ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 3, Fall 1982

ON SEMITOPOLOGICAL COMPACTIFICATIONS OF NON-ABELIAN GROUPS

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0. Introduction

Let G be a topological (or more generally semitopological) group. A compact semitopological semigroup S, together with a continuous homomorphism $\phi\colon G \to S$ is called a semitopological compactification of G if $\phi(G)$ is dense in S. Such compactifications appear as a natural object of study in harmonic analysis; in fact, the universal semitopological compactification, the weak almost periodic compactification, has been studied rather extensively (cf. Eberlein [7], de Leeuw–Glicksberg [9], [15], Berglund--Hofmann [1], Taylor [22], Burckel [5], just to mention a few.) However it seems that most attention has been given to properties of the function space associated with S, not to the structure of S itself. Comparatively few general theorems about the structure of S have been established; as the most important results we cite the following.

(i) The minimal ideal of S exists and is a compact topological group [21].
(ii) The multiplication of S is jointly continuous at all points (s, g) and (g, s), where $s \in S$ and $g \in \phi(G)$ [13].

Given the structure theory of compact semitopological semigroups, statement (i) is equivalent to:

(i′) The idempotents in the minimal ideal of S are central in S.

The first and central result of this paper can be considered as an extension of statement (i′): we show that if the group G is connected and locally compact then all idempotents in S are central. This result and the techniques of its proof give birth to various other structure theorems, which in some cases allow a very detailed description of the weak almost periodic compactification of a non-abelian connected locally compact group in terms of semitopological compactifications of abelian ones. In particular, we show that the remainder of the weak almost periodic compactification of the affine group $(\mathbf{R}^+, \cdot)$ Ⓢ $(\mathbf{R}, +)$ is isomorphic to the weak almost periodic compactification of the reals; this example also contradicts an erroneous statement by Burckel [5], that in the

Received August 20, 1980.

weak almost periodic compactification of a locally compact group G the closure of a normal subgroup H is isomorphic to the weak almost periodic compactification of H. (It is shown in the appendix that this statement becomes true if H is supposed to be central in G.) Furthermore, we deduce a theorem on the structure of the weak almost periodic compactification of a connected simple Lie group, which as a special case includes the recent result of Veech [23] that the weak almost periodic compactification of a non-compact connected simple Lie group with finite centrum must be a one-point compactification. Thus, the results of this paper provide further evidence for the general feeling that the structural complexity of the weak almost periodic compactification of a locally compact and connected group is mostly due to "abelian" features of the group; definitely non-abelian (and non-compact) groups are too "rigid" to be bent and plunged into "sophisticated" compact semitopological semigroups. (This philosophy does not apply to non-connected groups; for discrete groups the above statements are false (cf. 2.19).)

1. Preliminaries

1.1. In dealing with algebraic properties of semigroups we use the notation and terminology of Clifford–Preston [6], if not explicitly stated otherwise. Multiplication is usually denoted by simple juxtaposition; if A, B are subsets of a semigroup S then we write AB for the set $\{ab \mid a \in A,\ b \in B\}$; similarly $sA = \{sa \mid a \in A\}$ and $As = \{as \mid a \in A\}$ for any element $s \in S$.

If e, f are idempotents in a semigroup then we write $e \leq_L f$ if $ef = e$ and $e \leq_R f$ if $fe = e$; we write $e \leq f$ if $ef = fe = e$. Note that the relation "$\leq$" is a partial order whereas "$\leq_L$" and "$\leq_R$" are in general only pre-orders (i.e., they are reflexive and transitive but not necessarily antisymmetric). The set of all idempotents in a semigroup S is denoted with $E(S)$. The maximal subgroup $H(e)$ belonging to an idempotent e in a semigroup S is the set

$$\{x \in eSe \mid xeSe = eSex = eSe\},$$

i.e., the set of all elements in eSe which have an inverse with respect to e.

The identity of a semigroup—if it exists—is usually denoted with 1, without reference to the semigroup considered. The maximal subgroup belonging to 1 is called the group of units.

1.2. A semigroup is called *semitopological* if it is endowed with a Hausdorff topology rendering the left and right translations $x \to ax$ and $x \to xa$ continuous. If a semitopological semigroup is algebraically a group, lattice etc., then we speak of a semitopological group, lattice etc.

1.3. If A, G, H, ... are Lie groups then the associated Lie algebras are denoted with the corresponding gothic letters $\mathfrak{A}$, $\mathfrak{G}$, $\mathfrak{H}$, ... Let G be a Lie group, $\mathfrak{G}$ its Lie algebra. Following the general custom we write $(\operatorname{ad} x)y = [x, y]$. The adjoint representation $G \to \operatorname{Aut} \mathfrak{G}$ induces an action $G \times \mathfrak{G} \to \mathfrak{G}$,

$(g, x) \to g \cdot x$ and we have $\exp(g \cdot x) = g(\exp x)g^{-1}$, for all $g \in G$, $x \in \mathfrak{G}$. If x is an element of $\mathfrak{G}$ then we write $(\mathrm{Ad}\, x)y$ instead of

$$\exp x \cdot y = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}\, x)^n y.$$

All Lie algebras are assumed to have been equipped with a Euclidean norm, such that the norm of a subalgebra $\mathfrak{H}$ of a Lie algebra $\mathfrak{G}$ is induced by the norm of $\mathfrak{G}$.

1.4. We shall need the following version of Iwaswa's decomposition theorem (cf. Helgason [10], p. 234).

THEOREM. *Let G be a connected semisimple Lie group. Then there are closed subgroups K and H of G such that*

(i) *the adjoint representation $G \to \mathrm{Aut}\, \mathfrak{G}$ maps K onto a compact group*;

(ii) *H is simply connected and solvable*;

(iii) *the map $K \times H \to G$, $(k, h) \to kh$, is a diffeomorphism (thus a fortiori a homeomorphism).*

1.5. If T is a semitopological semigroup, $\phi\colon T \to S$ a continuous homomorphism from T to a compact semitopological semigroup S, then the pair (S, ϕ), is called a *semitopological compactification* of T if $\phi(T)$ is dense in S. If no ambiguity can occur then we omit the reference to the compactification map ϕ and speak of the "semitopological compactification S of T". (Notice that if T contains an identity, then ϕ must be identity-preserving, by continuity.)

Clearly, every subsemigroup H of a compact semitopological semigroup gives rise to a canonical semitopological compactification $H \to \bar{H}$. Throughout this paper the letter S will always indicate a semitopological compactification of a group G.

1.6. If S is a semitopological compactification of a semitopological group G then the *left action* of G on S, $G \times S \to S$, is defined by

$$(g, s) \to g \cdot s = \phi(g)s,$$

the *right action* $S \times G \to S$ by

$$(s, g) \to s \cdot g = s\phi(g).$$

These actions are (jointly) continuous, by a theorem of Lawson [13] (see 1.8. below). (It is the philosophy of this paper to study semitopological compactifications of groups just by investigating the properties of these actions.) Since no confusion is to be feared, we frequently omit the dot, thus simply writing gs instead of $\phi(g)s$, sg instead of $s\phi(g)$. It is easy to see that for g, $h \in G$, $s \in S$, we always have $(g \cdot s) \cdot h = g \cdot (s \cdot h)$; therefore, expressions such as $g^{-1}sg$ or gsh, g, $h \in G$, $s \in S$, are not ambiguous.

1.7. Every compact semitopological semigroup contains a minimal ideal (see Berglund–Hofmann [1], p. 67 for example). For a semitopological compactification of a group, the structure of the minimal ideal is particularly simple. The following theorem essentially goes back to Ryll–Nardzewsky [21], (cf. Berglund–Hofmann [1], p. 142).

THEOREM: *The minimal ideal of a semitopological compactification of a group is a compact topological group.*

(Clearly, the identity of the minimal ideal is a minimal idempotent, with respect to each of the relations $\leq_L$, $\leq_R$, $\leq$.)

1.8. By a famous theorem of Ellis [8], a locally compact semitopological group is already a topological group. This result is contained in the more recent theorem of J. D. Lawson [13], [14], given below.

THEOREM. *Let X be a compact Hausdorff space, T a compact semitopological semigroup with identity* 1. *Suppose that π: $T \times X \to X$ satisfies the following conditions.*

(a) *π is an action of T on X: $\pi(st, x) = \pi(s, \pi(t, x))$ for all $x \in X$, $s \in S$.*
(b) *$\pi(1, x) = x$ for all x in X.*
(c) *π is separately continuous; i.e., the maps*

$$X \to X,\ x \to \pi(t_0, x) \quad \text{and} \quad T \to X,\ t \to \pi(t, x_0)$$

are continuous for all $t_0 \in T$, $x_0 \in X$.
Then π is (jointly) continuous at all points (g, x), where $g \in H(1)$ and $x \in X$. In particular, every subgroup of a compact semitopological semigroup is a topological group.

1.9. The study of semitopological compactifications of groups is facilitated by the existence of universal semitopological compactifications.

