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MEROMORPHIC FUNCTIONS COVERING CERTAIN FINITE SETS AT THE SAME POINTS

BY

FRED GROSS AND CHUNG-CHUN YANG

1. Introduction

We suppose that two meromorphic functions f_1 and f_2 assume certain values at the same points (regardless of multiplicity). What can be said about the relationship between f_1 and f_2? In certain instances a great deal is known about the functions in question. For instance, Nevanlinna [4] showed that if the equations $f_i - a_j = 0$ have the same roots (regardless of multiplicity), for $i = 1, 2$ for each fixed j, where $a_1, a_2, \ldots, a_5$ are five distinct values (possibly including ∞), then $f_1 \equiv f_2$ or both f_1 and f_2 are constants. For rational functions only four distinct values will assure $f_1 \equiv f_2$ if f_1 and f_2 both are non-constant. In [2], the first author of the present paper studied a pair of meromorphic functions f and g such that $f(z_\alpha) \in S$ iff $g(z_\alpha) \in S$ for certain sets S of complex numbers and derived the corresponding relationships between f and g. In particular, he proved:

THEOREM A. *Let S_i $(i = 1, 2, 3)$ be distinct finite sets of complex numbers such that no set is equal to the union of the other two, and let T_i $(i = 1, 2, 3)$ be any finite sets of complex numbers having the same number of elements as S_i $(i = 1, 2, 3)$ respectively. Let $f(z)$ and $g(z)$ be two meromorphic functions satisfying the conditions*

$$E_f(S_i) = E_g(T_i), \quad i = 1, 2, 3,$$

and $E_{1/f}(\{0\}) = E_{1/g}(\{0\})$, where $E_h(S) = \bigcup_{a \in S}\{\xi \mid h(\xi) - a = 0\}$; a zero of multiplicity m being included m times. Then $f(z)$ and $g(z)$ are algebraically dependent.

With additional information about the sets S_i $(i = 1, 2, 3)$ several precise relationships between f and g were obtained in [2]. Here we also refer the reader to the book [1] for the above results and some other related topics. In this note, as a continuation of the previous studies, we characterize some of the sets S_i which have the property that any two meromorphic functions having the same preimage sets at those sets S_i must be identical. It is assumed that the reader is familiar with the definitions and basic properties of the quantities $T(r, f)$, $N(r, f)$ etc., employed in the Nevanlinna's value-distribution theory of meromorphic functions.

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II. Main results

THEOREM B. *For any positive integer $n > 1$ (resp. $n > 2$) there exists 4 (resp. 3) sets S_i, $i = 1, 2, 3, 4$, (resp. $i = 1, 2, 3$) each with n distinct elements such that any two non-constant meromorphic functions f_1 and f_2 satisfying $f_1(z_\alpha) \in S_i$ iff $f_2(z_\alpha) \in S_i$ for $i = 1, 2, 3, 4$ (resp. $i = 1, 2, 3$) must be identical.*

Proof. We may assume without loss of generality that f_1 and f_2 are transcendental. For otherwise, simply replace f_1, f_2 by $g(z)f_1(z)$ and $g(z)f_2(z)$ where g is some suitable transcendental entire function. For given distinct finite values $a_1, a_2, \ldots, a_n$, let $S = \{a_1, a_2, \ldots, a_n\}$. Let n_1 $(r, a_1, a_2, \ldots, a_n)$ be the number of elements in

$$\{z: |z| \leq r, \quad f_1(z) \in S \quad \text{and} \quad f_2(z) \in S\},$$

each z counted only once.

Let

$$N_1(r, a_1, a_2, \ldots, a_n) = \int_0^r \frac{n_1(t, a_1, \ldots, a_n) - n_1(0, a_1, \ldots, a_n)}{t} dt + n_1(0, a_1, \ldots, a_n) \log r.$$

Let

$$\begin{aligned} N_{12}(r, a_1, \ldots, a_n) = {} & N_1\left(r, \frac{1}{f_1 - a_1}\right) + N_1\left(r, \frac{1}{f_2 - a_1}\right) \\ & + N_1\left(r, \frac{1}{f_1 - a_2}\right) + N_1\left(r, \frac{1}{f_2 - a_2}\right) + \cdots \\ & + N_1\left(r, \frac{1}{f_1 - a_n}\right) + N_1\left(r, \frac{1}{f_2 - a_n}\right) \end{aligned}$$

$$N_1\left(r, \frac{1}{f - a}\right) = \int_0^r \frac{n_1(t, a) - n_1(0, a)}{t} dt + n_1(0, a) \log r,$$

where $n_1(t, a)$ is the number of distinct roots of $f(z) - a = 0$ in $|z| \leq t$.

