ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 3, Fall 1982

FINITE COMPLEXES AND INTEGRAL REPRESENTATIONS

BY

JAMES A. SCHAFER

I. If X and Y are finite CW complexes with isomorphic fundamental groups, one would like to know when X and Y are homotopically equivalent. One obvious necessary condition is that X and Y have the same Euler characteristic. This was shown to be sufficient for 2-complexes with cyclic fundamental groups of prime order by Cockcroft and Swan [6], and for arbitrary cyclic groups by Dyer and Sieradski [7]. On the other hand, Metzler [11] produced examples of finite 2 complexes with isomorphic finite abelian fundamental groups and the same Euler characteristic which were not of the same homotopy type. The major link of the geometry of the problem to algebra and a method of attack is given by a theorem of MacLane and Whitehead [19] and Wall [18] who show that two finite n dimensional complexes X and Y are the same homotopy type if and only if there exists an isomorphism

$$\theta\colon \pi_1(X, *) \to \pi_1(Y, *)$$

and the chain complex of the universal cover of X is chain homotopy equivalent to the chain complex of the universal cover of Y as $\pi_1(Y, *)$-complexes.

This theorem demonstrates the importance of the study of the chain homotopy types of free $\mathbf{Z}G$-complexes. This was done by W. Browning in his thesis and a series of papers [2], [3], [4], [5] for G a finite group. For certain groups he classifies the chain homotopy types of a fixed Euler characteristic by means of a certain finite abelian group. In Section II, we review briefly Browning's work and then in the following sections develop some exact sequences relating Browning's groups to more familiar objects of integral representation theory. We conclude with some calculations of these groups based on these sequences.

II. For the material in this section the reader is directed to Browning's thesis and papers [2], [3], [4], [5] for details.

G will always denote a finite group and Λ the integral group ring of G. B is an arbitrary Λ-lattice (i.e., finitely generated-torsion free as an abelian group).

Received September 28, 1980.

If

$$\mathbf{P}_*: P_m \xrightarrow{\partial_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\partial_0} B \longrightarrow 0$$

is a truncated finitely generated Λ-projective (Λ-stably free) resolution of B, i.e., exact except at P_m, the Euler characteristic of $\mathbf{P}_*$, $\chi(\mathbf{P}_*)$, will be the integer

$$rk_\Lambda P_m - rk_\Lambda P_{m-1} + \cdots + (-1)^m rk_\Lambda P_0, \quad \text{where } rk_\Lambda P_j = \frac{rk_{\mathbf{Z}} P_j}{|G|}.$$

Let $\beta^m(B, l)$ (resp. $F^m(B, l)$) denote the category of all truncated finitely generated Λ-projective (resp. Λ-stably free) resolutions of B of length m and Euler characteristic l. If $\mathbf{P}_*$, $\mathbf{Q}_* \in \beta^m(B, l)$, a map $[h]: \mathbf{P}_* \to \mathbf{Q}_*$ is a chain homotopy equivalence class of Λ-chain maps from the complex $P_m \to \cdots \to P_0$ to the complex $\mathbf{Q}_m \to \cdots \to \mathbf{Q}_0$ which induce the identity on B.

Let u be a finite set of primes of $\mathbf{Z}$ containing all those dividing the order $|G|$ of G. Denote by $\beta^m_u(B, l)$ (resp. $F^m_u(B, l)$) the category whose objects are the objects of $\beta^m(B, l)$ (resp. $F^m(B, l)$) localized at $\Lambda_u = \mathbf{Z}_u G$ and whose maps are Λ_u-chain homotopy classes of Λ_u-chain maps inducing the identity on B_u. Notice that this includes maps which are not localizations. Let $G^u_0(\Lambda)$ be the Grothendieck group of finite Λ-modules whose $\mathbf{Z}$-annihilators are relatively prime to u, i.e., those finite Λ-modules X such that $X_u = 0$, and relative to short exact sequences. If

$$[h]: \mathbf{P}_{*u} \to \mathbf{Q}_{*u}$$

is a Λ_u-homotopy equivalence class, then

$$H_m(h) = \bar{h}: H_m(\mathbf{P}_*)_u \to H_m(\mathbf{Q}_*)_u$$

is an isomorphism and is independent of the representative h. Choose $s \in \mathbf{Z}$ with $(s, u) = 1$ and such that $s\bar{h}: H_m(\mathbf{P}_*) \to H_m(\mathbf{Q}_*)$. Define

$$\lambda[h] = \langle H_m(\mathbf{Q}_*)/s\bar{h}H_m(\mathbf{P}_*)\rangle - \langle H_m(\mathbf{Q}_*)/sH_m(\mathbf{Q}_*)\rangle \in G^u_0(\Lambda).$$

This is well defined and gives a homomorphism λ from the groupoid

$$Eq_u(\beta^m_u(B, l)) = \{[h]: \mathbf{P}_{*u} \longrightarrow \mathbf{Q}_{*u} \mid h \text{ is a } \Lambda_u\text{-homotopy equivalence}\}$$

into $G^u_0(\Lambda)$. The image of λ is equal to $G^u_0(H_m(\mathbf{Q}_*))$, the subgroup of $G^u_0(\Lambda)$ generated by all $\langle X\rangle$ with $X_u = 0$ and X a quotient of $H_m(\mathbf{Q}_*)$. (If $\mathbf{P}_*$, $\mathbf{Q}_* \in \beta^m(B, l)$, then one can show that $G^u_0(H_m(\mathbf{P}_*)) = G^u_0(H_m(\mathbf{Q}_*))$.) Fix an arbitrary $\mathbf{P}_* \in \beta^m(B, l)$ and let

$$\mathrm{Aut}_u \mathbf{P}_* = \{[h]: \mathbf{P}_{*u} \to \mathbf{P}_{*u} \mid h \text{ a } \Lambda_u\text{-homotopy equivalence}\}.$$

Finally define

$$cl^{m+1}_u(G, B, l) = G^u_0(H_m(\mathbf{P}_*))/\lambda(\mathrm{Aut}_u \mathbf{P}_*).$$

If $E^m(G, B, l)$ (resp. $EF^m(G, B, l)$) denotes the isomorphism classes of $\beta^m(B, l)$ (resp. $F^m(B, l)$), then Browning proves the following.

(1) Given any $\mathbf{P}_* \in \beta^m(B, l)$, there exists a well-defined epimorphism

$$\langle\ \ ,\mathbf{P}_*\rangle_u\colon E^m(G, B, l) \to cl_u^{m+1}(G, B, l).$$

(2) $cl_u^{m+1}(G, B, l)$ is a finite abelian group.

(3) If $H_m(\mathbf{P}_*)$ is an Eichler module for some $\mathbf{P}_* \in \beta^m(B, l)$ and hence for all $\mathbf{P}_* \in \beta^m(B, l)$, then $\langle, \mathbf{P}_*\rangle_u\colon E^m(G, B, l) \to cl_u^{m+1}(G, B, l)$ is a bijection.

Note. Λ-lattice M is an Eichler module if the semi-simple rational algebra $\mathrm{End}_{QG}(QM)$ has no simple components of dimension 4 over its center which ramifies at some archimedean prime of the center [12].

(4) The map $t\colon G_0^u(\Lambda) \to \tilde{K}_0(\Lambda)$, given by $t\langle X\rangle = [P] - [Q]$ if $0 \to Q \to P \to X \to 0$ is exact and P is Λ-projective (hence Q also since $X_u = 0$), factors over $cl_u^{m+1}(G, B, l)$. If $h_u^{m+1}(G, B, l)$ denotes the kernel of $t_u\colon cl_u^{m+1}(G, B, l) \to \tilde{K}_0(\Lambda)$ and if $\mathbf{P}_* \in P^m(B, l)$ with $H_m(\mathbf{P}_*)$ Eichler, then

$$\langle, \mathbf{P}_*\rangle_u\colon EP^m(G, B, l) \to h_u^{m+1}(G, B, l)$$

is a bijection.