Let T be a semitopological semigroup. A bounded continuous function $f: T \to C$ is called *weakly almost periodic* if the translates $x \to f(tx)$, $t \in T$, of f form a weakly conditionally compact set. The multiplication of T extends canonically to the Gelfand space of the weakly almost periodic functions on T, thus inducing a semitopological compactification of T. This compactification is called the *weak almost periodic compactification* of T.

1.10. THEOREM. *The weak almost periodic compactification is universal.*

In other words, if T is a semitopological semigroup, (S, ϕ) its weak almost periodic compactification, then to every semitopological compactification there exists a continuous homomorphism γ rendering the diagram

$$\begin{tikzcd} T \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & S \arrow[d, "\gamma"] \\ & S' \end{tikzcd}$$

commutative. (cf. Berglund–Hofmann [1], p. 120 ff.) By this universality it is obvious that a property holds for all semitopological compactifications of a group if it holds for the weak almost periodic compactification and is preserved under continuous surjective homomorphisms.

1.11. If G is a locally compact topological group then the one-point compactification $G \cup \{\omega\}$ can be made into a compact semitopological semigroup; we only have to extend the multiplication of G by defining $\omega g = g\omega = \omega$. Thus if G is locally compact then it is imbedded homeomorphically onto an open subset of its weak almost periodic compactification and we may consider G as an open subset of its weak almost periodic compactification.

1.12. The following proposition is convenient for the study of semitopological compactifications of projective limits of Lie groups. The proof is left to the reader. (Hint: use Theorem 1.8.)

PROPOSITION. *Let S be a semitopological compactification of a topological group G. If N is a compact normal subgroup of G then $Ns = sN$ for all $s \in S$ (i.e., to every $n \in N$ there are $n', n'' \in N$ with $sn = n's$ and $ns = sn''$). The given multiplication of S induces a multiplication on the orbit space S/N, so that S/N becomes a compact semitopological semigroup. Also, the compactification map $\phi\colon G \to S$ induces a compactification map $\phi_{G/N}\colon G/N \to S/N$ and if ϕ is injective then so is $\phi_{G/N}$.*

We call $(S/N, \phi_{G/N})$ the quotient of (S, ϕ) by the compact normal subgroup N of G. This is not to be confused with the Rees quotient S/I, where I is a closed ideal (which is collapsed to a zero element by the quotient map).

1.13. We conclude this section with a remark on the idempotents in a compact semitopological semigroup.

PROPOSITION. *Let T be a compact semitopological semigroup with identity and suppose that its idempotents commute with each other. Then the idempotents of T form a complete lattice with respect to the ordering $\leq$, introduced in* 1.1.

Proof. Since the idempotents of T commute, they form a subsemigroup $E(T)$ of T. If A is a set of idempotents then the set

$$B = \{e \in T \mid e^2 = e,\ f = ef \text{ for all } f \in A\}$$

is closed in $E(T)$ (not necessarily in T). The set B is not empty since it contains 1. By Lemma 2.10 of [20], B must contain a minimal element a. Clearly $aa' \in E(T)$ and $aa' \leq a$ for all idempotents a' in A, so a is unique, $a = \sup A$. In the same way it follows that $\inf A$ exists, which finishes the proof.

2. The main theorem

2.1. In this section we prove the central result of this paper which was announced in the introduction. For the sake of technical simplicity it is convenient to prove a slightly stronger result.

THEOREM. *Let G be a locally compact topological group, S a semitopological compactification of G. Furthermore, suppose that the identity component G_0 of G is open in G and that there is a discrete central subgroup D with $G = G_0 D$. Then the following statements hold:*

(i) *All idempotents of S are central. (Thus $E(S)$ is a complete lattice, by* 1.13.)

(ii) *If, in addition, G_0 is solvable then $sG = Gs$ for all s in S (i.e., for all $s \in S$, $g \in G$ there are elements $g', g'' \in G$ with $sg = g's$, $gs = sg''$).*

2.2. Actually, (ii) is shown mainly to help prove the more important (i). The reason for admitting non-connected groups G will become clear later when we use an Iwasawa-type decomposition (2.14.) of G to establish (i) by applying a special case of (ii). Note that in (ii), $Gs = sG$ implies $sS = Ss$ for all s in S (by the separate continuity of the multiplication), so every left and every right ideal of S is a two-sided ideal of S. In terms of semigroup theory, this means that S is "nearly abelian" in the sense that all of Green's relations coincide. Note also that for an idempotent e, the relation $eS = Se$ implies that e is central.

At the end of this section it will be shown that for a rather large class of discrete groups the statements of Theorem 2.1 are not true.

2.3. To prove Theorem 2.1, we first show that we may assume that G_0 is a connected Lie group such that every compact normal subgroup is trivial. Since the connected locally compact topological group G_0 contains a compact normal subgroup N such that G_0/N is a Lie group, this reduction is contained in the lemma below. (Note that every normal subgroup of G_0 is also normal in G.)

LEMMA. *Let N be a compact normal subgroup of G and write $\kappa: S \to S/N$ for the canonical homomorphism between S and the quotient semigroup S/N. Then for an element s in S we have $sG = Gs$ if and only if $\kappa(sG) = \kappa(Gs)$. In particular, all idempotents in S are central in S if all idempotents in S/N are central in S/N.*

Proof. Since N is compact and normal in S we have $sN = Ns$ for all s in S (1.12); hence $\kappa(sG) = \kappa(Gs)$ implies $sG = GsN = NGs = Gs$.

2.4. By the remark in 2.3 we may assume that the group G is a Lie group and, since the properties (i) and (ii) of Theorem 2.1 are preserved under continuous surjective homomorphisms, that S is the weak almost periodic compactification of G. If G contains a compact normal subgroup of positive dimension and if Theorem 2.1 has been shown already for all groups of dimension less than dim G, then 2.3 implies that it also holds for G. Obviously, the theorem is trivial for dim $G = 0$, since then G (and therefore S) is abelian. Thus the following general assumptions will imply no loss of generality.

GENERAL ASSUMPTION. *Throughout the rest of this section we suppose that the following hold:*

(i) *G is a Lie group of dimension $n \geq 1$, such that $G = G_0 D$, where G_0 is the component of the identity in G and D is a discrete central subgroup.*

(ii) *S is the weak almost periodic compactification of G and G is considered as an open subgroup of S* (this is possible by 1.11).

(iii) *Theorem 2.1 is true for all groups with dimension less than n.*

2.5. We next observe some useful facts concerning the isotropy groups of the left or right action of G on S. For the sake of simplicity we henceforth use the following notation for left, right isotropy groups and their Lie algebras.

Notation. If s is an element of S then we define

$$F_L(s) = \{g \in G \mid gs = s\}, \qquad F_R(s) = \{g \in G \mid sg = s\};$$

the corresponding subalgebras of the Lie algebra $\mathfrak{G}$ of G are denoted with $\mathfrak{F}_L(s)$ and $\mathfrak{F}_R(s)$ respectively.

2.6. The general induction hypothesis in 2.4. allows the following reduction of the problem:

LEMMA. (i) *If e is an idempotent in S such that $\mathfrak{F}_R(e)$ contains a non-trivial ideal $\mathfrak{N}$ of $\mathfrak{G}$ then e is central in S.*

(ii) *If s is an element of S such that $\mathfrak{F}_R(s)$ contains a non-trivial ideal $\mathfrak{N}$ of $\mathfrak{G}$ and $\mathfrak{G}$ is solvable then $sG = Gs$.*

Proof. Write N for the closure in G of the normal subgroup of G which corresponds to $\mathfrak{N}$. Clearly, G/N also satisfies the assumption on G of Theorem 2.1. The closure $\bar{N}$ of N in S is a compact semitopological semigroup, its minimal ideal is a compact group contained in the maximal subgroup of some idempotent f. By continuity, $g^{-1}\bar{N}g = \bar{N}$ and therefore $g^{-1}fg = f$ for all $g \in G$ (every automorphism of $\bar{N}$ must leave invariant the minimal idempotent f). Now, $gf = fg$ for all $g \in G$ implies $sf = fs$ for all $s \in S$, so the map $S \to fS$, $s \to fs$ is a homomorphism. The minimal ideal $f\bar{N}$ of $\bar{N}$ is a compact normal subgroup of $H(f)$ and the quotient semigroup $fS/f\bar{N}$ is a semitopological compactification of the group G/N. Now dim $G/N <$ dim G, so by our general induction hypothesis 2.4, all idempotents in $fS/f\bar{N}$ are central in $fS/f\bar{N}$, hence the same is true in fS, by Lemma 2.3. It follows that $et = eft = ftef = te$ for all $t \in S$, which proves (i). The proof of (ii) is completely analoguous and therefore left to the reader.

2.7. The next lemma shows that the reduction indicated by Lemma 2.6. can be applied if e is an idempotent of S commuting with all of its conjugates $g^{-1}eg$, $g \in G$, and such that dim $F_R(e) > 0$.

