ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 4, Winter 1982

ASYMPTOTIC BEHAVIOR FOR THE FREE BOUNDARY OF PARABOLIC VARIATIONAL INEQUALITIES AND APPLICATIONS TO SEQUENTIAL ANALYSIS

BY

AVNER FRIEDMAN[1]

We consider a solution of a parabolic variational inequality in one space variable. The obstacle is the minimum of two functions, and the inhomogeneous term has a singularity as $t \downarrow 0$. It is shown that the free boundary consists of two curves initiating at a point on $t = 0$; their behavior as $t \downarrow 0$ is studied. An application is given to problems in sequential analysis with two or three hypotheses.

1. Introduction

Consider a parabolic variational inequality

$$\left.\begin{aligned} &u_t - \tfrac{1}{2}u_{xx} \le f \\ &u \le \Psi \\ &(u_t - \tfrac{1}{2}u_{xx} - f)(u - \Psi) = 0 \end{aligned}\right\} \quad \text{a.e. in } x \in \mathbf{R}^1,\ 0 < t < T, \tag{1.1}$$

with the initial condition

$$u(x, 0) = 0 \quad (x \in \mathbf{R}^1), \tag{1.2}$$

where

$$\Psi = \min\{\psi_1, \psi_2\}$$

and f, ψ_i are non-negative functions. The function f tends to ∞ as $t \to 0$, and

$$\psi_1(0, 0) = \psi_2(0, 0) = 0, \quad (\psi_1 - \psi_2)_x(0, 0) < 0. \tag{1.3}$$

We shall prove, under suitable conditions, that the free boundary consists of two curves Γ_i ($u = \psi_i$ on Γ_i) initiating at (0, 0), and we shall study their behavior as $t \to 0$.

Received January 14, 1981.

[1] This work is partially supported by a National Science Foundation grant.

In Sections 3 and 4, we deal with the case where, as $t \to 0$,

$$f(x, t) \sim \frac{\alpha}{t^p} \quad (\alpha > 0,\ p > \tfrac{1}{2}), \tag{1.4}$$

$$(\psi_1 - \psi_2)_x \sim -1, \quad -Ct^p \le (\psi_1 - \psi_2)_t \le 0, \quad (C > 0). \tag{1.5}$$

We shall prove (under some additional conditions on ψ_i) that Γ_i is given by $x = \zeta_i(t)$ and

$$\zeta_1(t) \sim \frac{t^p}{4\alpha}, \quad \zeta_2(t) \sim -\frac{t^p}{4\alpha}. \tag{1.6}$$

In Sections 5–8 we take, for simplicity,

$$f(x, t) = \alpha/t^p \tag{1.7}$$

and prove, under somewhat stronger assumptions on the ψ_i, that if $p > 1$ then

$$\begin{aligned} \zeta_1'(t) &= \frac{pt^{p-1}}{4\alpha} + O(t^{3p-2}), \\ \zeta_2'(t) &= -\frac{pt^{p-1}}{4\alpha} + O(t^{3p-2}) \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \zeta_1(t) &= \frac{t^p}{4\alpha} - \frac{pt^{3p-1}}{96\alpha^2} + O(t^{4p}), \\ \zeta_2(t) &= -\frac{t^p}{4\alpha} + \frac{pt^{3p-1}}{96\alpha^2} + O(t^{4p}). \end{aligned} \tag{1.9}$$

The problem of studying the asymptotic behavior of the free boundary under conditions such as (1.3)–(1.5) arise in some models of stochastic control with partial observation. In Section 2 we introduce such a model, taken from sequential analysis with two or three hypotheses, and show how the estimates (1.8), (1.9) can be applied.

Knerr [12] has considered the variational inequality (1.1), (1.2) in the half-space $x > 0$ with $\Psi \equiv 0$ and a Neumann condition $u_x = g$ on $x = 0$. He proved, under certain assumptions on f, g (similar to (1.4)) that

$$\frac{\zeta(t)}{t^p} \sim c \quad (c \text{ constant})$$

where $x = \zeta(t)$ is the free boundary.

Breakwell and Chernoff [1] have considered a special case of ψ_1, ψ_2 (with $\psi_1(x, t) = \psi_2(-x, t)$) and $f = \alpha/t^2$ and established an asymptotic series for $\zeta_1(t)$ (here $\zeta_2(t) = -\zeta_1(t)$).