(5) If (G) denotes the set of primes dividing the order of G, then for each u there exists an isomorphism

$$K_u\colon cl_u^{m+1}(G, B, l) \to cl_{(G)}^{m+1}(G, B, l) \equiv cl^{m+1}(G, B, l)$$

making the following diagram commute.

$$\begin{tikzcd}
& E^m(G, B, l) \arrow[dl, "{\langle\, , \mathbf{P}_*\rangle_u}"'] \arrow[dr, "{\langle, \mathbf{P}_*\rangle_{(G)}}"] & \\
cl_u^{m+1}(G, B, l) \arrow[rr, "K_u"] & & cl^{m+1}(G, B, l).
\end{tikzcd}$$

The aim of the remainder of the paper will be to develop some exact sequences relating $cl^{m+1}(G, l) \equiv cl_{(G)}^{m+1}(G, \mathbf{Z}, l)$ and $h^{m+1}(G, l) \equiv h_{(G)}^{m+1}(G, \mathbf{Z}, l)$ to more known groups and to use these sequences to make some calculations.

III. In this section we state and prove a theorem which gives a method of constructing chain maps of one truncated projective resolution to another. Throughout the section R will denote a subring of the rational numbers Q, $\mathbf{P}_*$ and $\mathbf{Q}_*$ will denote partial projective resolutions of R of length m. We will write M, N for $H_m(\mathbf{P}_*)$, $H_m(\mathbf{Q}_*)$ respectively. As usual, G is always a finite group.

We have the following:

(i) There exists a canonical isomorphism

$$k_{\mathbf{P}_*}\colon \mathrm{Ext}_{RG}^{m+1}(R, M) \underset{\rightarrow}{\approx} \hat{H}^0(G, R) \approx R/|G|R$$

such that $k_{\mathbf{P}_*}[\mathbf{P}_*] = [1] \in R/|G|R$.

Given any $f \in H_{RG}(M, N)$, define the degree of f by

$$\deg f = k_{\mathbf{Q}_*} f_* k_{\mathbf{P}_*}^{-1}[1] = k_{\mathbf{Q}_*} f_*[\mathbf{P}_*] \in R/|G|R.$$

Clearly $\deg (f \circ g) = \deg f \cdot \deg g$ whenever $f \circ g$ is defined.

(ii) If $\mathrm{Hom}_{RG}(\mathbf{P}_*, \mathbf{Q}_*)$ denotes the chain homotopy classes of RG-chain maps from $\mathbf{P}_*$ to $\mathbf{Q}_*$ which induce the identity on R ($\approx H_0(\mathbf{P}_*)$), then there exists a well-defined groupoid homomorphism

$$H\colon \mathrm{Hom}_{RG}(\mathbf{P}_*, \mathbf{Q}_*) \to \mathrm{Hom}_{RG}(H_m(\mathbf{P}_*), H_m(\mathbf{Q}_*))$$

given by $H([h]) = H_m(h)$.

THEOREM. *The sequence*

$$1 \longrightarrow \mathrm{Hom}_{RG}(\mathbf{P}_*, \mathbf{Q}_*)$$

$$\xrightarrow{H} \mathrm{Hom}_{RG}(H_m(\mathbf{P}_*), H_m(\mathbf{Q}_*)) \xrightarrow{\deg} R/|G|R$$

is exact and is natural with respect to inclusions $R \subset S \subseteq \mathbf{Q}$.

Proof. The last statement is clear since $R \subseteq S$ is a flat extension.

(a) $\deg \circ H = [1]$. If $h\colon \mathbf{P}_* \to \mathbf{Q}_*$ is a chain map inducing id on R, then (see [9], for example) if $f = H_m(h)$,

$$f_*[\mathbf{P}_*] = [\mathbf{Q}_*] \in \mathrm{Ext}^{m+1}(R, N)$$

and hence $\deg (f) = k_{\mathbf{Q}_*}[\mathbf{Q}_*] = [1]$.

(b) Suppose $f\colon M \to N$ and $\deg (f) = [1] \in R/|G|R$; i.e.,

$$f_*[\mathbf{P}_*] = [\mathbf{Q}_*] \in \mathrm{Ext}^{m+1}(R, N).$$

Since $\mathbf{P}_*$ is a partial projective resolution of R, we have the exact sequence

$$\mathrm{Hom}_{RG}(P_m, N) \xrightarrow{i*} \mathrm{Hom}_{RG}(M, N) \xrightarrow{\Delta} \mathrm{Ext}_{RG}^{m+1}(R, N) \longrightarrow 0$$

where $\Delta(h) = h_*[\mathbf{P}_*]$. Let $\tilde{g}\colon \mathbf{P}_* \to \mathbf{Q}_*$ be any lift of the identity and let $g = H_m(\tilde{g})\colon M \to N$. Then $\Delta(g) = [\mathbf{Q}_*]$, and, since $\Delta(f) = f_*[\mathbf{P}_*] = [\mathbf{Q}_*]$, there exists $u\colon P_m \to N$ such that $u \circ i = f - g$. Define $\bar{g}\colon \mathbf{P}_* \to \mathbf{Q}_*$ by $\bar{g}_k = g_k$ for $k < m$ and $\bar{g}_m = g_m + i' \circ u$ where $i'\colon N \to Q_m$. Then $\bar{g}$ is a chain map since

$$\bar{g}_{m-1}\partial_m = g_{m-1}\partial_m = \partial'_m g_m = \partial'_m(g_m + i'u) = \partial'_m \bar{g}_m$$

and

$$\bar{g}_m \circ i = (g_m + i' \circ u)i = g_m \circ i + i' \circ u \circ i = i'g + i'(f - g) = i'f.$$

Therefore $H_m(\bar{g}) = f$.

(c) We are left with showing that if $h_*, h'_*\colon \mathbf{P}_* \to \mathbf{Q}_*$ are chain maps (lifting id) and $H_m(h) = H_m(h')$ then h and h' are chain homotopic as maps $\mathbf{P}_* \to \mathbf{Q}_*$.

Now any two lifts of id: $R \to R$ are chain homotopic in the following sense. There exist maps $s_j\colon P_j \to Q_{j+1}$, $0 \le j \le m$, where we set $Q_{m+1} = N$ and $\partial'_{m+1} = i'$, such that

(i) $\partial'_1 s_0 = h_0 - h'_0$.
(ii) $\partial'_{j+1} s_j + s_{j-1}\partial_j = h_j - h'_j$, $1 \le j \le m$,
(iii) $H_m(h) - H_m(h') = s_m \circ i$.