LEMMA. *Let e be an idempotent in S with $eg^{-1}eg = g^{-1}ege$ for all $g \in G$. Then the identity component of $F_R(e)$ is a normal subgroup of G; i.e.,*

$$(F_R(e))_0 = (F_R(g^{-1}eg))_0 = g^{-1}(F_R(e))_0 g \quad \text{for all } g \in G.$$

Proof. We first recall the following proposition (cf. Ruppert [18], Proposition 2.3.; compare also the well-known results in Montgomery–Zippin [17], p. 241).

2.8. PROPOSITION. *Let G be a Lie group, Y a regular Hausdorff space and $Y \times G \to Y$, $(y, g) \to y \cdot g$, a jointly continuous action. Furthermore, let H be a closed connected subgroup of G,*

$$X = \{x \in Y \mid F(x) = \{g \in G \mid x \cdot g = x\} \supset H\}.$$

Then the set $\{x \in X \mid H$ is open in $F(x)\}$ is open in X (where X is provided with the topology inherited from Y).

Proof. Let $\mathfrak{G}$ be the Lie algebra of G, $\mathfrak{H}$ the Lie algebra of H. As a vector space, $\mathfrak{G}$ is the direct sum $\mathfrak{H} \oplus \mathfrak{H}'$ of $\mathfrak{H}$ with some complementary vector space $\mathfrak{H}'$. If ε is chosen sufficiently small, then, with

$$U_1 = \{h \in \mathfrak{H} \mid \|h\| < \varepsilon\}, \quad U_2 = \{h \in \mathfrak{H}' \mid \|h\| < \varepsilon\},$$

the map $U_1 \times U_2 \to G$, $(u, v) \to \exp u \cdot \exp v$, will be a homeomorphism onto an open neighborhood U of the identity 1 of G. Let $x \in X$ such that the component of the identity of $F(x)$ coincides with H. We may assume $U \cap F(x) = \exp U_1$. Define

$$A = \exp \{h \in \mathfrak{H}' \mid \varepsilon/4 \leq \|h\| \leq \varepsilon/2\}.$$

A is compact and $x \notin xA$, so we can find an open neighborhood V of x in X with $V \cap V \cdot A = \emptyset$. Let $v \in V$ and suppose $v \cdot u = v$ for some $u = \exp u_1 \exp u_2$, $u_1 \in U_1$, $\|u_2\| < \varepsilon/4$. Then, by assumption, $v \in X$ implies $v \cdot \exp u_1 = v$, hence $v \cdot u = v \cdot \exp u_2 = v$. Thus $v \cdot \exp u_2 = v$ implies $v \cdot \exp(nu_2) = v$ for all natural numbers n. Suppose $u_2 \neq 0$. Choosing

$$n = [\varepsilon \cdot 4^{-1} \|u_2\|^{-1}] + 1,$$

we get $\varepsilon/4 < \|n \cdot u_2\| < \varepsilon/2$ and $\exp(n \cdot u_2) \in A$, a contradiction to $V \cap V \cdot A = \emptyset$. Thus $u_2 = 0$, $u \in H$. The assertion follows.

2.9. *Proof of* 2.7 (*continued*). Applying the above proposition we can find a neighborhood U of e such that the identity components of $F_R(e)$ and $F_R(u)$ coincide for all u in $U \cap Se$. Choose a neighborhood V of the identity in G such that $eg^{-1}eg \in U$ for all $g \in V$. Clearly,

$$F_R(g^{-1}eg) = g^{-1}F_R(e)g \quad \text{and} \quad eg^{-1}eg = g^{-1}ege \in Se.$$

It follows that for all $g \in V$,

$$(F_R(g^{-1}eg))_0 \subset (F_R(eg^{-1}eg))_0 = (F_R(e))_0$$

and consequently $(F_R(g^{-1}eg))_0 = (F_R(e))_0$. Since G_0 is generated by every neighborhood of the identity, this implies the assertion of the lemma. (Note our general assumption (2.4), $G = G_0 \cdot D$!.)

2.10. The following lemma will take our reduction arguments a step further; in particular it implies that if A is a closed connected normal subgroup of G with Lie algebra $\mathfrak{A}$ then for any s in S either $sA = As$ or $\mathfrak{A} \cap \mathfrak{F}_R(s) \neq \{0\}$.

LEMMA. *Let $\langle g_n | n \in I \rangle$ be a net in G which converges to an element s in S. Suppose that A is a closed connected subgroup of G with corresponding Lie algebra $\mathfrak{A}$ and such that $g_n^{-1}Ag_n = A$ (hence $g_n^{-1} \cdot \mathfrak{A} = \mathfrak{A}$) for all n in I.*

(i) *If there is a non-zero element x of $\mathfrak{A}$ with* $\lim g_n^{-1} \cdot x = 0$ *then*

$$x \in \mathfrak{F}_L(s) \quad \textit{and} \quad \mathfrak{A} \cap \mathfrak{F}_R(s) \neq \{0\}.$$

(ii) *If there is an element x of $\mathfrak{A}$ with* $\lim \|g_n^{-1} \cdot x\| = \infty$ *then*

$$\mathfrak{A} \cap \mathfrak{F}_R(s) \neq \{0\}.$$

(iii) *If for every non-zero element x in $\mathfrak{A}$ there is a subnet*

$$\langle g_m | m \in I' \rangle \text{ of } \langle g_n | n \in I \rangle$$

such that $\lim g_m^{-1} \cdot x$ *exists and is not zero, then $sA = As$ (i.e., for every $s \in S$, $a \in A$ there exist elements a', $a'' \in A$ with $sa = a's$, $as = sa''$).* (It should be emphasized that there is always a subnet of $\langle g_n \rangle$ which satisfies the assumption of at least one of the statements (i)–(iii).)

Proof. (i) Since the action of G on S is jointly continuous we have, for every real number λ,

$$(\exp \lambda x)s = \lim\, (\exp \lambda x)g_n = \lim\, g_n(\exp \lambda(g_n^{-1} \cdot x)) = s \cdot 1 = s.$$

(Recall that $g_n^{-1}(\exp \lambda x)g_n = \exp \lambda(g_n^{-1} \cdot x)$.) Taking λ sufficiently small this implies $x \in \mathfrak{F}_L(s)$.

We next show that $\mathfrak{F}_R(s) \cap \mathfrak{A} \neq \{0\}$. Let B be the closure of the one-parameter group $\{\exp \lambda x \mid \lambda \in \mathbf{R}\}$ in S and let e be the minimal idempotent of B. Then e commutes with all elements of B. By continuity, $bs = s$ for all b in B, in particular $es = s$. Define $x_n = \|g_n^{-1} \cdot x\|^{-1}x$ and choose subnets $\langle g_m | m \in I' \rangle$ and $\langle x_m | m \in I' \rangle$ of the nets $\langle g_n | n \in I \rangle$ and $\langle x_n | n \in I \rangle$ respectively such that both of the limits $y = \lim g_m^{-1} \cdot x_m$ and $z = \lim e(\exp x_m)$ exist. Clearly, $\|y\| = 1$ and z is contained in the maximal subgroup $H(e)$. For all m and every real λ we have $e(\exp \lambda x_m) = e(\exp \lambda x_m)e$, since $\exp \lambda x_m$ lies in B. By Lawson's Theorem (cf. 1.8), multiplication is jointly continuous at (z, s) if it is restricted to

$eSe \times eS$ and the right action of G on S is jointly continuous. Therefore, for all real numbers λ,

$$s \exp \lambda y = \lim eg_m(\exp \lambda(g_m^{-1} \cdot x_m) = \lim (e \exp \lambda x_m)g_m$$
$$= \lim (e \cdot \exp \lambda x_m \cdot e)(eg_m) = zs = s.$$

As before, this implies $y \in \mathfrak{F}_R(s)$.

(ii) Define $x_n = \|g_n^{-1} \cdot x\|^{-1}x$ and choose a subnet

$$\langle g_m | m \in I' \rangle \text{ of } \langle g_n | n \in I \rangle$$

such that $y = \lim g_m^{-1} \cdot x_m$ exists. Then $\|y\| = 1$, and, similar to the proof of (i) (note that $\lim x_m = 0$),

$$s = \lim (\exp \lambda x_m)g_m = \lim g_m(\exp (g_m^{-1} \cdot \lambda x_m)) = s \cdot \exp \lambda y$$

for all real numbers λ, which implies the assertion.

(iii) By a standard argument, there is a subnet $\langle g_m | m \in I' \rangle$ such that $\rho(x) = \lim g_m^{-1} \cdot x$ exists for all x. Obviously, the mapping $x \to \rho(x)$ is an endomorphism of the Lie algebra $\mathfrak{A}$. Since $\rho(x) \neq 0$ for all non-zero x, we see that ρ is actually an automorphism. As in the above proofs we conclude

$$(\exp x)s = \lim(\exp x)g_m = \lim g_m \exp(g_m^{-1} \cdot x) = s \exp \rho(x).$$

Since A is generated by the elements $\exp x$, $x \in \mathfrak{A}$, and ρ is surjective, this implies $sA = As$, as asserted.