Now let us consider the sets

$$S_i = \{a_i, a_i + b, a_i + 2b, \ldots, a_i + (n - 1)b\},$$

where b is some fixed non-zero integer and $i = 1, 2, 3, \ldots, l$. Furthermore b may be chosen so that $S_{j_1} \cap S_{j_2} = \phi$ for $j_1 \neq j_2$.

By Nevanlinna's second fundamental theorem, for any k distinct numbers,

$$\begin{aligned}(k-2)[T(r,f_1)+T(r,f_2)] \leq \sum_{j=1}^{k}\Bigg[N_1\left(r,\frac{1}{f_1-b_j}\right) \\ + N_1\left(r,\frac{1}{f_2-b_j}\right)\Bigg] \\ + O(\log (rT(r,f_1)T(r,f_2))\end{aligned} \tag{1}$$

for all r outside a set of finite measure. With the points b_j taken to be the elements in S_i, $i = 1, 2, \ldots, l$, we have

$$\begin{aligned}&N_1(r, a_1, a_1+b, a_1+2b, \ldots, a_1+(n-1)b) + \cdots \\ &\quad + N_1(r, a_l, a_l+b, \ldots, a_l+(n-1)b) \\ &\leq N\left(r,\frac{1}{f_1-f_2}\right) + N\left(r,\frac{1}{f_1-f_2-b}\right) + N\left(r,\frac{1}{f_1-f_2-2b}\right) \\ &\quad + \cdots + N\left(r,\frac{1}{f_1-f_2-(n-1)b}\right) \\ &\leq n(T(r,f_1)+T(r,f_2)) + O(1).\end{aligned} \tag{2}$$

Now, if $f_1(z_\alpha) \in S_i$ iff $f_2(z_\alpha) \in S_i$ for $i = 1, 2, \ldots, l$, then, with $k = ln$,

$$\begin{aligned}(k-2)[T(r,f_1)+T(r,f_2)] &\leq \sum_{i=1}^{l} N_{12}(r, a_i, a_i+b, \ldots, a_i+(n-1)b) \\ &\quad + O(\log rT(r,f_1)T(r,f_2)). \\ &= 2\sum_{i=1}^{l} N_1(r, a_i, a_i+b, \ldots, a_i+(n-1)b) \\ &\quad + O(\log rT(r,f_1)T(r,f_2)).\end{aligned} \tag{3}$$

Thus, from (2) and (3), we have

$$(ln-2-2n)[T(r, f_1)+T(r, f_2)] \leq O(\log rT(r, f_1)T(r, f_2)) \tag{4}$$

for all r outside a set of finite measure. Hence

$$ln - 2 - 2n \leq 0. \tag{5}$$

Thus, for l sets with

$$l > \frac{2(n+1)}{n} = 2 + \frac{2}{n},$$

we must have $f_1 \equiv f_2 + c$ for some constant c. Since each S_i is finite, it follows that c must be zero. This also completes the proof of Theorem B.

For $n = 2$, the three sets $(a_1, a_1 + b)$, $(a_2, a_2 + b)$, $(a_3, a_3 + b)$ do not assure that $f_1 \equiv f_2$, so that for sets of this form the number 4 in Theorem B is sharp. We illustrate this with the example: let $g(z)$ be any non-constant meromorphic function and set

$$f(z) = \frac{g(z) + 3}{g(z) - 1}$$

with $S_1 = (-1 - 2\sqrt{2},\ 3 - 2\sqrt{2})$, $S_2 = (-1, 3)$, and $S_3 = (-1 + 2\sqrt{2},\ 3 + 2\sqrt{2})$.