In our case since $H_m(h) = H_m(h')$ we have $s_m \circ i = 0$. So if

$$K_m = \operatorname{coker}\{i\colon M \to P_m\},$$

there exists a map $\tilde{s}_m\colon K_m \to N$ such that $\tilde{s}_m \circ \partial_m = s_m\colon P_m \to N$. (Note:

$$M \xrightarrow{i} P_m \xrightarrow{\partial_m} K_m$$

is exact.) Now $K_m \subseteq P_{m-1}$, and I claim there exists a map $\rho\colon P_{m-1} \to N$ such that $\rho \mid K_m = \tilde{s}_m$. If $K_j = \operatorname{im} \partial_j\colon P_j \to P_{j-1}$, then $0 \to K_m \to P_{m-1} \to K_{m-1} \to 0$ is exact and it is sufficient to prove that

$$\operatorname{Ext}^1_{RG}(K_{m-1}, N) = (0).$$

Since each P_j is projective and $K_0 = R$, we see that

$$\operatorname{Ext}^1_{RG}(K_{m-1}, N) \simeq \operatorname{Ext}^m_{RG}(R, N) \simeq \hat{H}^m(G, N).$$

Since

$$\begin{array}{l} 0 \longrightarrow N \xrightarrow{i} Q_m \xrightarrow{\partial_m'} Q_{m-1} \longrightarrow \\ \qquad\qquad\qquad \cdots \longrightarrow Q_0 \longrightarrow R \longrightarrow 0 \end{array}$$

is exact with each Q_j projective, we have

$$\hat{H}^m(G, N) \approx \hat{H}^{-1}(G, R) \approx NR/IR = (0)$$

and the claim is proved.

To construct the homotopy from h to h', we let $\bar{s}_j = s_j$, $0 \le j < m - 1$, and let

$$\bar{s}_{m-1} = s_{m-1} + i'\rho\colon P_{m-1} \to Q_m.$$

Then, for $j < m - 2$, clearly

$$\partial'_{j+1}\bar{s}_j + \bar{s}_{j-1}\partial_j = h_j - h'_j,$$

while for $j = m - 1$,

$$\partial'_m \bar{s}_{m-1} + \bar{s}_{m-2}\,\partial_{m-1} = \partial'_m(s_{m-1} + i'\rho) + s_{m-2}\,\partial_{m-1} = h_{m-1} - h'_{m-1}$$

and for $j = m$,

$$\bar{s}_{m-1}\partial_m = (s_{m-1} + r'\rho)\partial_m = s_{m-1}\partial_m + i's_m = h_m - h'_m. \quad \blacksquare$$

Before we state some corollaries to this result we give an obvious but useful lemma.

LEMMA. *Let*

$$\mathbf{P}_*: 0 \to M \to P_m \to \cdots \to P_0 \to R \to 0$$

be a truncated projective resolution of R *and let* $a \in RG$ *be a central element. If* L_a: $M \to M$ *is left multiplication by* a, *then*

$$(L_a)_*: \mathrm{Ext}_{RG}^{m+1}(R, M) \to \mathrm{Ext}_{RG}^{m+1}(R, M)$$

is multiplication by $\varepsilon(a) \in R$.

Proof. Clearly,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \longrightarrow & P_m & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L_a} & & \downarrow{\scriptstyle L_a} & & \downarrow{\scriptstyle L_a} & & \downarrow{\scriptstyle L_a} & & \\ 0 & \longrightarrow & M & \longrightarrow & P_m & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

commutes so therefore

$$\begin{array}{ccc} k_{\mathbf{P}_*}: \mathrm{Ext}_{RG}^{m+1}(R, M) & \xrightarrow{\approx} & \hat{H}^0(G, M) \approx R/|G|R \\ \downarrow{\scriptstyle (L_a)_*} & & \downarrow{\scriptstyle (L_a)_*} \\ \mathrm{Ext}_{RG}^{m+1}(R, M) & \xrightarrow{\approx} & \hat{H}^0(G, M) \approx R/|G|R \end{array}$$

commutes, and $(L_a)_*$ obviously induces multiplication by εa on $R/|G|R$.

COROLLARY 1. *If* $\mathbf{P}_*$ *is a truncated projective resolution of* R, *then*

$$1 \longrightarrow \mathrm{Aut}_{RG}(\mathbf{P}_*) \xrightarrow{H} \mathrm{Aut}_{RG}(H_m(\mathbf{P}_*)) \xrightarrow{\deg} (R/|G|R)^* \longrightarrow 1$$

is an exact sequence of groups, where $(R/|G|R)^*$ *denotes the units of* $R/|G|R$.

Proof. This is immediate from the theorem and the fact that deg is onto by the lemma.

COROLLARY 2. *Let* $\mathbf{Z} \subseteq R \subseteq S \subseteq Q$; $\mathbf{P}_*$, $\mathbf{Q}_*$ *truncated projective resolutions of* R. *Suppose the natural map* τ: $R/|G|R \to S/|G|S$ *is mono. Then, given* $h \in \mathrm{Hom}_{SG}(S \otimes_R \mathbf{P}_*, S \otimes_R \mathbf{Q}_*)$, *there exists* $g \in \mathrm{Hom}_{RG}(\mathbf{P}_*, \mathbf{Q}_*)$ *with* $l \otimes g = h$ (*chain homotopy equivalent*) *if and only if there exists*

$$\bar{g} \in \mathrm{Hom}_{RG}(H_m(\mathbf{P}_*), H_m(\mathbf{Q}_*))$$

such that $l \otimes \bar{g} = H_m(h) \in \mathrm{Hom}_{SG}(S \otimes_R H_m(\mathbf{P}_*), S \otimes_R H_m(\mathbf{Q}_*))$. *Moreover, if* g *exists it is unique* (*up to chain homotopy*).

Proof. By the naturality of the exact sequence of the theorem we have

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathrm{Hom}_{RG}(\mathbf{P}_*, \mathbf{Q}_*) & \xrightarrow{H} & \mathrm{Hom}_{RG}(H_m\mathbf{P}_*, H_m\mathbf{Q}_*) & \xrightarrow{\deg} & R/|G|R \\
 & & \downarrow{\scriptstyle \mathrm{id}_s \otimes R} & & \downarrow{\scriptstyle \mathrm{id} \otimes R} & & \downarrow{\scriptstyle \tau} \\
1 & \longrightarrow & \mathrm{Hom}_{SG}(S \otimes_R \mathbf{P}_*, S \otimes_R \mathbf{Q}_*) & \xrightarrow{H} & \mathrm{Hom}_{SG}(S \otimes_R H_m(\mathbf{P}_*), S \otimes_R H_m(\mathbf{Q}_*)) & \xrightarrow{\deg} & S/|G|S
\end{array}$$

Since $H_m\mathbf{P}_*$ and $H_m\mathbf{Q}_*$ are R lattices, the middle map is a monomorphism and the last statement is immediate. The first statement is an immediate consequence of the commutativity of the diagram, the fact that τ is mono, and the fact that im $H = (\deg)^{-1}[1]$.

As a final application of the theorem we prove a result first deduced by Williams [20].

Let u be a finite set of primes containing all primes dividing the order of $|G|$. Let $R = \mathbf{Z}_u = \bigcap_{p \in u} \mathbf{Z}_{(p)}$.

COROLLARY 3. *If* $\mathbf{P}_*$, $\mathbf{Q}_*$ *are truncated projective* $\mathbf{Z}G$-*resolutions of* $\mathbf{Z}$ *with the same Euler characteristic, then there exists an* RG-*homotopy equivalence* h: $\mathbf{P}_{*u} \to \mathbf{Q}_{*u}$.

Proof. Since $\mathbf{P}_*$ and $\mathbf{Q}_*$ have the same Euler characteristic and since all projective RG-modules are free, we have, by Schanuel's lemma,·that

$$H_m(\mathbf{P}_*)_u \oplus (RG)^s \cong H_m(\mathbf{Q}_*)_u \oplus (RG)^s \quad \text{for some } s \in \mathbf{Z}.$$

Since RG is semi-local, this implies $H_m(\mathbf{P}_*)_u \simeq H_m(\mathbf{Q}_*)_u$. Let f: $M_u \to N_u$ be any isomorphism and let $\deg f = [r] \in (\mathbf{Z}/|G|\mathbf{Z})^* = (R/|G|R)^*$. Then $r^{-1}f$: $H_m(\mathbf{P}_*)_u \to H_m(Q_r)_u$ is an RG-isomorphism and $\deg(r^{-1}f) = [1]$. Therefore, by the theorem there exists an RG-chain map h: $\mathbf{P}_{*u} \to \mathbf{Q}_{*u}$ such that $H_m(h) = f/r$, an isomorphism. The following, I believe well-known, proposition completes the proof.