2.11. LEMMA. *Suppose* $s \in S$, $x \in \mathfrak{F}_R(s)$, $y \in \mathfrak{G}$. *If* $(\operatorname{ad} x)^2 y = 0$ *then* $\operatorname{ad} x \cdot y \in \mathfrak{F}_R(s)$.

Proof. We may suppose that $\operatorname{ad} x \cdot y$ lies in a Campbell–Hausdorff neighborhood so that $\operatorname{ad} x \cdot y \in \mathfrak{F}_R(s)$ if and only if $\exp (\operatorname{ad} x \cdot y) \in F_R(s)$. Since S is compact there is a net $\langle \lambda_n | n \in I \rangle$ of real numbers with $\lim \lambda_n = \infty$ and such that $t = \lim \exp \lambda_n x$ exists (in S). Write $g_n = \exp \lambda_n x$. Then

$$g_n^{-1} \cdot y = (\operatorname{Ad} (-\lambda_n x))y = y - \lambda_n \operatorname{ad} x \cdot y,$$

hence

$$\lim g_n^{-1} \cdot (-\lambda_n^{-1} y) = \lim (-\lambda_n^{-1} y + \operatorname{ad} x \cdot y) = \operatorname{ad} x \cdot y.$$

Since $\lim \exp \lambda_n^{-1} y = 1$ and since the left and right actions of G on S are jointly continuous we conclude that

$$s = st = s \lim (\exp (-\lambda_n^{-1} y))g_n = s \lim g_n g_n^{-1}(\exp (-\lambda_n^{-1} y))g_n$$
$$= st \exp (\operatorname{ad} x \cdot y) = s \exp (\operatorname{ad} x \cdot y).$$

The assertion follows.

2.12. COROLARY. *Suppose that in Lemma* 2.10, $\mathfrak{A} \cap \mathfrak{F}_R(s) = \{0\}$. *Then*

(i) $sA = As$,

(ii) $aF_R(s)a^{-1} = F_R(s)$ *for all* $a \in A$,

(iii) $ax = xa$ *for all* $a \in A$ *and all* $x \in F_R(s)$ *with* $x^{-1}Ax \subset A$.

Proof. (i) This follows immediately from Lemma 2.10.

(ii) Let $x \in F_R(s)$, $a \in A$. Then $sa = a's$ for some $a' \in A$ and therefore $saxa^{-1} = a'sxa^{-1} = a'sa^{-1} = saa^{-1} = s$; similarly, $a^{-1}xa \in F_R(s)$. Thus

$$aF_R(s)a^{-1} \subset F_R(s) \subset aF_R(s)a^{-1},$$

so

$$aF_R(s)a^{-1} = F_R(s).$$

(iii) Since A is generated by every neighborhood of the identity we may assume that a is contained in a symmetric neighborhood U of 1 in A with

$$x^{-1}UxU \cap F_R(s) = \{1\}.$$

But then $x^{-1}axa^{-1} = 1$ implies the assertion. (Note that $axa^{-1} \in F_R(s)$ by (ii).)

2.13. We now use the established results to prove the theorem for the case where G_0 is solvable.

LEMMA. *If* G_0 *is a solvable Lie group and* $s \in S$ *then* $sG = Gs$.

Proof. By Lemma 2.6, we may suppose that $\mathfrak{F}_R(s)$ does not contain a non-trivial ideal of $\mathfrak{G}$.

Let $\{\mathfrak{G}_i \mid i = 0, 1, 2, \ldots\}$ be the commutator chain of $\mathfrak{G}$; i.e., $\mathfrak{G}_0 = \mathfrak{G}$ and $\mathfrak{G}_{i+1} = [\mathfrak{G}_i, \mathfrak{G}_i]$ for $i \geq 0$. Since $\mathfrak{G}$ is solvable there is an m with $\mathfrak{G}_{m-1} \neq \mathfrak{G}_m = \{0\}$. We show successively that for $i = m$, $m-1$, $\ldots$ 0, the intersection $\mathfrak{F}_R(s) \cap \mathfrak{G}_i$ is trivial, $\mathfrak{F}_R(s) \cap \mathfrak{G}_i = \{0\}$. Having shown this assertion for $i = 0$, we can apply Corollary 2.12(i) to obtain the desired result.

The assertion is immediate for $i = m$. Assume now that it is true for all i with $m \geq i > j \geq 0$, and suppose x is a non-zero element of $\mathfrak{F}_R(s) \cap \mathfrak{G}_j$. Then $\operatorname{ad} x \cdot \mathfrak{G} \subset \mathfrak{G}_j$ and therefore $(\operatorname{ad} x)^2 \mathfrak{G} \subset \mathfrak{G}_{j+1}$. By induction, $\mathfrak{G}_{j+1} \cap \mathfrak{F}_R(s) = \{0\}$, hence Corollary 2.12 implies that $h = \exp x$ commutes with all elements of the subgroup corresponding to $\mathfrak{G}_{j+1}$, thus $\operatorname{ad} x \cdot \mathfrak{G}_{j+1} = \{0\}$. From this it follows $(\operatorname{ad} x)^3 \mathfrak{G} = \{0\}$ and by Lemma 2.11. $(\operatorname{ad} x)^2 \mathfrak{G} \subset \mathfrak{F}_R(s) \cap \mathfrak{G}_{j+1} = \{0\}$. Applying Lemma 2.11. once more, we see $\operatorname{ad} x \cdot \mathfrak{G} \subset \mathfrak{F}_R(s)$. It follows that $\mathfrak{F}_R(s) \cap \mathfrak{G}_j$ is an ideal of $\mathfrak{G}$, thus $\mathfrak{F}_R(s) \cap \mathscr{G}_j = \{0\}$, which completes the proof.

2.14. In order to get the full theorem from this partial result we need the following variant of Iwasawa's decomposition theorem.

LEMMA. *If* G *satisfies our general assumptions* (as formulated in 2.4.) *then it contains a closed connected solvable subgroup* A, *a discrete central subgroup* Z *and a compact subset* K *such that* $G = K \cdot Z \cdot A$ (*i.e.*, $Z \cdot A$ *is syndetic in* G) *and such that the maximal compact subgroup of* A *is central in* G.

Proof. Since, by 2.4, $G = G_0 \cdot D$, D being central and discrete, we may suppose that G is connected.

Let G_1 be the quotient of G by its centrum, $G_1 = G/\text{centrum } G$. Then G_1 has a finite-dimensional faithful representation (induced by the adjoint representation of G); therefore, by a well-known theorem (cf. Hochschild [11], p. 245, Théorème 4.3.), there is a normal simply connected solvable subgroup B and a reductive subgroup H such that G_1 is the semidirect product $B \circledS H$. Thus it is sufficient to show that every reductive Lie group H can be written as the product $H = A_2 \cdot K_2$, A_2 being a simply connected and solvable subgroup of H, K_2 being a compact subgroup of H. But the centrum of H is a compact subgroup K_3 and the quotient group H/K_3 is semisimple with trivial centrum (cf. Hochschild [11], p. 245, Théorème 4.4.); therefore this assertion follows from the global version of Iwasawa's decomposition theorem given in 1.4.

2.15. Since the set K in the above decomposition is compact, we see easily that to every element s in S there are elements k and k' in K such that $k^{-1}s$ and sk' lie in the closure of $A \cdot Z$. (If $s = \lim k_n b_n$, where $\langle k_n \rangle$, $\langle b_n \rangle$ are suitable nets in K and in $A \cdot Z$ respectively, such that $k = \lim k_n$ and $b = \lim b_n$ exist, then, by the continuity of the action of G on S, we get $k^{-1}s = b$. Similarly we can write $s = \lim (k'_n b'_n)^{-1}$ and find $b' \in \overline{A \cdot Z}$ and $k' \in K$ with $sk' = b'$.).

2.16. LEMMA. *Let $G = K \cdot A \cdot Z$ be the decomposition of G given in* 2.14 *and let e be an idempotent in S. Then $es = se$ for every element s in the closure of $A \cdot Z$.*

Proof. By the preceding remark, there are elements k, k^{-1} in K such that ek' and $k^{-1}e$ lie in the closure of $A \cdot Z$. By Lemma 2.13 we have $ek'AZ = AZek'$ and $k^{-1}eAZ = AZk^{-1}e$, hence $eAZ = eAZe = AZe$ and the assertion follows by continuity.

