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SINGULARITY WITH RESPECT TO STRATEGIC MEASURES[1]

BY

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1. Introduction

By a *probability* on a nonempty set X is meant a finitely additive probability measure defined on all subsets of X. A *conditional probability* on a nonempty set Y given X is a mapping from X to the set of probabilities on Y. A *strategy* σ on the cartesian product $X \times Y$ is a pair (σ_0, σ_1) where σ_0 is a probability on X and σ_1 is a conditional probability on Y given X. Each such strategy σ determines a probability on $X \times Y$ which is also denoted by σ and is defined by

$$\sigma(S) = \int \sigma_1(x)(S_x)\sigma_0(dx) \tag{1.1}$$

where $S \subset X \times Y$ and, for each $x \in X$, $S_x = \{y: (x, y) \in S\}$. Probabilities which arise from strategies in this fashion are called *strategic*.

For the rest of this note, let $X = Y = \{1, 2, \ldots\}$ and let α and β be probabilities on X. Define a probability μ on $X \times Y$ by the formula

$$\mu(S) = \int \alpha(S^y)\beta(dy) \tag{1.2}$$

for $S \subset X \times Y$ and $S^y = \{x: (x, y) \in S\}$ for every $y \in Y$. Notice that μ is *reverse strategic* in the sense that it is strategic when the coordinates are interchanged. If α and β are countably additive, then μ is also strategic as follows from Fubini's Theorem or a general result on the existence of conditional distributions. Our major result states that the situation is quite different for diffuse measures. (A probability α on X is *diffuse* or *purely finitely additive* if $\alpha(\{x\}) = 0$ for every $x \in X$.)

THEOREM. *If α and β are diffuse, then μ is singular with respect to every strategic measure.*

Lester Dubins [2] proved this result for the special case when α and β assume only the values 0 and 1, and thereby exhibited the first example of a probability which could not be approximated by strategic measures.

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A probability v on $X \times Y$ is *nearly strategic* if it lies in the variation norm closure of the strategic measures. The following result is an easy consequence of a theorem of Dubins [2, Proposition 2].

PROPOSITION. *If α or β is countably additive, then μ is nearly strategic.*

The converses to both the theorem and the proposition are true. To see this, write

$$\alpha = p\alpha_1 + \bar{p}\alpha_2, \quad \beta = q\beta_1 + \bar{q}\beta_2,$$

where p, $q \in [0, 1]$, $\bar{p} = 1 - p$, $\bar{q} = 1 - q$, α_1 and β_1 are countably additive probabilities, and α_2 and β_2 diffuse. (To obtain the decomposition for α, for example, set $p = \sum_{x \in X} \alpha(\{x\})$ and, for $A \subset X$, set

$$\alpha_1(A) = p^{-1} \sum_{x \in A} \alpha(\{x\})$$

if $p \neq 0$ and let α_1 be an arbitrary countably additive probability if $p = 0$.) Let μ_{ij} be the probability on $X \times Y$ defined by formula (1.2) when α and β are replaced there by α_i and β_j, respectively. Then

$$\begin{aligned} \mu &= pq\mu_{11} + p\bar{q}\mu_{12} + \bar{p}q\mu_{21} + \bar{p}\bar{q}\mu_{22} \\ &= (1 - \bar{p}\bar{q})v + \bar{p}\bar{q}\mu_{22} \end{aligned}$$

where v is defined by the second equality. By the Theorem, μ_{22} is singular with respect to every strategic measure. By the proposition, the remaining μ_{ij}'s are nearly strategic and, by a result of Armstrong and Sudderth [1, Theorem 1], v, being a convex combination of nearly strategic measures, is itself nearly strategic. Furthermore, this decomposition of μ into nearly strategic and singular parts is essentially unique [1, Corollary 1]. Thus, if μ is nearly strategic, then $\bar{p}\bar{q} = 0$ which proves the converse to the Proposition. Likewise, if μ is singular with respect to every strategic measure, then $\bar{p}\bar{q} = 1$ which proves the converse to the theorem.

The next section presents the proof of the theorem. A final section contains a few remarks and open questions.