The methods in both [1] and [12] do not provide any estimates on $\zeta_i'(t)$ as $t \to 0$. In Section 2 we shall give more information on the results of [1].

2. Applications to sequential analysis

2.1. *Two hypotheses.* Let $z(t)$ be a one-dimensional stochastic process given by

$$dz(t) = y\,dt + \sigma\,dw(t), \quad z(0) = 0, \tag{2.1}$$

where $w(t)$ is a normalized Brownian motion, $\sigma > 0$, and

$$y = \mu_0 + \xi;$$

ξ is a normal variable $N(0, \sigma_0^2)$ independent of the Brownian motion and μ_0 is a real number. Denote by $\mathscr{F}_t$ the σ-field generated by $z(\lambda)$, $0 \le \lambda \le t$.

We impose two composite hypotheses:

H_1. Accept that $y > 0$.
H_2. Accept that $y \le 0$.

Let δ be a variable taking values $\delta = 1$ if H_1 is accepted and $\delta = 2$ if H_2 is accepted. We define the risk function

$$W(\mu, \delta) = \begin{cases} k\,|\mu| & \text{if } \delta = 1,\ \mu \le 0 \quad \text{or if } \delta = 2,\ \mu > 0, \\ 0 & \text{in all other cases,} \end{cases}$$

where k is a positive constant. We assume that the cost of observation of the process $z(t)$ is c, per unit time ($c > 0$). Then the total cost of observation and accepting a hypothesis is

$$J_{\mu_0,\sigma_0}(\tau, \delta) = E[c\tau + W(\mu, \delta(\omega))] \tag{2.2}$$

where $\tau = \tau(\omega)$ is an $\mathscr{F}_t$ stopping time and $\delta = \delta(\omega)$ is $\mathscr{F}_\tau$ measurable. The sequential analysis problem is to find τ^*, δ^* such that

$$J_{\mu_0,\sigma_0}(\tau^*, \delta^*) = \min_{\tau,\delta} J_{\mu_0,\sigma_0}(\tau, \delta). \tag{2.3}$$

Set

$$\phi(u) = \frac{e^{-u^2/2}}{\sqrt{2\pi}}, \tag{2.4}$$

$$\Phi(u) = \int_{-\infty}^{u} \phi(v)\,dv, \tag{2.5}$$

$$\psi(u) = \begin{cases} \phi(u) + u\Phi(u) & \text{if } u \le 0, \\ \phi(-u) - u\Phi(-u) & \text{if } u > 0, \end{cases} \tag{2.6}$$

$$s = \frac{1}{t/\sigma^2 + 1/\sigma_0^2} \equiv p(t), \tag{2.7}$$

$$\Psi(x, s) = \sqrt{s}\,\psi(x/\sqrt{s}). \tag{2.8}$$

As shown in [10], if u is the solution of the variational inequality

$$\left.\begin{aligned} &u_s - \tfrac{1}{2}u_{xx} \le \frac{c\sigma^2}{s^2} \\ &u(x, s) \le k\Psi(x, s) \\ &\left(u_s - \tfrac{1}{2}u_{xx} - \frac{c\sigma^2}{s^2}\right)(u - k\Psi) = 0 \end{aligned}\right\} \quad \text{a.e. in } x \in \mathbf{R}^1,\ s > 0. \tag{2.9}$$

with

$$u(x, 0) = 0, \quad x \in \mathbf{R}^1, \tag{2.10}$$

then

$$u(\mu_0, \sigma_0^2) = \inf_{(\tau, \delta)} J_{\mu_0, \sigma_0}(\tau, \delta), \tag{2.11}$$

and the optimal decision (τ^*, δ^*) is given as follows:

Denote by S the *stopping set* (or coincidence set)

$$S = \{(x, s);\ u(x, s) = \Psi(x, s)\}$$

and by C the *continuation set* (or non-coincidence set)

$$C = \{(x, s);\ u(x, s) < \Psi(x, s)\}.$$

Then τ^* is the hitting time of S by $(\tilde{y}(s), s)$ and $\delta^* = 1$ if $\tilde{y}(s) > 0$, $\delta^* = 2$ if $\tilde{y}(s) \le 0$, where

$$\tilde{y}(s) = \hat{y}(t) \quad \text{(recall (2.7))}$$

and

$$\hat{y}(t) = E[y \mid \mathscr{F}_t]$$

is the filter of y.