2.17. We finally prove statement (i) of Theorem 2.1.

Proof of 2.1(i). Let e be any idempotent in S. It is obviously sufficient to show that $g^{-1}eg = e$ for all g in G. Let $g \in G_0$ and $k, k' \in K$ such that $k^{-1}e$ and ek' are contained in the closure of $A \cdot Z$. Then by 2.16 (with $g^{-1}eg$ instead of e),

$$ek'g^{-1}eg = g^{-1}egek' \quad \text{and} \quad k^{-1}eg^{-1}eg = g^{-1}egk^{-1}e;$$

therefore

$$g^{-1}ege = e(g^{-1}eg)e = eg^{-1}eg.$$

By Lemma 2.7, this implies that $\mathfrak{F}_R(e)$ is an ideal of $\mathfrak{G}$. If $\mathfrak{F}_R(e) = \{0\}$ then the assertion follows from 2.10, if not then it follows from Lemma 2.6.

2.18. Corollary. *Under our general assumption* 2.4, *there exists a closed connected solvable subgroup* A, *a discrete central subgroup* Z *and a compact subset* K *of* G *such that*:

(i) *the maximal compact subgroup of* A *is central in* G;
(ii) *all idempotents of* S *are contained in the closure of* $A \cdot Z$;
(iii) $G = K \cdot A \cdot Z$.

Proof. Lemma 2.14 provides subgroups A, Z and a compact subset K such that (iii) and (i) are satisfied. Let e be an idempotent in S and choose an element k in K and a net $\langle a_n \mid n \in I \rangle$ of elements in $A \cdot Z$ with $e = \lim k \cdot a_n$. Since e is central in S the elements $eka_n = ka_ne$, $n \in I$, lie in $H(e)$. Passing to a suitable subnet if necessary, we may assume that $s = \lim a_n^{-1}$ exists. By the result of Lawson (cf. 1.8), $H(e)$ is a topological group, so $e = \lim eka_n$ implies

$$e = \lim (eka_n)^{-1} = \lim a_n^{-1}k^{-1}e = sk^{-1}e.$$

The closure of $A \cdot Z$ is a subsemigroup of S containing both s and $k^{-1}e$, so it must also contain e. The proof is completed.

2.19. We conclude this section with an example showing that the statements of Theorem 2.1 do not hold for all discrete groups G having a subnormal subgroup H with $g^{-1}Hg \cap H$ finite for some $g \in G$.

Example. Let G *be a discrete group*, $\phi: G \to S_G$ *the weak almost periodic compactification map*; *let* H *be a subnormal subgroup and* g *an element of* G *such that the intersection* $g^{-1}Hg \cap H$ *is a finite set.* (Such objects G, H, g can be constructed easily; for example, let G be the discretization of the affine group $(\mathbf{R}, +)\circledS(\mathbf{R}^+, \cdot)$, let H be the countable subgroup $\{(z, 1) \mid z \text{ is an integer}\}$, and $g = (0, \pi)$.) *Then* $e \neq g^{-1}eg$ *for every idempotent* e *in the closure of* $\phi(H)$ *in* S_G.

Proof. Let $e^2 = e \in \overline{\phi(H)}$. Then e lies in the closure of $\phi(H \backslash g^{-1}Hg)$, since $H \cap g^{-1}Hg$ is finite. By Corollary 4.6 of the appendix, the closures of the sets $\phi(H \backslash g^{-1}Hg)$ and $\phi(g^{-1}Hg)$ are open in S_G and therefore disjoint, since $\phi(H \backslash g^{-1}Hg) \cap \phi(g^{-1}Hg) = \emptyset$. (Note that ϕ is injective!) Thus e cannot lie in the closure of $\phi(g^{-1}Hg)$, whereas $g^{-1}eg \in \overline{\phi(g^{-1}Hg)}$. This proves the assertion.

3. Some structure theorems

3.1. The results and techniques of the preceding chapter can be applied to obtain more insights into the structure of semitopological compactifications of locally compact connected groups. A special feature of such compactifications is an intrinsic tendency towards being "nearly abelian". That is to say, they have properties one would expect only for the abelian case, (Theorem 2.1 could be cited as a typical example) and structural questions can often be reduced to the abelian case. It seems that the more a connected group G differs from an abelian one the less sophisticated is its weakly almost periodic compactification, the less can it be "fitted" into a "complicated" semitopological semigroup; this view is also supported by the results of Ruppert [18][19], [20]. As we shall see below the groups which "differ most" from abelian groups, namely the connected simple Lie groups with finite centrum, have only trivial (one-point) semitopological compactifications or are already compact.

As an introduction to this complex of ideas we start with a simple (but characteristic) example.

3.2. *Example. The weak almost periodic compactification of the affine group* $G = (\mathbf{R}, +) \circledS (\mathbf{R}^+, \cdot)$. Write S for the weak almost periodic compactification of G. The adjoint representation of G is a topological isomorphism of G onto a closed subgroup of the automorphism group of the Lie algebra $\mathfrak{G}$ of G, hence for every net $\langle g_n \mid n \in I \rangle$ in G with no convergent subnet there is an $x \in \mathfrak{G}$ with either $\lim g_n \cdot x = \infty$ or $\lim g_n \cdot x = 0$. By 2.10, it follows that $\mathfrak{F}_R(s) \neq \{0\}$ for every s in the remainder $S \backslash G$ of the compactification S; by Theorem 2.1, $\mathfrak{F}_R(s)$ has to be an ideal of $\mathfrak{G}$. But the only non-trivial ideal of $\mathfrak{G}$ is the subalgebra $\mathfrak{N}$ corresponding to the normal subgroup $N = (\mathbf{R}, +)$ of G. Denote by e the minimal idempotent of the closure $\bar{N}$ of N in S. Then e is actually a zero element of $\bar{N}$ and $S \backslash G = eS$. By the universality of the weak almost periodic compactification we have an identity preserving homomorphism $\alpha \colon eS \to S_{G/N}$ mapping S onto the weak almost periodic compactification $S_{G/N}$ of G/N. On the other hand, a similar homomorphism $\beta \colon S_{G/N} \to eS$ exists between S_N and eS; and $\alpha\beta$, $\beta\alpha$ are the respective identity mappings. (A more general version of this statement is given in Proposition 3.11 below.). It follows that $S \backslash G$ is isomorphic to the weak almost periodic compactification of the reals (since $G/N \cong (\mathbf{R}^+, \cdot) \cong (\mathbf{R}, +)$).

3.3 *Remark.* It has been claimed in the book of Burckel [5, p. 52, Theorem 3.17] that in the weak almost periodic compactification S of a locally compact topological group G the S-closure of a closed normal subgroup N of G must be isomorphic to the weak almost periodic compactification of N. (For G abelian (or for N open in G) this had been shown already by deLeeuw-Glicksberg [15].) However, Burckel's proof of this assertion has a gap right in the beginning; in fact, he makes use of a theorem (Theorem 3.16 in his notation) which was proved only for the abelian case. The above 3.2 provides a counterexample: the closure of N is just $N \cup \{e\}$ which obviously cannot be isomorphic to the weak almost periodic compactification of N (e.g., the minimal ideal of the weak almost periodic compactification of N is isomorphic to the Bohr compactification of N—which certainly contains more than one point). It will be shown in the appendix that the assertion becomes true if N is asumed to be central.

3.4. The most striking example supporting our view that "definitely non-abelian" groups must have "comparatively simple" weak almost periodic compactifications is the case of a connected simple Lie group with finite centrum.

Example. The weak almost periodic compactification S of a connected simple Lie group G with finite centrum. If G is compact, then obviously $G = S$. Assume G to be non-compact and let $s \in S \backslash G$. We show first that $\mathfrak{F}_R(s) \neq \{0\}$. We may assume by 2.14 and 2.15 that s lies in the closure of a simply connected solvable subgroup A of G, such that $G = KA$, where K is a maximal compact subgroup of G. Since G is simple, the adjoint representation $G \to \operatorname{Aut} \mathfrak{G}$ maps A onto a closed subgroup of $\operatorname{Aut} \mathfrak{G}$ (cf. 3.7 below, for example). Thus the assumptions of Lemma 2.10 (i) or (ii) are satisfied and it follows that $\mathfrak{F}_R(s) \neq \{0\}$.

If $F_R(s)$ were compact then $\mathfrak{A} \cap \mathfrak{F}_R(s) = \{0\}$, hence by Corollary 2.12 (ii), $a^{-1}F_R(s)a = F_R(s)$ for all $a \in A$. But G is a simple Lie group, so this would imply

$$F_R(s) \subset \bigcap\{a^{-1}Ka \mid a \in A\} = \bigcap\{g^{-1}Kg \mid g \in G\} = \text{centrum } G;$$

thus $F_R(s)$ would be discrete, a contradiction.