2. Proof of the theorem

The proof uses a slightly more general notion of strategic measure. Suppose σ_0 is a probability on X and, for every x, $\sigma_1(x)$ is a finitely additive measure defined on all subsets of X which has total mass less than or equal to one. Then (σ_0, σ_1) is a *generalized strategy* and the measure σ of (1.1) is *generalized strategic*.

What will be shown is that given such a σ and given $\varepsilon > 0$, there is a set $S \subset X \times Y$ such that $\sigma(S) < \varepsilon$ and $\mu(S) = 1$. Three cases will be considered.

Case 1. For all $x \in X$, $\sigma_1(x)$ is diffuse.

Take $S = \{(x, y): x > y\}$. It is easy to check that $\sigma(S) = 0$ and $\mu(S) = 1$.

Case 2. For all $x \in X$, $\sigma_1(x)$ is countably additive.

This is the most involved of the three cases and the proof takes several steps. For each $y \in Y$, let

$$\varepsilon_y = \int \sigma_1(x)(\{y\})\alpha(dx), \tag{2.1}$$

and set $Y_1 = \{y: \varepsilon_y = 0\}$, $Y_2 = \{y: \varepsilon_y > 0\}$ so that Y is the disjoint union of Y_1 with Y_2. Two sets S_1 and S_2 will be constructed so that for $i = 1, 2$,

$$S_i \subset X \times Y_i, \quad \sigma(S_i) < 2\varepsilon, \quad \mu(S_i) = \mu(X \times Y_i).$$

Then $S = S_1 \cup S_2$ will satisfy

$$\sigma(S) < 4\varepsilon, \quad \mu(S) = 1,$$

which will complete the proof for this case.

First S_1 will be defined. For each $y \in Y_1$ and every $\delta > 0$,

$$\alpha\{x: \sigma_1(x)(\{y\}) < \delta\} = 1.$$

Thus, if

$$A_y = \{x: \sigma_1(x)(\{y\}) < \varepsilon/2^y\} \quad \text{and} \quad S_1 = \bigcup_{y \in Y_1} (A_y \times \{y\}),$$

then, for each $y \in Y_1$, $S_1^y = A_y$ and $\alpha(S_1^y) = 1$. Hence,

$$\mu(S_1) = \int_{Y_1} \alpha(S_1^y)\beta(dy) = \beta(Y_1) = \mu(X \times Y_1).$$

However, for every x, the x-section $S_{1,x}$ is

$$\{y \in Y_1: \sigma_1(x)(\{y\}) < \varepsilon/2^y\}$$

which, by the countable additivity of $\sigma_1(x)$, has $\sigma_1(x)$-measure less than or equal to $\sum_{y \in Y_1} \varepsilon/2^y < 2\varepsilon$. Hence,

$$\sigma(S_1) = \int \sigma_1(x)(S_{1,x})\sigma_0(dx) \leq \int 2\varepsilon\sigma_0(dx) = 2\varepsilon.$$

The following lemma is used in the construction of S_2.

LEMMA. $\sum_{y \in Y} \varepsilon_y \leq 1$.

Proof. For every $n \in Y$,

$$\begin{aligned}\sum_{y \leq n} \varepsilon_y &= \int \left\{\sum_{y \leq n} \sigma_1(x)(\{y\})\right\} \alpha(dx) \\ &\leq \int \sigma_1(x)(Y)\alpha(dx) \\ &\leq 1.\end{aligned}$$

By the lemma, there is an n such that

$$\sum_{y>n} \varepsilon_y < \varepsilon. \tag{2.2}$$

Let $\{K_y, y > n\}$ be a sequence of positive numbers such that

$$\lim_{y\to\infty} K_y = \infty, \quad \sum_{y>n} K_y \varepsilon_y \le 1. \tag{2.3}$$

The existence of such a sequence is well known and easy to verify.

