ (here $C_c(G)$ is identified with $C_c(G) \otimes \mathbf{C}$ in the usual way) with respect to the inner product given on $f, g \in C_c(G)$ by

$$\langle f, g\rangle = \int_H g^* * f(t)\, d\alpha_H(t),$$

where $g^*(t) = \overline{g(t^{-1})}$. The action is given, for $s \in G$, by

$$\mathrm{Ind}_H^G\,(1)(s)(f)(t) = f(s^{-1}t).$$

The first lemma is well known, but we were unable to find an appropriate reference in the literature. A complete proof may be found in [8, Lemma 5.1].

LEMMA 1. $\ker(\mathrm{Ind}_H^G\,(1)) = \{f \in C_0(\hat{G}) : f(\sigma) = 0, \textit{for all } \sigma \in H^\perp\}$

Sketch of the Proof. G/H is also a locally compact abelian group and we may choose a Haar measure, μ, on G/H such that

$$\int_{G/H} \int_H f(st)\, d\alpha_H(t)\, d\mu(\dot{s}) = \int_G f(s)\, d\alpha_G(s)$$

for every $f \in C_c(G)$.

It is not difficult to show that $\mathrm{Ind}_H^G(1)$ is unitarily equivalent to a representation R on $L^2(G/H, \mu)$. Namely, if $s \in G$ and $f \in L^2(G/H)$,

$$R(s)(f)(\dot{r}) = f(s^{-1} \cdot \dot{r}).$$

However, if $g \in C_c(G)$ then $R(g) = 0$ if and only if

$$R(g)(f) = 0 \quad \text{for every } f \in L^2(G/H) \cap L^1(G/H).$$

The latter holds if and only if $(R(g)(f))^\wedge(\sigma) = 0$ for every $\sigma \in H^\perp = (G/H)^\wedge$. A simple computation shows $(R(g)(f))^\wedge(\sigma) = \hat{f}(\sigma)\hat{g}(\sigma)$. Since f is arbitrary, it follows from the above remarks that $R(g) = 0$ if and only if $\hat{g}(\sigma) = 0$ for every $\sigma \in H^\perp$. QED

The proof of the next lemma is actually based on a special case of the well-known fact that $H \mapsto \mathrm{Ind}_H^G(1)$ is continuous when Fell's inner hull kernel topology is put on the space of representations of G [2]. For convenience, we give a short self-contained proof here. The reader may find the proof interesting as it may be extended easily to cover more general settings (cf. For example, [8, Lemma 4.9]).

LEMMA 2. *If $\{H_\beta\}_{\beta \in \Lambda}$ converges to H in Σ and $\sigma \in H^\perp$, then there is a subnet, $\{H_\gamma\}_{\gamma \in \Lambda'}$, with $\sigma_\gamma \in H_\gamma^\perp$ and σ_γ converging to σ in $\hat{G}$.*

Proof. If the lemma were false, then there would be an open neighborhood of σ, $\mathscr{U}$, and a subnet, $\{H_\gamma\}_{\gamma \in \Lambda'}$, such that $H_\gamma^\perp \cap \mathscr{U} = \phi$ for every γ.

Let $g, h \in C_c(G)$ and let $\langle g, h \rangle_\gamma$ denote the inner product in the space of $\tau_\gamma = \mathrm{Ind}_{H_\gamma}^G(1)$. That is,

$$\langle g, h \rangle_\gamma = \int_{H_\gamma} h^* * g(t)\, d\alpha_{H_\gamma}(t).$$

Since the $\{d\alpha_{H_\gamma}\}$ are a continuous choice of Haar measures, $\langle g, h \rangle_\gamma$ converges to

$$\int_H h^* * g(t)\, d\alpha_H(t) = \langle g, h \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the space of $\tau = \mathrm{Ind}_H^G(1)$. If $f \in C_c(G)$ as well, then since $\tau_\gamma(f)(g) = f * g$, a similar argument shows $\langle \tau_\gamma(f)(g), h \rangle_\gamma$ converges to $\langle \tau(f)(g), h \rangle$. Now, since $\|g\|_\gamma = \langle g, g \rangle_\gamma^{1/2}$ converges to $\|g\| = \langle g, g \rangle^{1/2}$ for each $g \in C_c(G)$ by the above, it is easy to see that $\langle \tau_\gamma(f)(g), h \rangle_\gamma$ converges to $\langle \tau(f)(g), h \rangle$ for every $f \in C^*(G)$.

Let $f \in C^*(G)$ be such that f is 1 at σ and zero off $\mathscr{U}$. There are $g, h \in C_c(G)$ such that $\langle \tau(f)g, h \rangle \neq 0$. However, by Lemma 1, $\tau_\gamma(f) = 0$ for every γ. Thus $\langle \tau_\gamma(f)(g), h \rangle_\gamma$ converging to $\langle \tau(f)(g), h \rangle$ implies that the latter is zero. This contradiction completes the proof. QED

Proof of the theorem. Suppose $\sigma \in H^\perp$. By Lemma 2 we may, passing to a subset if necessary, find $\sigma_\gamma \in H_\gamma^\perp$ which converge to σ. If $\sigma \notin C^\perp$, then there exists a compact neighborhood of σ, K, such that $K \cap C^{\perp} = \phi$. However, since the $H_{\gamma}^{\perp}$ converge to $C^{\perp}$, the $H_{\gamma}^{\perp}$ must eventually also satisfy $H_{\gamma}^{\perp} \cap K = \phi$. This contradiction completes the proof. QED

Now, let $C^*(G, \Omega)$ be the C^*-algebra associated with the locally compact transformation group, (G, Ω) (see [8] for further definitions and references). For $s \in G$ and $x \in \Omega$, let the image of the s-action on x be denoted by $s \cdot x$. Recall that the stability group at x, S_x, is $\{s \in G: s \cdot x = x\}$. Notice that even though the map of $G \times \Omega \to \Omega$, defined by $(s, x) \to s \cdot x$, is jointly continuous, the natural map from Ω to Σ, $x \mapsto S_x$, often fails to be continuous.