It is reasonable to ask: Can one exhibit two pairs $S_1 = (a_1, a_2)$, $S_2 = (b_1, b_2)$ such that $f_1(z_\alpha) \in S_i$ iff $f_2(z_\alpha) \in S_i$ implies $f_1 \equiv f_2$ for any two non-constant entire functions? In this connection, we present the following result for the class of entire functions of finite order.

THEOREM C. *Let $S_1 = (a_1, a_2)$, $S_2 = (b_1, b_2)$ be two pairs of distinct elements with $a_1 + a_2 = b_1 + b_2$ but $a_1 a_2 \neq b_1 b_2$. Suppose that there are two non-constant entire functions f and g of finite order such that $E_f(S_i) = E_g(S_i)$ for $i = 1$, 2. Then either $f \equiv g$ or $f + g \equiv a_1 + a_2$ or*

$$f(z) = \frac{c}{2} \pm \left[\frac{a_1 a_2 - b_1 b_2}{2} e^{-p}\right]^{1/2} \quad \textit{and} \quad g(z) = \frac{c}{2} \pm \left[\frac{a_1 a_2 - b_1 b_2}{2} e^{p}\right]^{1/2},$$

where $c = a_1 + a_2$ and $p(z)$ is a polynomial.

Proof. According to the assumptions, we have

$$(g - a_1)(g - a_2) = e^{p(z)}(f - a_1)(f - a_2) \tag{6}$$

and

$$(g - b_1)(g - b_2) = e^{q(z)}(f - b_1)(f - b_2), \tag{7}$$

where $p(z)$ and $q(z)$ are polynomials. The difference of the above two identities yields

$$\begin{aligned} a_1 a_2 - b_1 b_2 &= e^{p(z)}(f^2 - (a_1 + a_2)f + a_1 a_2) \\ &\quad - e^{q(z)}(f^2 - (b_1 + b_2)f + b_1 b_2) \\ &= f^2(e^{p(z)} - e^{q(z)}) + c(e^{q(z)} - e^{p(z)})f \\ &\quad + a_1 a_2 e^{p(z)} - b_1 b_2 e^{q(z)} \end{aligned} \tag{8}$$

where $c = a_1 + a_2 = b_1 + b_2$. Let $e^{p(z)} - e^{q(z)} \equiv E(z)$, then (8) becomes

$$E(z)f^2 - cE(z)f + a_1 a_2 e^{p(z)} - b_1 b_2 e^{q(z)} + (b_1 b_2 - a_1 a_2) \equiv 0. \tag{9}$$

Since f is entire, the discriminant of the above quadratic equation (in f) must be a complete square of an entire function unless $e^p - e^q \equiv 0$. Clearly, if $e^p \equiv e^q$, then from (8) it follows that $e^q \equiv e^p \equiv 1$. This case leads to the assertion right away.

Now suppose that $E(z) \not\equiv 0$, then the discriminant of (9) is

$$\begin{aligned} H^2(z) &\equiv [cE(z)]^2 - 4E(z)[a_1 a_2 e^{p(z)} - b_1 b_2 e^{q(z)} + (b_1 b_2 - a_1 a_2)] \\ &\equiv E(z)[c^2 E(z) - 4\{a_1 a_2 e^{p(z)} - b_1 b_2 e^{q(z)} + (b_1 b_2 - a_1 a_2)\}] \\ &\equiv E(z)[e^{p(z)}\{(c^2 - 4a_1 a_2) - (c^2 - 4b_1 b_2)e^{q(z)-p(z)}\} \\ &\quad - 4(b_1 b_2 - a_1 a_2)] \end{aligned} \tag{10}$$

where $H(z)$ is an entire function.