Let R be a ring with 1; (A_*, d), (B_*, d') chain complexes over R and f: $A_* \to B_*$ a chain map. The mapping cone of f is defined by $C(f)_0 = B_0$, $C(f)_q = A_{q-1} \oplus B_q$, $q \geq 1$; $\partial_1 = (f_0, d'_1)$,

$$\partial_{q+1}\begin{pmatrix} -d_q & 0 \\ f_q & d'_{q+1} \end{pmatrix}: C(f)_{q+1} \to C(f)_q.$$

PROPOSITION. *Consider the following statements.*

(i) *$C(f)$ has a contracting homotopy.*
(ii) *f is a homotopy equivalence.*
(iii) *$f_*: H_*(A) \to H_*(B)$ is an isomorphism.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and if A and B are complexes of projective modules, then* (iii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). let $\bar{s}_q: C(f)_q \to C(f)_{q+1}$ be the contracting homotopy and write

$$\bar{s}_0 = \begin{pmatrix} g_0 \\ t_0 \end{pmatrix}, \quad \bar{s}_q = \begin{pmatrix} -s_{q-1} & g_q \\ w_{q-1} & t_q \end{pmatrix}, \quad q \geq 1$$

where $g_q: B_q \to A_q$, $\partial_j: A_j \to A_{j+1}$, $t_j: B_j \to B_{j+1}$, $w_j: A_j \to B_{j+2}$. Then an easy calculation shows that $g: B \to A$ is a chain map and $s: gf \simeq \mathrm{id}$, $t: fg \simeq \mathrm{id}$.

(ii) $\Rightarrow$ (iii) This is immediate from the standard exact sequence relating $H_*(A)$, $H_*(B)$ and $H_*(C(f))$.

(iii) $\Rightarrow$ (i) If A and B are complexes of projectives, then $C(f)$ is a complex of projectives and if $f_*: H_*(A) \to H_*(B)$ is an isomorphism, then $C(f)$ is exact and so has a contracting homotopy.

IV. In this section we develop the exact sequences relating Browning's groups, $cl^{m+1}(G, l)$ and $h^{m+1}(G, l)$ to more familiar objects of integral representation theory.

Let M be a $\Lambda = \mathbf{Z}G$ lattice and u a finite set of prime ideals in $\mathbf{Z}$ containing all primes dividing the order of G. Let $\Omega = \mathrm{End}\,(M)$. According to Bass [1], there exists an exact localization sequence

$$(1) \qquad K_1(\Omega) \to K_1(\Omega_u) \xrightarrow{\partial} G_0^u(\Omega) \xrightarrow{\Delta} K_0(\Omega) \to K_0(\Omega_u)\ .$$

where $G_0^u(\Omega)$ is the Grothendieck group based on finite Ω-modules whose $\mathbf{Z}$-annihilator is relatively prime to u;

$$\partial[\alpha] = \langle \Omega^n / s\alpha\Omega^n \rangle - \langle \Omega^n / s\Omega^n \rangle$$

where $\alpha: \Omega_u^n \to \Omega_u^n$ is an isomorphism and $s\alpha: \Omega^n \to \Omega^n$, $(s, u) = 1$. Moreover, $\Delta\langle X \rangle = [P] - [Q]$ where $0 \to Q \to P \to X \to 0$ is exact and P is Ω-projective.

Let $G_0^u(\Lambda)(M) =$ Grothendieck group generated by finite Λ-modules whose $\mathbf{Z}$-annihilator is relatively prime to u and which are quotients of M. (See Section II, where this is denoted $G_0^u(M)$). M is a Λ-Ω-bimodule where Ω acts on the right of M and if ${}_\Omega\mathscr{M}$, ${}_\Lambda\mathscr{M}$ denotes the category of left Ω, Λ modules respectively, then $M \otimes_\Omega: {}_\Omega\mathscr{M} \to {}_\Lambda\mathscr{M}$. Since $\mathrm{ann}_{\mathbf{Z}}\,(M \otimes_\Omega U) \supseteq \mathrm{ann}_{\mathbf{Z}}\, U$ we see

$$M \otimes_\Omega: G_0^u(\Omega) \to G_0^u(\Lambda)(M).$$

Note. (1) $M \otimes_\Omega$ preserves exact sequences since if $0 \to A \to B \to C \to 0$ is exact and $a \in \mathbf{Z}$ is a common annihilator, then localizing at a does not change the sequence and M_a is Λ_a projective since Λ_a is a maximal order. (2) If U is a simple Ω-module, then U is a quotient of Ω, hence $M \otimes_\Omega U$ is a quotient of M. Since

$$GL_n(\Omega_u) \simeq \mathrm{Aut}_{\Lambda_u}(M^n),$$

we can define a map $\theta\colon K_1(\Omega_u) \to G_0^u(\Lambda)(M)$ by

$$\theta[\alpha] = \langle M^n/s\alpha M^n\rangle - \langle M^n/sM^n\rangle, \quad (s, u) = 1, \; s\alpha\colon M^n \to M^n.$$

If

$$0 \to \Omega^n \xrightarrow{\alpha} \Omega^n \to \Omega^n/\alpha\Omega^n \to 0$$

is exact then

$$\begin{tikzcd}
M \otimes_\Omega \Omega^n \arrow[r, "1\otimes\alpha"] \arrow[d, "\simeq"'] & M \otimes_\Omega \Omega^n \arrow[r] \arrow[d, "\simeq"] & M \otimes_\Omega \Omega^n/\alpha\Omega^n \arrow[d, "\simeq"] \\
M^n \arrow[r, "\bar{\alpha}"] & M^n \arrow[r] & M^n/\bar{\alpha}M^n
\end{tikzcd}$$

commutes, so we have the proposition.

PROPOSITION 1. *The diagram*

$$\begin{tikzcd}
K_1(\Omega_u) \arrow[rr, "\partial"] \arrow[dr, "\theta"'] & & G_0^u(\Omega) \arrow[dl, "M\otimes_\Omega"] \\
& G_0^u(\Lambda)(M) &
\end{tikzcd}$$

commutes.

PROPOSITION 2. $M \otimes_\Omega\colon G_0^u(\Omega) \to G_0^u(\Lambda)(M)$ *is an isomorphism.*

Proof. Let u' be the complementary set of primes to u, then

$$G_0^u(\Omega) = \bigoplus_{p \in u'} G_0^u(\hat{\Omega}_p) \quad \text{and} \quad G_0^u(\Lambda)(M) = \bigoplus_{p \in u'} G_0^u(\hat{\Lambda}_p)(\hat{M}_p),$$

where $\hat{\Omega}_p$, $\hat{\Lambda}_p$ are the completions of Ω, Λ at the prime p and $\hat{M}_p = \hat{\Lambda}_p \otimes_\Lambda M$. If Γ is a maximal order containing Λ, then $\hat{\Lambda}_p = \hat{\Gamma}_p$ for all $p \in u'$ since u contains all primes dividing $|G|$. So we may assume Λ is a maximal order. If $QG \simeq \oplus A_i$, where the A_i are simple algebras corresponding to the idempotents e_i, then $\Lambda = \oplus \Lambda_i$ where $\Lambda_i = \Lambda e_i$ is a maximal order in A_i and $M = \oplus M_i$, $M_i = (\Lambda e_i)M$. Moreover, if $f\colon M_i \to M_j$ and $i \neq j$, then, since $e_i M_i = M_i$, $e_i M_j = 0$, we have $f = 0$ so

$$\Omega = \mathrm{End}_\Lambda(M) = \bigoplus_i \mathrm{End}_{\Lambda_i}(M_i) = \bigoplus_i \Omega_i.$$

So we may assume Λ is a maximal order in a simple algebra A.