It will be necessary to assume that the natural maps from G/S_x to $G \cdot x$ are homeomorphisms for each x. Recall that this hypothesis will be automatically met if (G, Ω) is second countable and Ω/G is T_0 [4].

Combining the last theorem with the results in [8], it is possible to prove the following.

THEOREM. *Suppose that G is abelian and the maps of G/S_x onto $G \cdot x$ are homeomorphisms for each $x \in \Omega$. Then the spectrum of $C^*(G, \Omega)$ is Hausdorff in the Jacobson Topology if and only if the map $x \mapsto S_x$ is continuous from Ω to Σ, and Ω/G is Hausdorff.*

Proof. First suppose that $C^*(G, \Omega)^\wedge$ is Hausdorff. The argument in [8, Proposition 4.16] shows that Ω/G must be T_0. One can see without too much difficulty that the arguments in [8, Theorem 4.11 and Lemma 4.5(2)] show that there is a continuous injective map of Ω/G into Prim $C^*(G, \Omega)$, even if $C^*(G, \Omega)$ is not assumed to be quasi-regular. The hypothesis of quasi-regularity is used only to insure the existence of a map from Prim $C^*(G, \Omega)$ to Ω/G. Thus, Ω/G must be Hausdorff, since Prim $C^*(G, \Omega)$ is.

Moreover, by [5, Corollary 19], $C^*(G, \Omega)$ must be quasi-regular, and in fact, the assumption on the natural maps from G/S_x to $G \cdot x$ implies that $C^*(G, \Omega)$ is EH-regular [5, Proposition 20 and definitions p. 223].

It now follows from [8, Theorem 5.5] that $C^*(G, \Omega)^\wedge$ is homeomorphic to the quotient topological space obtained from $\Omega/G \times \hat{G}$ by identifying $(G \cdot x, \omega)$ and $(G \cdot y, \sigma)$ if and only if $G \cdot x = G \cdot y$ and $\omega S_x^{\perp} = \sigma S_x^{\perp}$. Suppose $x_\alpha \to x$ in Ω, but $S_{x_\alpha} \not\to S$. Since Σ is compact, it may be assumed that $S_{x_\alpha} \to C$. Moreover, as in the proof of [8, Lemma 4.9], $C \subseteq S_x$. By the previous theorem, $S_{x_\alpha}^{\perp} \to C^{\perp} \supseteq S_x^{\perp}$. Therefore, there exist $\sigma_\alpha \in S_{x_\alpha}^{\perp}$ converging to $\sigma \in C^{\perp}$ with $\sigma|_{S_x} \neq 1$, the trivial character. Since $(G \cdot x_\alpha, \sigma_\alpha)$ and $(G \cdot x_\alpha, 1)$ have the same class in the quotient, it follows that $\{(G \cdot x_\alpha, \sigma_\alpha)\}$ converges to both $(G \cdot x, 1)$ and $(G \cdot x, \sigma)$. Since these are district elements in the quotient space, $C^*(G, \Omega)^\wedge$ could not be Hausdorff.

On the other hand, if Ω/G is Hausdorff, then, as above, $C^*(G, \Omega)$ is EH-regular and Prim $C^*(G, \Omega)$ is homeomorphic to the quotient described above. Moreover, by [8, Proposition 3.2], $C^*(G, \Omega)$ is C. C. R, and in particular, Prim (G, Ω) is homeomorphic to the spectrum.

Suppose that $[G \cdot x_\alpha, \sigma_\alpha]$ is a net in the quotient converging to both $[G \cdot x, \sigma]$ and $[G \cdot y, \omega]$. By [8, Corollary 5.6], it may be assumed that $(x_\alpha, \sigma_\alpha) \to (x, \sigma)$ in $\Omega \times \hat{G}$ and that there are $s_\alpha \in G$ as well as $\omega_\alpha \in S_{x_\alpha}$ such that $(s_\alpha \cdot x_\alpha, \omega_\alpha \sigma_\alpha) \to (y, \omega)$. Since Ω/G is Hausdorff, $G \cdot x = G \cdot y$. But the ω_α must converge to $\omega\sigma^{-1}$. If $y \to S_y$ is continuous then $S_{x_\alpha}^{\perp} \to S_x^{\perp}$ and $\omega\sigma^{-1} \in S_x^{\perp}$. In particular, $(G \cdot x, \sigma)$ and $(G \cdot y, \omega)$ represent the same element in the quotient. QED

Remark. It follows from this proof and the remarks preceeding the theorem that if (G, Ω) is second countable, then the hypothesis on the maps from G/S_x to $G \cdot x$ can be dropped entirely from the statement of the theorem.

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