Let

$$K(z) \equiv c^2 E(z) - 4(a_1 a_2 e^{p(z)} - b_1 b_2 e^{q(z)} + b_1 b_2 - a_1 a_2).$$

Now we shall treat two cases separately: case (i) $\deg q < \deg p$, case (ii) $\deg q = \deg p$. We treat case (i) first. In case (i), if $q(z)$ is not a constant, then e^p would have both e^q and

$$[(c^2 - 4b_1 b_2)e^q - 4(a_1 a_2 - b_1 b_2)]/(c^2 - 4a_1 a_2)$$

as completely ramified deficient functions. We also note that these two deficient functions can not be identically the same. Thus, according to the extension of Nevanlinna's second fundamental theorem for deficient functions (see [3, p. 47] for example) we have a contradiction. Hence $q(z)$ must be a constant, say $q(z) \equiv q_0$ for some constant. Then the two constants e^{q_0} and

$$[(c^2 - 4b_1 b_2)e^{q_0} - 4(a_1 a_2 - b_1 b_2)]/(c^2 - 4a_1 a_2)$$

must be equal (since $e^p - A$, A a non-zero constant has all, except finitely many, of its zeros simple). It follows that $4(e^{q_0} - 1)(b_1 b_2 - a_1 a_2) = 0$. Since $b_1 b_2 \neq a_1 a_2$, we must have $e^{q_0} = 1$. Hence from (6) or (7) one derives $f \equiv g$ or $f + g \equiv c = a_1 + a_2$ as claimed. Now suppose that case (ii) holds. First of all, if both p and q are constants, then these two constants must differ by $2n\pi i$ for some integer n. Otherwise, by taking the difference of the two identities (6) and (7), one would derive $d_0 f^2 + d_1 f + d_2 \equiv 0$ for some constants d_o ($\neq 0$), d_1, and d_2. This is certainly impossible for any non-constant entire function f. Once p and q are constants and can differ only by $2n\pi i$, then again, by taking the difference of (6) and (7), we can conclude that $e^p = e^q \equiv 1$ and the assertions follow from this. Thus, we shall concentrate on the case that p, q both are non-constant with $\deg q = \deg p$. Two subcases may arise: subcase (I) $\deg (q - p) < \deg p$ and subcase (II) $\deg (q - p) = \deg p$. In subcase (I), suppose that $q - p \not\equiv$ constant (also note that $c^2 - 4b_1 b_2 \neq 0$), then we see from the last equation of (10) that both the functions e^{q-p} and

$$L(z) \equiv 4(b_1 b_2 - a_1 a_2)/[(c^2 - 4a_1 a_2) - (c^2 - 4b_1 b_2)e^{q-p}]$$

are completely ramified deficient functions of e^P. This is again impossible as we have argued in case (i). Now, suppose $q - p =$ constant. Set $t = e^{q-p}$. We only need to consider $t \neq 1$ for the case $t = 1$ has been settled in the very beginning of the proof. Thus

$$E(z) \equiv e^P(1 - e^{q-p}) \equiv e^p(1 - t)$$

is a zero free factor of $H^2(z)$. We shall treat two cases separately:

$$\text{case (a)} \quad c^2 - 4a_1a_2 - (c^2 - 4b_1 b_2)t = 0$$

and

$$\text{case (b)} \quad c^2 - 4a_1 a_2 - (c^2 - 4b_1 b_2)t \neq 0.$$

Suppose that case (a) holds. Then, by taking the quotient of identities of (6) and (7), one can derive easily that $t = -1$,

$$f(z) = \frac{c}{2} \pm \left[\left(\frac{a_1 a_2 - b_1 b_2}{2}\right) e^{-p(z)}\right]^{1/2}$$

and

$$g(z) = \frac{c}{2} \pm \left[\left(\frac{a_1 a_2 - b_1 b_2}{2}\right) e^{p(z)}\right]^{1/2}.$$

Now, suppose that case (b) holds; then

$$e^{P(z)}\{[(c^2 - 4a_1 a_2) - (c^2 - 4b_1 b_2)t]\} - 4(b_1 b_2 - a_1 a_2)$$

has to be a complete square of a certain entire function. Clearly, this is impossible, since

$$4(b_1 b_2 - a_1 a_2) \neq 0.$$

Finally, we need to treat subcase (II): deg $(q - p) =$ deg p. In this case, let z_0 be a root of $e^{q-p} - 1 = 0$, i.e., $e^{q(z_0)} = e^{p(z_0)}$. Then it follows from (8) that $e^{p(z_0)} = e^{q(z_0)} = 1$. Thus we have two non-constant polynomials $p(z)/2\pi i$ and $[q(z) - p(z)]/2\pi i$ of the same degree, and $p/2\pi i$ assumes an integral value whenever $[q - p]/2\pi i$ does. Then, according to a known result [5, p. 411],