3.5. The above almost trivial consequence of Theorem 2.1 and Lemma 2.10 was established first by Veech [23] using a different method. In fact he proved a slightly more general version which we state below, as Theorem 3.6. Before doing so we recall that to every connected semisimple Lie group we can find closed connected normal subgroups $G_1, G_2, \ldots, G_k$, which are simple Lie groups, and a discrete central subgroup N of the direct product

$$G_1 \times G_2 \times \cdots \times G_k,$$

such that G is isomorphic to the quotient

$$G_1 \times G_2 \times \cdots \times G_k/N.$$

3.6. THEOREM. *Let G be a connected semisimple Lie group with finite centrum, S its weak almost periodic compactification. Assume that*

$$G \cong G_1 \times G_2 \times \cdots \times G_k/N$$

is the above representation of G and define G_i^* to be the one-point compactification $G_i \cup \{m_i\}$ (m_i acting as a zero) if G_i is not compact and $G_i^* = G_i$ otherwise ($i = 1, 2, \ldots, k$). Then S is isomorphic to the quotient $G_1^* \times G_2^* \times \cdots \times G_k^*/N$. (Note that N is finite since the centrum of G is finite.)

Proof. We may assume that none of the groups G_i, $i = 1, 2, \ldots, k$, is compact and that the centrum of G is trivial, so that G is isomorphic to the direct product $G_1 \times G_2 \times \cdots \times G_k$. By 3.4, we know that for every factor G_i of G we have $G_i^* \cong \bar{G}_i = G_i \cup \{m_i\}$. Furthermore, taking over the respective arguments of 3.4, we infer that for every s in $S \backslash G$ the identity component of $F_R(s)$ is not compact. If $s \in S \backslash G$ and e is the minimal idempotent in $\overline{F_R(s)}$ then $F_R(s) \supset F_R(e)$ and $F_R(e)$ contains one of the factors G_i, hence $s = m_i s$ for some i.

The theorem will be proved if we have shown that the map

$$\bar{G}_1 \times \bar{G}_2 \times \cdots \times \bar{G}_k \to S, \qquad (x_1, x_2, \ldots, x_k) \to x_1 x_2 \cdots x_k,$$

is jointly continuous, because of the universality of S. We proceed by induction. The inclusion map $\bar{G}_i \to S$ is obviously continuous for every factor G_i. Assume that the map

$$\bar{H} \times \bar{G}_l \to S, \ (x, y) \to xy,$$

is jointly continuous for every factor G_l and every normal subgroup H which is the product of less than $j > 1$ factors $G_i \neq G_k$. Now suppose that H is the product of exactly j (distinct) factors $G_i \neq G_l$. We may assume that indexes are choosen so that

$$H = G_1 G_2 \cdots G_j.$$

Since both the left and the right action of G on S are continuous, the induction hypothesis implies that the map

$$\bar{H} \times \bar{G}_l \to S, \ (x, y) \to xy$$

is jointly continuous at all points (x, y) except possibly at the point (m, m_l), where $m = m_1 m_2 \cdots m_j$. Let

$$\langle h_n | n \in I \rangle, \langle g_n | n \in I \rangle$$

be nets in $\bar{H}$, $\bar{G}$ respectively, which are defined over the same domain I. Passing over to suitable subnets if necessary, we may assume that $s = \lim h_n g_n$ exists. We have to show $s = mm_l$. If $s \in G$ then $g_n h_n \in G$ for all sufficiently large b (since G is open in S), hence

$$m_l s^{-1} = \lim g_n(g_n^{-1} h_n^{-1}) = \lim h_n^{-1} \in \bar{H}.$$

But by the induction hypothesis,

$$m_l \notin s\bar{H} = s\bar{G}_1 \cdot \bar{G}_2 \cdots \bar{G}_j,$$

a contradiction. Thus $s \in \overline{H \cdot G_l} \backslash HG_l$, so we may apply the observation in the first paragraph of this proof to find an index i, $1 \leq i \leq j$ or $i = l$, with $s = m_i s$. If $i = l$ then

$$s = sm_i = \lim h_n g_n m_l = \lim h_n m_l = mm_l,$$

as required. If $1 \leq i \leq j$, say $i = 1$, then we apply the induction hypothesis to conclude that the map

$$m_1 \cdot G_2 \cdot G_3 \cdots G_j \times G_l \to S, \ (x, y) \to xy$$

is jointly continuous. Now $m_1 \bar{H} = m_1 G_2 \cdot G_3 \cdots G_j$, so

$$s = m_1 s = \lim m_1 h_n g_n = (\lim m_1 h_n) \lim g_n = m_1 m m_l = mm_l.$$

The proof is completed.

3.7. We now want to apply the ideas used above to deduce a general theorem about isotropy groups. As is to be expected, we shall use Lemma 2.10, and therefore have to look first what happens if the image of G under the adjoint representation is not closed. The following lemma is an easy application of well-known facts.

LEMMA. *Let G be a connected Lie group, $\alpha: G \to \operatorname{Aut} \mathfrak{G}$ its adjoint representation. Furthermore, let $G = K \cdot A \cdot Z$ be the decomposition of 2.14, where A is a closed connected solvable subgroup (whose maximal compact subgroup is central in G), Z a discrete central subgroup and K a compact subset of G. Then there exists a closed vector subgroup V of A such that:*

(i) *the closure of $\alpha(V)$ (in the usual topology of* Aut $\mathfrak{G}$ *as a sub-group of $GL(\mathfrak{G})$) is compact;*

(ii) $\overline{\alpha(G)} = \alpha(G)\overline{\alpha(V)}$.

Proof. Let A' be the commutator subgroup of A. Then $\alpha(A')$ is the commutator subgroup of both $\alpha(A)$ and $\overline{\alpha(A)}$, since the commutator subgroup of a Lie group coincides with the commutator subgroup of any of its dense analytic subgroups (cf. Hochschild [11], p. 210, for example). Now the commutator subgroup of a linear Lie group is closed, (Hochschild [11], p. 246) hence $\alpha(A')$ is closed in Aut $\mathfrak{G}$. Let N be a closed connected subgroup of A with $A' \subset N$ and such that $\alpha(N)$ is closed in Aut $\mathfrak{G}$; suppose that N is maximal with respect to this property. Since A is solvable and the maximal compact subgroup of A is central in G we conclude that $N/N \cap$ centrum G is simply connected. Because, by definition, $\alpha(N)$ is closed in Aut $\mathfrak{G}$, it is isomorphic to $N/N \cap$ centrum G and therefore simply connected. Moreover N is normal in A, since $A' \subset N$. Consider the quotient map

$$\kappa: \overline{\alpha(A)} \to \overline{\alpha(A)}/\alpha(N).$$

Let R be a one-parameter subgroup of A not contained in N and suppose that $\overline{\kappa(\alpha(R))}$ is not compact. Since a one-parameter subgroup of a locally compact group is either closed or has compact closure (cf. Hochschild [11], p. 212, for example), this would imply that

$$\alpha(N \cdot R) = \kappa^{-1}\kappa(\alpha(R))$$

is closed and simply connected (otherwise $\kappa(\alpha(R))$ would be compact), hence the identity component of $\alpha^{-1}(\alpha(N \cdot R))$ would be a group satisfying the conditions on N, a contradiction to the assumed maximality of N.

It follows that $\overline{\alpha(A)}/\alpha(N)$ is compact, so $\overline{\alpha(A)}$ is the semidirect product of $\alpha(N)$ with some torus group T (cf. Hochschild [11], p. 155, for example).

Let $g \in A$. There are elements $n \in N$ and $t \in T$ with $\alpha(g) = \alpha(n)t$, so

$$t = \alpha(gn^{-1}) \in \alpha(A)$$

implies

$$\alpha(g) \in \alpha(N \cdot (\alpha^{-1}(T) \cap A)), \quad g \in N \cdot (\alpha^{-1}(T) \cap A).$$

Thus,

$$A = N \cdot (\alpha^{-1}(T) \cap A) \quad \text{and} \quad T = \overline{\alpha(\alpha^{-1}(T) \cap A)}.$$

Let H be the identity component of $\alpha^{-1}(T) \cap A$, $\mathfrak{H}$ the corresponding subalgebra of $\mathfrak{G}$. The image of $\alpha^{-1}(T) \cap A$ under α is connected (since $A/N = (\alpha^{-1}(T) \cap A)/A \cap \text{centrum } G$ is); thus

$$\alpha^{-1}(T) \cap A = H \cdot (A \cap \text{centrum } G)$$

and therefore

$$\overline{\alpha(A)} = \alpha(N)\overline{\alpha(\alpha^{-1}(T) \cap A)} = \alpha(N)\overline{\alpha(H)}.$$

The group H is nilpotent, since $\alpha(H)$ is abelian. But a linear action of a torus group is semisimple, hence the action of $\alpha(H)$ on $\mathfrak{G}$ is semisimple and therefore trivial, because $\mathfrak{G}$ is nilpotent. Thus H is a closed connected abelian subgroup of G; it can be written as the direct product of a vector group V with some torus group T'. Plainly, $\overline{\alpha(H)} = \overline{\alpha(V)}\alpha(T')$, so $\overline{\alpha(A)} = \alpha(A)\overline{\alpha(H)} = \alpha(A)\overline{\alpha(V)}$. Consequently,

$$\overline{\alpha(G)} = \alpha(K)\overline{\alpha(A)} = \alpha(G)\overline{\alpha(V)},$$

and the assertion follows.