Consider $\mathrm{ann}_\Lambda M \subseteq \Lambda$. This is a 2-sided ideal in Λ so $Q(\mathrm{ann}_\Lambda M) = (0)$ or A since A is simple. If $Q(\mathrm{ann}_\Lambda M) = (0)$ then M is faithful since $\mathrm{ann}_A (QM) \subseteq Q(\mathrm{ann}_\Lambda M)$ and M is a lattice. Since Λ is a maximal order M is projective and hence a progenerator. Therefore by the Morita equivalence

$$M \otimes_\Omega \colon G_0^u(\Omega) \to G_0^u(\Lambda) = G_0^u(\Lambda)(M)$$

is an isomorphism.

If $Q(\mathrm{ann}_\Lambda M) = A = Q\Lambda$ then $Q(\Lambda/\mathrm{ann}_\Lambda M) = (0)$ and there exists $n \in \mathbf{Z}$ such that $n1_\Lambda \in \mathrm{ann}_\Lambda M$; i.e., $nM = 0$. But M is a lattice so $M = (0)$ and therefore $\Omega = \mathrm{End}_\Lambda (M) = (0)$. Hence

$$M \otimes_\Omega \colon G_0^u(\Lambda) \to G_0^u(\Lambda)(M)$$

is an isomorphism since both groups are zero. ∎

Let E_M be the category consisting of all Λ-modules N which are direct summands of $M^{(s)}$ for some s, and let D_M be the Grothendieck group of this category based on split exact sequences. It is well known that $M \otimes_\Omega -$, $\mathrm{Hom}_\Lambda (M, -)$ provide a pair of inverse isomorphisms $K_0(\Omega) \to D_M$, (Reiner [13]). Define $\sigma\colon G_0^u(\Lambda)(M) \to D_M$ as follows: If U is simple, $\sigma[U] = [M] - [N]$ where $0 \to N \to M \to U \to 0$ is exact. It is then clear that

$$\begin{tikzcd} G_0^u(\Omega) \arrow[r, "\Delta"] \arrow[d, "M\otimes_\Omega", "\simeq"'] & K_0(\Omega) \arrow[d, "M\otimes_\Omega", "\simeq"'] \\ G_0^u(\Lambda)(M) \arrow[r, "\sigma"] & D_M \end{tikzcd}$$

commutes.

Combining these propositions and the exact localization sequence of Bass we obtain the following sequence (compare to Roggenkamp [14, p. 165]):

$$K_1(\Omega) \to K_1(\Omega_u) \xrightarrow{\theta} G_0^u(\Lambda)(M) \xrightarrow{\sigma} \tilde{g}(M) \to 0 \tag{2}$$

where

$$\tilde{g}(M) = \ker i\colon D_M \to D_{M_u} = \bigoplus_{p \in u'} D_{M_p}.$$

Note. $\tilde{g}(M) = \{\langle N_1\rangle - \langle N_2\rangle \mid N_2 \in E_M, N_1 \vee N_2\}$. $N_1 \vee N_2$ means N_1 and N_2 are in the same genus. By a result of Roiter [15] and Jacobinski [8], there exists $N \vee M$ such that $M \oplus N_2 \simeq N \oplus N_1$; i.e., $[N_1] - [N_2] = [M] - [N]$. Therefore $\tilde{g}(M)$ is the "reduced genus group of M" consisting of $\{[M] - [N] \mid M \vee N\}$ and addition given by

$$([M] - [N_1]) + ([M] - [N_2]) = [M] - [N_3] \quad \text{where } N_1 \oplus N_2 \simeq M \oplus N_3.$$

THEOREM. *Let $\mathbf{P}_*$ be a truncated projective resolution of $\mathbf{Z}$ of length m and Euler characteristic l and $M = H_m(\mathbf{P}_*)$. Then there exists an exact sequence*

$$K_1(\mathrm{End}_\Lambda M) \xrightarrow{\det} (\mathbf{Z}/|G|)^* \xrightarrow{\tilde{\theta}} cl^{m+1}(G, l) \xrightarrow{\rho} \tilde{g}(M) \longrightarrow 0.$$

Proof. Recall, we have an exact sequence

$$\mathrm{Aut}_{\Lambda_u}(\mathbf{P}_{*u}) \xrightarrow{\lambda} G_0^u(\Lambda)(M) \xrightarrow{\pi} cl^{m+1}(G, l) \to 0.$$

Since u is finite, Ω_u is semi-primary and hence

$$\mathrm{Aut}_{\Lambda_u} M_u = GL(1, \Omega_u) \to K_1(\Omega_u)$$

is onto. Moreover, it is obvious that $\lambda = \theta \cdot H$. Hence we have defined ρ by the following diagram:

$$\begin{tikzcd}
\mathrm{Aut}_{\Lambda_u}(\mathbf{P}_{*u}) \arrow[r, "\lambda"] \arrow[d, "H"] & G_0^u(\Lambda)(M) \arrow[r, "\pi"] \arrow[d, equal] & cl^{m+1}(G, l) \arrow[r] \arrow[d, "\rho"] & 0 \\
\mathrm{Aut}_{\Lambda_u} M_u \arrow[r, "\theta"] & G_0^u(\Lambda)(M) \arrow[r, "\sigma"] & \tilde{g}(M) \arrow[r] & 0.
\end{tikzcd}$$

The map $\tilde{\theta}$ is induced by $\pi \circ \theta$. This follows since, in the diagram below, the top row is exact, $\theta \cdot H = \lambda$ and $\pi \cdot \lambda = 0$. One

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Aut}_{\Lambda_u}(\mathbf{P}_{*u}) \arrow[r, "H"] \arrow[dr, "\lambda"'] & \mathrm{Aut}_{\Lambda_u} M_u \arrow[r, "\deg"] \arrow[d, "\theta"] & (\mathbf{Z}/|G|)^* \arrow[r] \arrow[ddl, dotted, "\tilde{\theta}"] & 1 \\
& & G_0^u(\Lambda)(M) \arrow[d, "\pi"] & & \\
& & cl^{m+1}(G, l) & &
\end{tikzcd}$$

easily checks that $\tilde{\theta}[r] = \pi\langle M/rM\rangle$.

Let $[\alpha] \in K_1(\mathrm{End}_\Lambda M)$ be represented by $\alpha \in GL(n, \Omega) = \mathrm{Aut}_\Lambda M^n$. Then,

$$\begin{tikzcd}
\mathrm{Ext}^{m+1}(1, \alpha)\colon \mathrm{Ext}_\Lambda^{m+1}(\mathbf{Z}, M^{(n)}) \arrow[r] \arrow[d, "\approx"] & \mathrm{Ext}_\Lambda^{m+1}(\mathbf{Z}, M^{(n)}) \arrow[d, "\approx"] \\
(\mathbf{Z}/|G|)^{*(n)} \arrow[r, "\alpha^*"] & (\mathbf{Z}/|G|)^{*(n)}
\end{tikzcd}$$

is an automorphism, so lies in $GL(n, \mathbf{Z}/|G|)$; hence $\det \alpha^* \in (\mathbf{Z}/|G|)^*$. Moreover, it is clear that $\det(\alpha_* \oplus 1) = \det \alpha_*$ so

$$\det\colon K_1(\mathrm{End}_\Lambda M) \to (\mathbf{Z}/|G|)^*$$

given by $\det[\alpha] = \det \alpha_*$ is well defined.