The filter satisfies

$$d\hat{y}(t) = \frac{p(t)}{\sigma}\, d\hat{w}(t), \quad \hat{y}(0) = \mu_0$$

where $\hat{w}(t)$ is a certain Brownian motion adapted to $\mathscr{F}_t$.

If we set

$$\psi_1(x, s) = \psi_2(-x, s) = \Psi(x, s)$$

and replace u by ku, then the variational inequality (2.9), (2.10) reduces to the variational inequality (1.1), (1.2) with $\alpha = c\sigma^2/k$. The present functions ψ_i satisfy

$$L\psi_i = 0, \quad (\psi_1 - \psi_2)_x \equiv -1, \quad (\psi_1 - \psi_2)_t \equiv 0.$$

Since it is not practical to carry out the sampling of the process $(\tilde{y}(s), s)$ for arbitrarily small s (i.e., for arbitrarily large times t), it is important to know the error incurred when we arbitrarily stop sampling at some small value of s. This error can be determined in terms of the error incurred in the computation of $u(x, s)$, or in the computation of the location of the free boundary for small s.

The asymptotic formulas to be derived in this paper (Theorems 7.4, 8.1) require much weaker assumptions on Ψ than in the present case. Consequently, if we specialize those theorems, taking also $p = 2$, we conclude that

$$\zeta_1(s) = \frac{s^2}{4\alpha} - \frac{s^5}{48\alpha^2} + O(s^8), \tag{2.12}$$

$$\zeta_1'(s) = -\frac{s}{2\alpha} + O(s^4). \tag{2.13}$$

For this problem of sequential analysis Breakwell and Chernoff [1] have derived an asymptotic series

$$\zeta_1(s) \sim \frac{s^2}{4\alpha}\left(1 + \sum_{j=1}^{\infty} \gamma_j \frac{s^{3j}}{j}\right) \tag{2.14}$$

and computed the first five γ_j. Their method, which is entirely different than ours, is based on comparison with problems having suitably perturbed cost functions J; they employ specific solutions of the heat equation adapted to the special form of f and Ψ.

It is reasonable to expect that our method, which is based on repeated bootstrap arguments, can yield additional terms of an asymptotic expansion than already derived in (8.1)–(8.4), but the calculations become rather complicated. The method of Breakwell and Chernoff yields more easily the terms in the asymptotic expansion (for special choices of f, ψ_i). On the other hand that method does not provide any information on the derivative of the free boundary.

Remark 2.1. The proof in the appendix in [1] relies on probabilistic considerations involving discretized stochastic control problems, and can be simplified by appealing to the maximum principle. Thus, setting

$$L_0 w \equiv \tfrac{1}{2} w_{xx} + \frac{x}{t} w_x + w_t,$$

it suffices to prove for v_r, D_r that

$$v_r \le D_r, \quad L_0 v_r + 1 \ge 0, \quad \text{and} \quad v_r < D_r \text{ between } \pm\rho_r.$$

Only the last inequality is not immediate. One can prove it by applying the maximum principle between $x = 0$ and ρ_r to $(v_r - D_r)_x$ (since $L(v_r - D_r)_x = 0$ in this region) to deduce that $(v_r - D_r)_x < 0$ and, consequently, that $v_r - D_r < 0$ between $x = 0$ and ρ_r (the proof between $x = 0$ and $-\rho_r$ is the same).

2.2. *Three hypotheses.* We shall now consider a case where the ψ_i are not symmetric. We take the same process (2.1), but introduce three hypotheses:

H_1. Accept that $y > a$.
H_2. Accept that $-a < y < a$.
H_3. Accept that $y < -a$.

Here, a is a given positive number.

Let δ be a variable taking the value j if H_j is accepted. We define the risk function for accepting the wrong hypothesis:

$$\begin{aligned} W(y, 1) &= k(a - y) && \text{if } y < a, \\ W(y, 3) &= k(y + a) && \text{if } y > -a, \\ W(y, 2) &= k(y - a) && \text{if } y > a, \\ W(y, 2) &= k(-y - a) && \text{if } y < -a, \end{aligned}$$

and

$$W(y, j) = 0 \quad \text{in all other cases;}$$

here k is a given positive number.