(Note that the above proof also shows that for $G = H$, with H the solvable subgroup of the Iwasawa decomposition 1.4 of a non-compact semisimple Lie group, we have $\overline{\alpha(H)} = \alpha(H)$ by the well-known properties of semisimple Lie groups (cf. Helgason [10], p. 222 ff., for example).

3.8. LEMMA. *Let S be a semitopological compactification of a Lie group G which satisfies the assumptions of Theorem* 2.1 (*i.e.*, $G = G_0 . D$, *where G_0 is the identity component and D is a central discrete subgroup of* G); *write $\phi\colon G \to S$ for the compactification map. If the identity component of the right isotropy group $F_R(s)$ of an element $s \in S$ is compact then $s \cdot G = G \cdot s$.*

Proof. Suppose s is an element in S such that the identity component $(F_R(s))_0$ of $F_R(s)$ is compact. Let $G = K \cdot A \cdot Z$ be the decomposition of Lemma 2.14. By the remark in 2.15 there is an element $k \in K$ such that $k^{-1} \cdot s$ is contained in the closure of $\phi(A \cdot Z)$; clearly $F_R(k^{-1}s) = F_R(s)$ and $s \cdot G = G \cdot s$ if and only if $k^{-1}s \cdot G = G \cdot k^{-1}s$. Thus we may assume that $s \in \overline{\phi(A \cdot Z)}$. The maximal compact subgroup T of A is central in G; thus, taking the quotients G/T and S/T (cf. Lemma 2.3), we may also assume that T is trivial, A is simply connected. Since $(F_R(s))_0$ is compact, we have, by Lemma 2.10, $\mathfrak{F}_R(s) \cap \mathfrak{A} = \{0\}$ and, by Lemma 2.12(ii),

$$a^{-1}F_R(s)a = F_R(s) \quad \text{for all } a \in A.$$

(Note that this also implies $a^{-1}(F_R(s))_0 a = (F_R(s))_0$ for all $a \in A$.) It is readily seen from the proof of 2.14 that $K \cdot Z$ is a closed subgroup of G with $K \cdot Z/Z$ compact. Thus the identity component $(K \cdot Z)_0$ of $K \cdot Z$ can be written $(K \cdot Z)_0 = V \cdot K_1$, where V is a vector subgroup which is central in $K \cdot Z$.

(This follows from the well-known fact that the simply connected covering group of a compact connected Lie group is the direct product of a central vector group with a compact semisimple Lie group.) Therefore, $k^{-1}K_1 k = K_1$ for all $k \in K$ and consequently

$$K_0 = \bigcap \{a^{-1}K_1 a \mid a \in A\} = \bigcap \{g^{-1}K_1 g \mid g \in G\}$$

is a compact normal subgroup of G_0 which contains $(F_R(s))_0$. Passing over to G/K_0 and S/K_0 we may suppose that $(F_R(s))_0 = \{1\}$; i.e., $\mathfrak{F}_R(s) = \{0\}$. The assertion follows now from Lemma 2.12(i) (taking $\mathfrak{G}$ instead of $\mathfrak{A}$).

3.9. THEOREM. *Let S be a semitopological compactification of a Lie group G which satisfies the assumption of Theorem* 2.1. (*In other words, $G = G_0 \cdot D$, where G_0 is the identity component and D is a central discrete subgroup of G.*) *Write $\phi\colon G \to S$ for the compactification map and C for the centrum of G. Then G contains a vector subgroup V with the following properties:*

(i) *$s \cdot G = G \cdot s$ for all $s \in \overline{\phi(V \cdot C)}$.*

(ii) *If the identity component N of the right isotropy group $F_R(s)$ of an element $s \in S$ is compact and $N \cap$ centrum G is finite, then $s \in G \cdot \overline{\phi(VC)}$.*

Moreover we have:

(iii) *If the image of G under the adjoint representation $\alpha\colon G \to$ Aut $\mathfrak{G}$ is closed then V can be taken to be trivial* $(=\{1\})$.

Proof. Let $G = K \cdot A \cdot Z$ be the decomposition of 2.14. By Lemma 3.7 there is a closed vector subgroup V of A such that

$$\overline{\alpha(G)} = \alpha(G)\overline{\alpha(V)}$$

and such that $\overline{\alpha(V)}$ is a torus group. It is plain that V can be chosen trivial if $\alpha(G)$ is closed in Aut $\mathfrak{G}$, so we only have to show (i) and (ii).

(i) Let s be an element in $\overline{\phi(VC)}$, $\langle v_n \mid n \in I\rangle$ a net in VC such that $s = \lim \phi(v_n)$. Then for every $x \in \mathfrak{G}$ we have

$$\begin{aligned} s \cdot \exp x &= \lim \phi(v_n) \cdot \exp x \\ &= \lim \phi(v_n(\exp x)v_n^{-1})\phi(v_n) \\ &= \lim \phi(\exp(v_n \cdot x)) \cdot \phi(v_n) \\ &= \lim \exp(v_n \cdot x) \cdot \phi(v_n). \end{aligned}$$

Since $\overline{\alpha(VC)} = \overline{\alpha(V)}$ is compact, we can find a suitable subnet

$$\langle v_m \mid m \in I'\rangle \text{ of } \langle v_n \rangle$$

such that $\langle v_m \cdot x\rangle$ converges to some element $y \in \mathfrak{G}$. By the continuity of the left action of G on S it follows that $s \cdot \exp x = \exp y \cdot s$; hence $s \cdot G \subset G \cdot s$, since G is generated by the set $\exp \mathfrak{G}$ and C is the centrum. In the same way we see that $G \cdot s \subset s \cdot G$.

(ii) Suppose that s is an element in S which satisfies the assumption of (ii). Then, by Lemma 3.8, $s \cdot G = G \cdot s$, hence N is normal in G. Since the identity component of the centrum of N is a normal torus subgroup of G and every normal torus group of a connected Lie group is central, it follows that the centrum of N is discrete. Thus N is semisimple and the theorem of Levi-Malcev implies that G contains a closed analytic subgroup H with $G = HN$ and such that $H \cap N$ is finite. If we can prove the assertion for H and $\overline{\phi(H)}$ then it also follows for G and S; thus we may suppose that $N = \{1\}$. But then Lemma 2.10 implies that there is a net $\langle g_n | n \in I \rangle$ in G such that $\lim \phi(g_n) = s$ and such that $\langle \alpha(g_n) \rangle$ converges to some automorphism $\rho \in \operatorname{Aut} \mathfrak{G}$. By Lemma 3.7, we can write $\rho = \lim \alpha(g)\alpha(v_n)$, where $\langle v_n \rangle$ is a suitably chosen net in V, and $g \in G$; we may assume that $\langle v_n \rangle$ is defined over the same domain I as $\langle g_n \rangle$. Now $\lim \alpha(g_n v_n^{-1}) = \alpha(g)$, hence we can find central elements c_n such that $\lim g_n v_n^{-1} c_n^{-1} = g$. By the continuity of the left action of G on S; we have

$$s = \lim g_n = \lim g_n v_n^{-1} c_n^{-1}(v_n c_n) = g \cdot \lim v_n c_n.$$

The assertion follows.

3.10. COROLLARY. *If in the above theorem the centrum of G is trivial and G is mapped onto a closed set under the adjoint representation α: $G \to \operatorname{Aut} \mathfrak{G}$, then* $S \backslash G = \bigcup \{eS | e^2 = e \in S \backslash \{1\}\}$.

Proof. Pick $s \in S \backslash G$. By 3.9, $F_R(s)$ is not compact, so the minimal idempotent e in $\overline{F_R(s)}$ is not equal to 1 and $es = s$. The assertion follows.

3.11. COROLLARY. *Let S be the weak almost periodic compactification of a non-compact connected simple Lie group G. Then $S = \bar{C} \cdot G \cup \{m\}$, where C is the centrum of G and m is a zero element.* (By Theorem 4.1 of the appendix, $\bar{C}$ is isomorphic to the weak almost periodic compactification of C.)

Proof. From Lemma 2.14 we can see that the Iwasawa decomposition of G implies a decomposition $G = K \cdot A \cdot C$, where K is a compact subset, A a closed simply connected solvable subgroup of G. Moreover, $A \cap C = \{1\}$. Pick an element $s \in \overline{AC} \backslash A\bar{C}$. By Theorem 3.9, and since $S = K \cdot \overline{A \cdot C}$, it suffices to show that s is a zero element of S. Let $\langle a_n | n \in I \rangle$, $\langle c_n | n \in I \rangle$ be nets in A, C respectively, such that $\lim a_n c_n = s$. The adjoint representation α: $A \to \operatorname{Aut} \mathfrak{A}$ maps A onto a closed subgroup of $\operatorname{Aut} \mathfrak{A}$, hence $\mathfrak{F}_R(s) \cap \mathfrak{A} \neq \{0\}$, by Corollary 2.12. Let B be the subgroup of G which corresponds to the subalgebra $\mathfrak{F}_R(s) \cap \mathfrak{A}$ of the Lie algebra $\mathfrak{G}$ of G. Let e denote the minimal idempotent of $\bar{B}$. Since $e \in \bar{A}$, we have $\mathfrak{F}_R(e) \cap \mathfrak{A} \neq \{0\}$, hence $F_R(e) = \mathfrak{G}$, because $\mathfrak{F}_R(e)$ is an ideal and $\mathfrak{G}$ is simple. It follows $sS = se\bar{G} = \{se\} = \{s\}$ and the proof is completed.