If $\varepsilon_y > 0$, then

$$\alpha(\{x\colon \sigma_1(x)(\{y\}) \ge \varepsilon_y(1 + K_y\varepsilon)\}) \le (1 + K_y\varepsilon/2)^{-1} \tag{2.4}$$

because otherwise, by (2.1),

$$\varepsilon_y \ge \varepsilon_y(1 + K_y\varepsilon)(1 + K_y\varepsilon/2)^{-1} > \varepsilon_y.$$

For each $y \in Y_2$, set

$$A_y = \{x\colon \sigma_1(x)(\{y\}) < \varepsilon_y(1 + K_y\varepsilon)\}.$$

Then, by (2.4), for $y > n$ and $y \in Y_2$,

$$\alpha(A_y) \ge 1 - (1 + K_y\varepsilon/2)^{-1}$$

and so, by (2.3),

$$\alpha(A_y) \to 1 \quad \text{as } y \to \infty. \tag{2.5}$$

Define

$$S_2 = \bigcup_{\substack{y\in Y_2\\ y>n}} (A_y \times \{y\}).$$

Because of (2.5) and the diffuseness of β,

$$\mu(S_2) = \int_{Y_2} \alpha(A_y)\beta(dy) = \beta(Y_2) = \mu(X \times Y_2).$$

Also, for each x, the x-section $S_{2,x}$ is

$$\{y \in Y_2\colon y > n, \sigma_1(x)(\{y\}) < \varepsilon_y(1 + K_y\varepsilon)\},$$

which, by the countable additivity of $\sigma_1(x)$, has $\sigma_1(x)$-measure less than or equal to

$$\sum_{y>n} \varepsilon_y(1 + K_y\varepsilon) = \sum_{y>n} \varepsilon_y + \varepsilon \sum_{y>n} K_y\varepsilon_y \le 2\varepsilon,$$

by (2.2) and (2.3). Hence, $\sigma(S_2) \le 2\varepsilon$.

This completes the argument for Case II.

Case 3. σ_1 is arbitrary.

For every $x \in X$ and every $A \subset X$, let

$$\sigma_1^c(x)(A) = \sum_{y \in A} \sigma_1(x)(\{y\})$$

and let

$$\sigma_1^d(x)(A) = \sigma_1(x)(A) - \sigma_1^c(x)(A).$$

Thus $\sigma_1^c(x)$ is countably additive and $\sigma_1^d(x)$ is diffuse for each x.

Case I applies to the generalized strategy $\sigma^d = (\sigma_0, \sigma_1^d)$ to yield $S_d \subset X \times Y$ such that $\sigma^d(S_d) < \varepsilon/2$ and $\mu(S_d) = 1$. Case II applies to the generalized strategy $\sigma^c = (\sigma_0, \sigma_1^c)$ to yield $S_c \subset X \times Y$ such that $\sigma^c(S_c) < \varepsilon/2$ and $\mu(S_c) = 1$.

Set $S = S_c \cap S_d$. Then $\mu(S) = 1$ and

$$\sigma(S) = \sigma^c(S) + \sigma^d(S) \leq \sigma^c(S_c) + \sigma^d(S_d) < \varepsilon.$$

The proof of Theorem 1 is now complete.

3. Remarks

One might be misled by the results presented so far to think that reverse strategic measures are strategic only if they are countably additive. Here is a simple counterexample.

Example. Write $X = \bigcup_{n=1}^{\infty} X_n$ where the X_n are disjoint, infinite sets. For every $y \in Y$, let $\alpha(y)$ be a diffuse probability on X_y. Define the reverse strategic measure μ by

$$\mu(S) = \int \alpha(y)(S^y)\beta(dy) \tag{3.1}$$

for $S \subset X \times Y$. Then, as is almost obvious, μ is also the measure induced by the strategy (σ_0, σ_1) where σ_0 is the marginal of μ on X and $\sigma_1(x)$ is point mass at y when $x \in X_y$.

Additional example of diffuse measures on $X \times Y$ which are strategic in both directions are in Heath and Sudderth [3, Theorem 3]. However, there is as yet no satisfactory theorem which characterizes those reverse strategic measures which are also strategic.

Addendum. Mr. S. Ramakrishnan has pointed out to us that our proposition can be improved to say that, if α or β is countably additive, then μ is strategic. This follows immediately from the fact that, if α or β is countably additive, then

$$\iint f\, d\alpha\, d\beta = \iint f\, d\beta\, d\alpha$$

for every bounded, real-valued f defined on $N \times N$. This fact is easy to verify; just approximate the countably additive measure by a linear combination of point masses.

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