This defines det, $\tilde{\theta}$, ρ. Exactness at $\tilde{g}(M)$ is obvious.

(a) $\rho \cdot \tilde{\theta} = 0$. If $[r] \in (\mathbf{Z}/|G|)^*$ then $\rho\tilde{\theta}[r] = \rho\pi\theta(r)$ where $r\colon M_u \to M_u$ is the automorphism given by multiplication by r, but $\rho\pi\theta(r) = \sigma\theta(r) = 0$.

(b) $\ker \rho \subseteq \operatorname{im} \tilde{\theta}$. Let $x \in cl^{m+1}(G, l)$, $\rho x = 0$ and let $x = \pi y$, $y \in G_0^u(\Lambda)(M)$. Then $\rho x = \rho\pi y = \sigma y = 0$. So there exists $\bar{h} \in \operatorname{Aut}_{\Lambda_u} M_u$ with $\theta(\bar{h}) = y$. Hence there exists a chain map

$$h\colon \mathbf{P}_{*u} \to \mathbf{P}_{*u} \quad \text{with } H_m(h) = \bar{h}/\deg \bar{h}.$$

Suppose $s\bar{h}\colon M \to M$, then ($r = \deg \bar{h}$)

$$s\bar{h} = \frac{rs\bar{h}}{r}\colon M \to M,$$

so $\lambda[h] = \langle M/s\bar{h}M\rangle - \langle M/rsM\rangle$ while

$$y = \theta(\bar{h}) = \langle M/s\bar{h}M\rangle - \langle M/sM\rangle.$$

Since $\langle M/rsM\rangle = \langle M/rM\rangle + \langle M/sM\rangle$, we have $\lambda[h] = y - \langle M/rM\rangle$ and therefore

$$x = \pi y = \pi\langle M/rM\rangle = \tilde{\theta}[r].$$

(c) $\tilde{\theta} \circ \det = 0$. Let $[\alpha] \in K_1(\operatorname{End}_\Lambda M)$ be represented by

$$\alpha \in GL(n, \Omega) = \operatorname{Aut}_\Lambda M^n.$$

The following diagram commutes

$$\begin{tikzcd}
K_1(\Omega) \arrow[r, "i"] \arrow[rr, bend left, "\det"] & K_1(\Omega_u) \arrow[dr, "\det"] \arrow[dd, bend right, "\theta"'] & \\
& \operatorname{Aut}_{\Lambda_u} M_u \arrow[u, "j"] \arrow[r, "\deg"] \arrow[d, "\theta"] & (\mathbf{Z}/|G|)^x \arrow[ddl, "\tilde{\theta}"] \\
& G_0^u(\Lambda)(M) \arrow[d, "\pi"] & \\
& cl^{m+1}(G, l) &
\end{tikzcd}$$

and $\theta \circ i = 0$ from the sequence (2). Now, $\tilde{\theta} \circ \det[\alpha] = \tilde{\theta} \det i[\alpha]$. Let $i[\alpha] = jh$. Then $\det jh = \deg h = \det \circ\, i[\alpha]$, and

$$\tilde{\theta} \circ \det[\alpha] = \tilde{\theta}(\deg h) = \bar{u}\theta(h) = \bar{u}\theta(h) = \bar{u}\theta jh = \bar{u}\theta i[\alpha] = 0$$

since $\theta i = 0$.

(d) Ker $\tilde{\theta} \subseteq$ im det. Suppose $\tilde{\theta}[r] = 0$; i.e., $\langle M/rM \rangle \in$ im λ. Then there exists

$$\bar{h}: \mathbf{P}_{*u} \to \mathbf{P}_{*u}$$

such that $\langle M/rM \rangle = \langle M/shM \rangle - \langle M/sM \rangle h = H_m(\bar{h})$. h is an automorphism of M_u and $\theta(h) = \lambda[\bar{h}] = \langle M/rM \rangle = \theta(r)$. From sequence (2), there exists $[\alpha] \in K_1(\Omega)$ such that $i[\alpha] = h^{-1} \circ r$. But

$$\det [\alpha] = \det i[\alpha] = \deg (h^{-1} \circ r) = \deg h^{-1} \circ r = r$$

since $\deg h = 1$, because $h = H(\bar{h})$ and $\deg \circ \bar{H} = 1$. ∎

Recall from Section 2, there exists an exact sequence

$$0 \to h^{m+1}(G, l) \to cl^{m+1}(G, l) \xrightarrow{t} \tilde{K}_0(\Lambda),$$

where t is induced from the map

$$G_0^u(\Lambda)(M) \xrightarrow{t} \tilde{K}_0(\Lambda)$$

given by projectively resolving $\langle X \rangle \in G_0^u(\Lambda)(M)$; i.e.,

$$t\langle X \rangle = [P] - [Q]$$

where $0 \to Q \to P \to X \to 0$ is exact. Consider the map $t \circ \tilde{\theta}$. From the above description of t and the fact that $\tilde{\theta}[r] = \pi\langle M/rM \rangle$, we see that $t \circ \tilde{\theta}[r] = [P] - [Q]$ if $0 \to Q \to P \to M/rM \to 0$ is exact and P is projective. Let $\Sigma = \sum_{g \in G} g$ denote the norm element of Λ, and

$$SW_\Lambda: (\mathbf{Z}/|G|)^* \to K_0(\Lambda)$$

the homomorphism which maps $r \in (\mathbf{Z}/|G|)^*$ to the class of the projective ideal (r, Σ) generated by r and Σ in Λ.

PROPOSITION 3. $t \circ \tilde{\theta} = (-1)^{m+1} SW_\Lambda$.

Proof. Since $(\mathbf{Z}/|G|)^*$ is generated by the primes not dividing $|G|$, it is enough to show these two maps are the same for all primes $q \nmid |G|$. Since $0 \to M \to P_m \to \cdots \to P_0 \to \mathbf{Z} \to 0$ is exact and Λ_q is a maximal order,

$$0 \to \hat{M}_q \to (\hat{P}_m)_q \to \cdots \to (\hat{P}_0)_q \to \hat{\mathbf{Z}}_q \to 0$$

splits completely and so $\hat{M}_q \oplus (\hat{P}_{m-1})_q \oplus \cdots \simeq (\hat{P}_m)_q \oplus \cdots$. Dividing out the ideal generated by q, we have

$$M/qM \oplus P_{m-1}/qP_{m-1} \oplus \cdots \simeq P_m/qP_m \oplus \cdots .$$

This gives

$$\langle M/qM \rangle = (-1)^{m+1}\langle \mathbf{Z}/q\mathbf{Z} \rangle + \langle P_m/qP_m \rangle - \langle P_{m-1}/qP_{m-1} \rangle + \cdots$$

in $G_0^u(\Lambda)$. Therefore,

$$t\langle M/qM\rangle = (-1)^{m+1}t\langle \mathbf{Z}/q\mathbf{Z}\rangle + t\langle P_m/qP_m\rangle - t\langle P_{m-1}/qP_{m-1}\rangle + \cdots.$$

But $t\langle P_j/qP_j\rangle = 0$ since

$$0 \to P_j \xrightarrow{q} P_j \to P_j/qP_j \to 0$$

is exact. It is well known that one obtains the class of (r, Σ) by projectively resolving $\mathbf{Z}/r\mathbf{Z}$; hence the result. ▮