$$p(z) \equiv t[q(z) - p(z)] + s2\pi i$$

for some integers t and s. Thus, $e^{p(z)} \equiv e^{t[(q(z) - p(z)]}$. Moreover, we may assume without loss of generality that t is a positive integer. Let

$$e^{q(z) - p(z)} \equiv F(z).$$

Then (10) becomes

$$\begin{aligned} H^2(z) &\equiv F^t(1 - F)[F^t(C_1 - C_2 F) + C_3] \\ &\equiv F^t(1 - F)(d_0 F^{t+1} + d_1 F^t + d_2), \end{aligned}$$

where d_0 ($\neq 0$), d_1, and d_2 ($\neq 0$) are constants. Since F^t is zero free, we require

$$(1 - F)(d_0 F^{t+1} + d_1 F^t + d_2)$$

to be a complete square of a certain entire function. Now the derivative (with respect to F) of the factor

$$M(F) \equiv d_0 F^{t+1} + d_1 F^t + d_2$$

is

$$(t+1)d_0 F^t + t d_1 F^{t-1} = F^{t-1}\{(t+1)d_o F + t d_1\}.$$

Since $d_2 \neq 0$, the only possible multiple root of the factor $M(F)$ arises from the factor

$$(t+1)d_0 F + t d_1,$$

which is a linear function. Thus $M(F)$ can have one multiple root with multiplicity equal to 2. There is one other factor of $M(F)$ which is linear, hence $M(F)$ is of degree 3 in F. It follows that $t = 2$. Furthermore, it can easily be shown that $d_0 + d_1 + d_2 = 0$ and $F - 1$ is a factor of M; consequently the multiple factor of M by a simple calculation is $(F + 2)^2$. Solving (9) for f we get

$$\begin{aligned} f &= \frac{cE(z) \pm \sqrt{H^2(z)}}{2E(z)} \\ &= \frac{c}{2} \pm \frac{(-d_0)^{1/2} F(F-1)(F+2)}{2E(z)} \\ &= \frac{c}{2} \pm \frac{(-d_0)^{1/2} F(F-1)(F+2)}{2F^2(1-F)} \\ &= \alpha_0 + \alpha_1 F^{-1} = \alpha_0 + \alpha_1 e^{-(q-p)} \end{aligned}$$

for some constants α_0 and α_1. Substituting this into (8) would lead to an impossibility. This completes the proof of the theorem.

Remarks. (i) The above argument is applicable to two meromorphic functions satisfying $E_g(\{\infty\}) = E_f(\{\infty\})$.

(ii) We suspect that the assertion remains valid for entire functions of arbitrary growth.

By means of a linear transformation one can generalize Theorem C. Let $S_1 = \{a, b\}$ and $S_2 = \{c, d\}$ be two disjoint pairs each with distinct elements. Let L denote a linear transformation on the plane such that $LS = \{Lx, Ly\}$, whenever $S = \{x, y\}$ x, y are points in the finite plane. We claim that it is always possible to find a transformation L: $L(z) = Az + B$ such that it maps a, b into $-a_1$, $a_1 + c_0$ $(c_0 \neq 0)$ respectively and c, d into $-c_1$, $c_1 + c_0$ respectively for some a_1 and c_1 and given c_0. For the equations

$$Aa + B = -a_1, \quad Ab + B = a_1 + c_0$$

$$Ac + B = -c_1, \quad Ad + B = c_1 + c_0$$

have a solution for A, B, a_1, and c_1 provided that the determinant

$$D = \begin{vmatrix} a & 1 & 1 & 0 \\ b & 1 & -1 & 0 \\ c & 1 & 0 & 1 \\ d & 1 & 0 & -1 \end{vmatrix} \neq 0.$$

Now,

$$\begin{aligned} D &= -\begin{vmatrix} a & 1 & 1 \\ b & 1 & -1 \\ c & 1 & 0 \end{vmatrix} - \begin{vmatrix} a & 1 & 1 \\ b & 1 & -1 \\ d & 1 & 0 \end{vmatrix} \\ &= -[(b-c) + (a-c) + (b-d) + (a-d)] \\ &= -2(b+a) + 2(c+d) \end{aligned}$$

which is not equal to zero as long as $a + b \neq c + d$.