4. Appendix

4.1. THEOREM. *Let G be a locally compact topological group, H a closed central subgroup. Then the inclusion $H \to G$ induces an isomorphic imbedding $S_H \to S_G$ of the weak almost periodic compactification S_H of H into the weak almost periodic compactification S_G of G.*

Proof. By the universality of the weak almost periodic compactification, it is sufficient to show that there is a semitopological compactification S of G such that the inclusion $H \to G$ extends to an isomorphic imbedding $i: S_H \to S$. We construct S as follows.

Define an equivalence relation $\sim$ on the direct product $S_H \times G$ by

$$(a, b) \sim (c, d) \text{ if and only if } (a, b) = (ch, dh^{-1}) \text{ for some } h \in H.$$

Since H is central and closed in G, the relation $\sim$ is a closed congruence; it is also open since the map $(x, y) \to (xh, yh^{-1})$ is a homeomorphism for every $h \in H$. Thus the quotient $S_1 = S_H \times G/\sim$ is a locally compact semitopological semigroup. Denote the equivalence class of an element $(a, b) \in S_H \times G$ with $[a, b]^\sim$. The map $G \to S_1, g \to [1, g]^\sim$, maps G isomorphically onto an open and dense subset of S_1. Also, the map $S_H \to S_1$, $s \to [s, 1]^\sim$ is an isomorphic and homeomorphic imbedding. If S_1 is already compact then we may take $S = S_1$ and the proof is completed. Suppose S_1 is not compact. Then we define S to be the one-point compactification $S_1 \cup \{\omega\}$, where ω acts as a zero element. We only have to show that if a net

$$\langle [x_n, y_n]^\sim \mid n \in I \rangle$$

in S has no convergent subnet then, for every element $(a, b) \in S_H \times G$, the nets

$$\langle [ax_n, by_n]^\sim \mid n \in I \rangle \quad \text{and} \quad \langle [x_n a, y_n b]^\sim \mid n \in I \rangle$$

cannot converge. If $[c, d^\sim = \lim [ax_n, by_n]^\sim$ exists in S_1, then there are elements $h_n \in H$ with $\lim ax_n h_n = c$, $\lim by_n h_n^{-1} = d$. But then

$$\lim y_n h_n^{-1} = b^{-1} d$$

and we may assume that $u = \lim x_n h_n$ exists (since S_H is compact), so

$$\lim [x_n, y_n]^\sim = \lim [x_n h_n, yh_n^{-1}]^\sim = [u, b^{-1} d]^\sim$$

exists in S_1. The proof for $\lim [x_n a, y_n b]^\sim$ is completely analogous.

4.2. COROLLARY. *If H is a closed central subgroup of a locally compact group G then every weakly almost periodic function on H can be extended to a weakly almost periodic function on G.*

Proof. This follows immediately by the Stone-Weierstraß theorem.

4.3. *Remark.* The statements of 4.1 and 4.2 remain valid if we drop the assumption that H is closed. This follows from Corollary 4.7 of de Leeuw-Glicksberg [15]. Note that the method in the proof of 4.1 can be used only if H is central; otherwise the relation $\sim$ would not be a congruence.

4.4. The following theorem is a generalization of a theorem in de Leeuw-Glicksberg [15] to not necessarily locally compact groups.

THEOREM. *Let G be a Hausdorff topological group, N an open normal subgroup. Then the canonical isomorphic imbedding $i\colon N \to G$ induces an isomorphic imbedding $i^*\colon S_N \to S_G$ of the weak almost periodic compactification S_N of N into the weak almost periodic compactification S_G of G. Moreover, i^* maps S_N onto an open subset of S_G.*

Proof. As in 4.1, it is sufficient to construct a semitopological compactification S of G such that the image of N under the compactification map $G \to S$ is dense in an open subsemigroup of S which is isomorphic to S_N. Since every automorphism of the topological group N has an extension to its weak almost periodic compactification S_N, the kernel $\phi^{-1}(1)$ of the compactification map $\phi\colon N \to S_N$ must be invariant under every automorphism $n \to g^{-1}ng$, $g \in G$; thus $\phi^{-1}(1)$ is a normal subgroup of G. We may therefore assume that $\phi^{-1}(1)$ is trivial and, since N is open in S, that N is topologically isomorphic to $\phi(N)$. For notational simplicity, we identify N with $\phi(N)$, so that N is considered as a dense subgroup of S_N.

Let H denote the quotient group G/N and let $\kappa\colon G \to H$ be the quotient map. Since H is discrete, there is a continuous (but, in general, not homomorphic) section $\gamma\colon H \to G$ to κ, i.e., a map with $\kappa(\gamma(h)) = h$. The topological group G is isomorphic to the group which is defined on the product space $N \times H$ by the multiplication rule

$$\begin{aligned}(n, h)(n', h') &= (n\gamma(h)n'(\gamma(h))^{-1}\gamma(h)\gamma(h')(\gamma(hh'))^{-1}, hh') \\ &= (n\gamma(h)n'\gamma(h')(\gamma(hh'))^{-1}, hh'),\end{aligned}$$

for all $n, n' \in N$; $h, h' \in H$. (An isomorphism is given by the map

$$g \to (g(\gamma(\kappa(g)))^{-1}, \kappa(g)).)$$

We extend this rule to the space $S_0 = S_N \times H$ in the obvious way:

$$(s, h)(s, h') = (s\gamma(h)n'\gamma(h')\gamma(hh')^{-1}, hh')$$

for all $s, s' \in S_N$; $h, h' \in H$. It is readily seen that this multiplication is separately continuous, hence associative. Also, the canonical imbedding

$$S_N \to S_0, \quad s \to (s, 1)$$

maps S_N onto a compact open subset of the locally compact space S_0. If $\langle (s_\alpha, h_\alpha) \mid \alpha \in D \rangle$ is a net in S_0 with no convergent subnet then none of the nets

$$\langle (s, h)(s_\alpha, h_\alpha) \rangle \text{ and } \langle (s_\alpha, h_\alpha)(s, h) \rangle, \quad (s, h) \in S_0,$$

can have a convergent subnet. (This follows from the fact that a net $\langle (s_\alpha, h_\alpha) \rangle$ has a convergent subnet if and only if h_α is constant for all sufficiently large α—at this point the argument would break down if N were not open in G.)

Thus the one-point compactification $S = S_0 \cup \{\omega\}$, obtained from S_0 by adjoining a zero element ω, is a compact semitopological semigroup. The natural imbedding

$$G \to S, \quad g \to (g(\gamma(\kappa(g)))^{-1}, \kappa(g)),$$

has all of the properties required above. This finishes the proof.

4.5. COROLLARY. *Let G, N be as in Theorem 4.4. If f: $N \to \mathbf{C}$ is a weakly almost periodic function on N, then the extension*

$$\hat{f}\colon G \to \mathbf{C}, \quad \hat{f}(g) = f(g) \text{ if } g \in N, \quad \hat{f}(g) = 0 \text{ otherwise},$$

is an almost periodic function on G.

Proof. By Theorem 4.4, $i^*(N)$ is an open subsemigroup of S_G and isomorphic to S_N. It follows that the function

$$u\colon G \to \mathbf{C}, \quad z(g) = 1 \text{ if } g \in N, \quad u(g) = 0 \text{ otherwise},$$

is weakly almost periodic on G, and that f has a weakly almost periodic extension $f^*\colon G \to C$ to G. Clearly, $\tilde{f}(g) = f^*(g)u(g)$ for all $g \in G$, so $\tilde{f}$ is weakly almost periodic.

4.6. COROLLARY. *Let G be a discrete topological group, H a subnormal subgroup. Then the inclusion $i\colon H \to G$ induces an isomorphic imbedding $i^*\colon S_H \to S_G$ of the weak almost periodic compactification of H into the weak almost periodic compactification of G; moreover, $i^*(S_H)$ is open in S_G.*

Proof. Let $H \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_n = G$ be a subnormal series. Then the statement follows from 4.4 by induction. (Details are left to the reader.)

Added in proof. In the meantime the author was able to show that in statement (ii) of Theorem 2.1 the assumption that G_0 is solvable can be dropped. Thus $sG = Gs$ for every element s in a semitopological compactification of a connected locally compact topological group G. Also, the author noticed that P. Milnes has already shown (Pacific J. Math., vol. 56 (1975), pp. 187–193) that Theorem 4.4 of the appendix still holds if N is only assumed to be open (not necessarily normal).

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