Let $\det_M$ denote image $\det\colon K_1(\mathrm{End}_\Lambda M) \to (\mathbf{Z}/|G|)^*$ and let $SW(\Lambda)$ denote the kernel $SW_\Lambda\colon (\mathbf{Z}/|G|)^* \to \tilde{K}_0(\Lambda)$. Since $\tilde{\theta} \circ \det = 0$ we see $\det_M \subseteq SW(\Lambda)$ and we see easily from the last theorem that we have the following exact

$$(*)\quad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & SW(\Lambda)/\det_M & \xrightarrow{\tilde{\theta}} & h^{m+1}(G, l) & \xrightarrow{\rho} & \ker \bar{t} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathbf{Z}/|G|)^*/\det_M & \xrightarrow{\tilde{\theta}} & cl^{m+1}(G, l) & \xrightarrow{\rho} & \tilde{g}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle SW_\Lambda} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \bar{t}} & & \\
0 & \longrightarrow & \mathrm{Im}\, SW_\Lambda & \longrightarrow & \tilde{K}_0(\Lambda) & \xrightarrow{\pi} & \tilde{K}_0(\Lambda)/\mathrm{Im}\, SW_\Lambda & \longrightarrow & 0
\end{array}$$

where $\bar{t}$ is the obvious induced map. An easy chase of the definitions shows $\bar{t}$ is given as follows. Let $[M] - [N] \in \tilde{g}(M)$. Then by Roiters lemma, since $M \vee N$, there exists an embedding $\phi\colon N \to M$ with $\mathrm{ann}_{\mathbf{Z}}$ (coker ϕ) relatively prime to $|G|$. If

$$0 \to Q \to P \to \mathrm{coker}\, \phi \to 0$$

is exact and P is projective then $\bar{t}([M] - [N]) = \pi([P] - [Q])$.

Let F_M be the subgroup of $\tilde{g}(M)$ given by

$$\{[M] - [N] \mid M \vee N,\ M \oplus F \simeq N \oplus F,\ F \text{ free}\}.$$

This is easily seen to be a subgroup. We record for use in the next section the following proposition.

PROPOSITION 4. $\ker \bar{t} \subseteq F_M$.

Proof. Suppose $\bar{t}([M] - [N]) = 0$. Then, if

$$0 \to N \to M \to X \to 0 \quad \text{and} \quad 0 \to Q \to P \to X \to 0$$

are exact, where $(\mathrm{ann}_{\mathbf{Z}} X, u) = 1$ and P is projective we must have

$$[P] - [Q] \in \mathrm{Im}\, SW_\Lambda.$$

It follows that for some $r \in (\mathbf{Z}/|G|)^*$ there exists an exact sequence $0 \to Q' \to P' \to M/rM \to 0$ with P' projective such that $[P] - [Q] = [P'] - [Q']$. Hence there exists a projective module S such that

$$P \oplus Q' \oplus S \simeq P' \oplus Q \oplus S.$$

On the other hand, since $M \to M \to M/rM$ is also exact and M is a lattice we have $M \oplus Q' \simeq M \oplus P'$ and $M \oplus Q \simeq N \oplus P$, (Roggenkamp [14, Chapter VII]). Hence

$$N \oplus P \oplus P' \oplus S \simeq M \oplus Q \oplus P' \oplus S \simeq M \oplus P \oplus Q' \oplus S \simeq M \oplus P \oplus P' \oplus S.$$

Choosing a complement to $P \oplus P' \oplus S$ gives the result. ▌

V. We will use the exact sequences of the last section to make some calculations of the groups $h^{m+1}(G, l)$ and $cl^{m+1}(G, l)$. We will assume throughout this section that there exists a stably free resolution of $\mathbf{Z}$ of length m and Euler characteristic l.

THEOREM 1. *Suppose there exists a periodic (stably free) resolution of* $\mathbf{Z}$ *of length m and (necessarily) Euler characteristic* 0, *then*

$$cl^{m+1}(G, 0) \approx (\mathbf{Z}/|G|)^*/(\pm 1) \quad \textit{and} \quad h^{m+1}(G, 0) \approx SW(\Lambda)/(\pm 1).$$

Proof. Since $M = \mathbf{Z}$ it is obvious that $\tilde{g}(M) = 0$ and $\operatorname{End}_\Lambda(M) \approx \mathbf{Z}$ which gives the results. ▌

We can say somewhat more in the case of a cyclic group. Swan has shown that the ideal (r, Σ) is free if and only if there exists a unit $u \in \Lambda/(\Sigma)$ such that $\varepsilon u = [r]$. Hence the following well-known result is useful [17].

LEMMA 1. *If* $G = \mathbf{Z}/n$, *then* $\varepsilon\colon (\Lambda/(\Sigma))^* \to (\mathbf{Z}/|G|)^*$ *is onto.*

COROLLARY. *If* $G = \mathbf{Z}/n$, *then* $h^{\text{even}}(G, 0) \approx (\mathbf{Z}/|G|)^*/(\pm 1)$.

Note. This agrees with the classification of m dimensional lens spaces.

COROLLARY. *Let* $G = Q_8$. *Then* $h^{4s+1}(G, 0) = (0)$.

Proof. It is known in this case (Martinet [10]) that $(3, \Sigma)$ is the non-zero element of $\tilde{K}_0(Q_8) \approx \mathbf{Z}/2$.

THEOREM 2. *Suppose there exists a periodic (stably free) resolution of length m, and* $(\Lambda/(\Sigma))^* \to (\mathbf{Z}/|G|)^*$ *is onto, then*

$$cl^m(G, 1) \simeq \tilde{g}(\Lambda/(\Sigma)) \quad \textit{and} \quad h^m(G, 1) = (0).$$

Proof. We may assume (by dualizing) that $F_m = \Lambda$. Hence there exists a complex of length $m-1$ of Euler characteristic 1 and $M \approx \Lambda/(\Sigma)$. By hypothesis, det: $K_1(\mathrm{End}_\Lambda M) \to (\mathbf{Z}/|G|)^*$ is onto, and hence

$$cl^m(G, 1) \approx g \sim (M), \quad h^m(G, 1) \approx \ker \bar{t}.$$

By Proposition 4 of Section IV, $\ker \bar{t} \subseteq F_M$. But if $[M] - [M_1] \in F_M$ then

$$M \oplus \Lambda \approx M_1 \oplus \Lambda$$

and, since $M = \Lambda/(\Sigma)$ and $M \vee M_1, \Sigma \cdot M_1 = (0)$, we have

$$M \oplus \Lambda/(\Sigma) \approx M_1 \oplus \Lambda/(\Sigma);$$

and therefore, $[M] - [M_1] = [M] - [M] = 0$ in $\tilde{g}(M)$.

COROLLARY (Dyer-Sieradski [7]). *If $G = \mathbf{Z}/n$, then $h^{\mathrm{odd}}(G, l) = (0)$.*

Proof. This is immediate from Lemma 1 and the above result.

As our last example we consider the case where there exists a truncated projective resolution of $\mathbf{Z}$ of length m and Euler characteristic $l-1$, ($\beta^m(G, l-1) \neq 0$).

PROPOSITION 1. *$\beta^m(G, l-1) \neq \phi$ if and only if there exists $P_* \in \beta^m(G, l)$ with $M = H_m(\mathbf{P}_*) \simeq \bar{M} \oplus \Lambda$ for some $\bar{M}$.*

Note. The proposition is also true if we use $\mathbf{F}^m(G, l)$ instead of $\beta^m(G, l)$.