One can easily verify that Theorem C can be stated in a more general form as follows.

THEOREM D. *Let $S_1 = \{a_1, a_2\}$, $S_2 = \{b_1, b_2\}$ be any two disjoint pairs of complex numbers with $a_1 a_2 \neq b_1 b_2$. Suppose that there are two non-constant entire functions f and g of finite order such that $E_f(S_i) = E_g(S_i)$ for $i = 1, 2$. Then either $f(z) \equiv Ag(z) + B$ for some constants A, B or*

$$f(z) = c_1 + c_2 e^{p(z)} \quad \textit{and} \quad g(z) = c_1 + c_2 e^{-p(z)}$$

for some polynomial $p(z)$, and constants, c_1 and c_2.

Remark. If $a_1 + a_2 \neq b_1 + b_2$, and $f(z) \equiv Ag(z) + B$ for some constants A and B, then an elementary analysis will show it is necessarily that $f \equiv g$.

Finally Theorem B can be generalized as follows.

THEOREM E. *Let T be a set with $n - 1$ distinct numbers and let $S_1, S_2, \ldots, S_l$ be pairwise disjoint finite sets of distinct complex numbers such that the difference between the numbers in each of the sets S_i $(i = 1, 2, \ldots, l)$ is in T. Let m_i be the number of elements in S_i. If f_1 and f_2 are any two non-constant meromorphic functions such that*

$$f_1(z_\alpha) \in S_i \quad \textit{iff} \quad f_2(z_\alpha) \in S_i \quad \textit{for } i = 1, 2, \ldots, l$$

and if

$$m_1 + m_2 + \cdots + m_l > 2(n+1),$$

then $f_1 \equiv f_2$.

Proof. Arguing as in Theorem B, we obtain

$$\begin{aligned} &N_1(r, a_{11}, a_{12}, \ldots, a_{1m_l}) + \cdots + N_1(r, a_{l1}, a_{l2}, \ldots, a_{lm_l}) \\ &\quad \le N\left(r, \frac{1}{f_1 - f_2}\right) + N\left(r, \frac{1}{(f_1 - f_2) - A_1}\right) + \cdots \\ &\quad + N\left(r, \frac{1}{(f_1 - f_2) - A_{n-1}}\right), \\ &\quad < n[T(r, f_1) + T(r, f_2)] + O(1). \end{aligned}$$

Now if $f_1(z_\alpha) \in S_i$ iff $f_2(z_\alpha) \in S_i$ for $i = 1, 2, \ldots, l$ then with $k = m_1 + m_2, \ldots + m_l$,

$$\begin{aligned} (k-2)[T(r, f_1) + T(r, f_2)] &\le \sum_{i=1}^{l} N_{12}(r, a_{i1}, a_{i2}, \ldots, a_{im_i}) \\ &\quad + O(\log rT(r, f_1)T(r, f_2)) \\ &= 2\sum_{i=1}^{l} N_1(r, a_{i1}, a_{i2}, \ldots, a_{im_i}) \\ &\quad + O(\log rT(r, f_1)T(r, f_2)). \end{aligned} \tag{11}$$

Thus, from (11), we have

$$\begin{aligned} &(m_1 + m_2 + \cdots + m_l - 2 - 2n)[T(r, f_1) + T(r, f_2)] \\ &\qquad \le O(\log rT(r, f_1)T(r, f_2)). \end{aligned} \tag{12}$$

Since we may assume that f_1 and f_2 are transcendental, thus (12) is impossible to hold if

$$m_1 + m_2 + \cdots + m_l - 2 - 2n > 0 \quad \text{or} \quad m_1 + m_2 + \cdots + m_l > 2(n+1)$$

unless $f_1 - f_2 \equiv c$, c a constant. Again c has to be zero. Theorem E is thus proven.

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UNIVERSITY OF MARYLAND, BALTIMORE COUNTY
CATONSVILLE, MARYLAND
NAVAL RESEARCH LABORATORY
WASHINGTON, D.C.