Proof. If $\bar{\mathbf{P}}_* \in \beta^m(G, l-1)$, let $P_j = \bar{P}_j$, $j < m$, $P_m = \bar{P}_m \oplus \Lambda$ and $\partial_m = (\partial_m, 0)$, then $\mathbf{P}_* \in \beta^m(G, l)$ and $H_m(\mathbf{P}_*) = H_m(\bar{P}_*) \oplus \Lambda$. Conversely if $\mathbf{P}_*$ exists, then

$$0 \longrightarrow M \oplus \Lambda \xrightarrow{k} P_m \xrightarrow{\delta_m} B_m \longrightarrow 0$$

is exact. Consider

$$(*) \qquad 0 \to \Lambda \xrightarrow{j} P_m \to P_m/\Lambda \to 0 \quad \text{where } j = k \circ i.$$

Since $0 \to M \to P_m/\Lambda \to B_m \to 0$ is exact, P_m/Λ is $\mathbf{Z}$ torsion free and therefore $(*)$ splits, since Λ is weakly injective. Hence P_m/Λ is projective (stably free if P_m is) and

$$\mathbf{P}_*\colon M \to P_m/\Lambda \to P_{m-1} \to \cdots \to \mathbf{Z} \to 0$$

has $\bar{\mathbf{P}}_* \in \beta^m(G, l-1)$.

LEMMA 2. *Suppose $H_m(\mathbf{P}_*) = M \oplus \Lambda$. Then*

$$K_1(\mathrm{End}\,(H_m \mathbf{P}_*)) \xrightarrow{\det} (\mathbf{Z}/|G|)^*$$

is onto.

Proof. If

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \xi \end{pmatrix} \in \mathrm{Aut}_\Lambda (M \oplus \Lambda)$$

then, since

$$\begin{tikzcd} \mathrm{Ext}^{m+1} (\mathbf{Z}, M) \oplus \mathrm{Ext}^{m+1} (\mathbf{Z}, \Lambda) \arrow[r, phantom, "\approx"] \arrow[d, "\begin{pmatrix} \alpha_* & \beta_* \\ \gamma_* & \delta_* \end{pmatrix}"] & \mathrm{Ext}^{m+1} (\mathbf{Z}, M \oplus \Lambda) \arrow[d, "\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_*"] \\ \mathrm{Ext}^{m+1} (\mathbf{Z}, M) \oplus \mathrm{Ext}^{m+1}(\mathbf{Z}, \Lambda) \arrow[r, phantom, "\approx"] & \mathrm{Ext}^{m+1}(\mathbf{Z}, M \oplus \Lambda) \end{tikzcd}$$

commutes and $\mathrm{Ext}_\Lambda^{m+1} (\mathbf{Z}, \Lambda) = (0)$, we have

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_* = \det \alpha_*.$$

Let $[r] \in (\mathbf{Z}/|G|)^*$, $(r, |G|) = 1$. Choose $a, b \in \mathbf{Z}$ such that $ar + s|G| = 1$, then

$$\begin{pmatrix} r & -b \\ |G| & a \end{pmatrix} \in \mathrm{Aut}_\Lambda (M \oplus \Lambda) \quad \text{and} \quad \det \begin{pmatrix} r & -b \\ |G| & a \end{pmatrix} = \det r = [r]. \quad \blacksquare$$

THEOREM 3. (i) *If $\beta^m(G, l-1) \neq \phi$, then $cl^{m+1}(G, l) \simeq \tilde{g}(M)$.*
(ii) *If $F^m(G, l-1) \neq \phi$, then $h^{m+1}(G, l) = (0)$.*

Proof. (i) By Proposition 1, there exists

$$\mathbf{P}_* \in \beta^m(G, l) \quad \text{with } H_m(\mathbf{P}_*) = \bar{M} \oplus \Lambda.$$

From Lemma 2, det is onto and hence $cl^{m+1}(G, l) \simeq \tilde{g}(\bar{M} \oplus \Lambda)$.

(ii) We may assume $M = \bar{M} \oplus \Lambda$. If $[M] - [M_1] \in F_M$ then

$$M \oplus \Lambda \simeq M_1 \oplus \Lambda.$$

Since $\Lambda | M$ we may conclude $[M] - [M_1] = [\Lambda] - [\Lambda] = 0$. Hence $F_M = 0$ and so $\ker \bar{t} = (0)$. This gives (ii).

In all the examples computed so far, it turns out that $\ker \bar{t} = (0)$. So we end this paper with the following question: Is $h^{m+1}(G, l)$ always a subquotient of $(\mathbf{Z}/|G|)^*$ or more precisely is $\ker \bar{t}$ always equal to zero?

REFERENCES

1. H. BASS, *Algebraic K-Theory*, Benjamin, New York, 1968.
2. W. BROWNING, *Homotopy types of certain finite CW complexes with finite fundamental group*, Ph.D. thesis, Cornell University, Ithaca, New York, 1979.
3. ———, *Pointed lattices over finite groups*, Forschungsinstitut für Mathematik, ETH, Zürich, 1979.
4. ———, *Truncated projective resolutions over a finite group*, Forschungsinstitut für Mathematik, ETH, Zürich, 1979.

5. ———, *Finite CW-complexes of cohomological dimension* 2 *with finite abelian* π_1, Forschungsinstitut für Mathematik, ETH, Zürich, 1979.
6. W. H. COCKCROFT and R. G. SWAN, *On the homotopy type of certain two dimensional complexes*, Proc. London Math. Soc. (3), vol. 41 (1961), pp. 194–202.
7. M. L. DYER and A. SIERADSKI, *Trees of homotopy types of* 2 *dimensional CW-complexes*, Comment. Math. Helv., vol. 48 (1973), pp. 31–44.
8. H. JACOBINSKI, *Genera and direct decomposition of lattices over orders*, Acta Math., vol. 121 (1968), pp. 1–29.
9. S. MACLANE, *Homology*, Academic Press, New York, 1963.
10. J. MARTINET, *Modules sur l'algèbre du groupe quaternionien*, Ann. Sci. École Norm. Sup. 4^e serie, t.4 (1971), pp. 399–409.
11. W. METZLER, *Über den Homotopietyp Zweidimensionaler CW-Komplexes und Elementar-transformationen bei definierende Relatumen*, J. Reine Angew. Math., vol. 285 (1976), pp. 7–23.
12. I. REINER, *Maximal orders*, Academic Press, New York, 1975.
13. I. REINER and K. W. ROGGENKAMP, *Integral representations*, Lecture Notes in Math. No. 744, Springer-Verlag, New York, 1979.
14. K. W. ROGGENKAMP, *Lattices over Orders II*, Lecture Notes in Math. No. 147, Springer-Verlag, New York, 1970.
15. A. V. ROITER, *Categories with division and integral representations*, Dokl. Akad. Nauk. SSSR, vol. 153 (1963), pp. 46–48.
16. R. G. SWAN and E. G. EVANS, *K theory of finite groups and orders*, Lecture Notes in Math. No. 149, Springer-Verlag, New York, 1970.
17. R. G. SWAN, *Periodic resolutions for finite groups*, Ann. of Math., vol. 72 (1960), pp. 267–291.
18. C. T. C. WALL, *Finiteness conditions for CW complexes, II*, Proc. Roy. Soc. Ser. A, vol. 295 (1966), pp. 129–139.
19. J. H. C. WHITEHEAD and S. MACLANE, *On the* 3 *type of a complex*, Proc. Nat. Acad. Sci. U.S.A., vol. 36 (1950), pp. 41–48.
20. J. S. WILLIAMS, *Free presentations and relation modules of finite groups*, J. Pure Appl. Algebra, vol. 3 (1973), pp. 203–217.

UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND
ÅRHUS UNIVERSITET
ÅRHUS, DENMARK