We take the cost of observation of the process $z(t)$ to be c per unit time ($c > 0$). We are interested in the cost function (2.2) and in the problem (2.3).

As in 2.1 this problem can be reduced (see [10]) to a variational inequality (2.9), (2.10), but now

$$\Psi(x, s) = \min_{1 \le j \le 3} \psi_j(x, s),$$

where

$$\begin{aligned} \psi_1(x, s) &= \sqrt{s}\,\phi\left(\frac{a - x}{\sqrt{s}}\right) + (a - x)\Phi\left(\frac{a - x}{\sqrt{s}}\right), \\ \psi_3(x, s) &= \psi_1(-x, s), \\ \psi_2(x, s) &= \sqrt{s}\,\phi\left(\frac{x - a}{\sqrt{s}}\right) + (x - a)\Phi\left(\frac{x - a}{\sqrt{s}}\right) \\ &\quad + \sqrt{s}\,\phi\left(\frac{a + x}{\sqrt{s}}\right) - (a + x)\Phi\left(-\frac{a + x}{\sqrt{s}}\right), \end{aligned}$$

and ϕ, Φ are defined by (2.4), (2.5). The relations (2.3) and (2.11) are valid, τ^* is the hitting time by $(\tilde{y}(s), s)$ of the set $\{u = \Psi\}$, and $\delta^* = j$ if $(\tilde{y}(s), s)$ lies on $\{u = \psi_j\}$.

One verifies that $L\psi_j = 0$, $\psi_1(a, 0) = \psi_2(a, 0) = 0$, and

$$\frac{\partial}{\partial x}(\psi_1 - \psi_2) = -1 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-(a+x)/\sqrt{t}} e^{-\lambda^2/2}\, d\lambda, \tag{2.15}$$

$$\frac{\partial}{\partial t}(\psi_1 - \psi_2) = -\frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{t}} e^{-(a+x)^2/(2t)}. \tag{2.16}$$

Since the problem is symmetric withrespect to the t-axis and $u(x, s) = u(-x, s)$, it suffices to study the variational inequality in $x > 0$, adding the boundary condition $u_x(0+0, t) = 0$. We recall [10] that the free boundary does not intersect the curves where $\psi_i = \psi_j$ for $i \neq j$; in particular, it does not intersect the t-axis. For this reason it suffices to verify the conditions (3.4), (3.5) uniformly in $x > 0$ (rather than $x \in \mathbf{R}^1$); these conditions follow immediately from (2.15), (2.16).

All the other (local) conditions of Theorem 8.1 are satisfied about the point $(a, 0)$ (instead of $(0, 0)$) with $p = 2$ and r, q, σ and ν being any positive numbers; in fact, the powers of t in (5.1)–(5.5) can be replaced by $O(e^{-b/t})$ for any $0 < b < 2a$. We conclude that

$$\zeta_1(s) = a + \frac{s^2}{4\alpha} - \frac{s^5}{48\alpha^2} + O(s^8), \quad \zeta_2(s) = a - \frac{s^2}{4\alpha} + \frac{s^5}{48\alpha^2} + O(s^8),$$

$$\zeta_1'(s) = \frac{s}{2\alpha} + O(s^4), \quad \zeta_2'(s) = -\frac{s}{2\alpha} + O(s^4).$$

The curve $x = \gamma(s)$ (where $\psi_1 = \psi_2$) is monotone decreasing in some interval $0 < s < s^*$, and

$$\gamma(s^*) = 0, \quad -\infty < \gamma'(s^*) < 0,$$

as easily verified. The curve $x = \zeta_2(s)$ is montone decreasing in some interval $0 \leq s \leq s^{**}$ ($s^{**} < s^*$) and $\zeta_2(s^{**}) = 0$, and the curve $x = \zeta_1(s)$ is monotone increasing for s in some small interval $0 < s < \tilde{s}$. We do not know whether $\zeta_1(s)$ is monotone increasing for all $s > 0$.

Remark 2.2. The methods of this paper should extend to the case where

$$(\psi_1 - \psi_2)_x = -|x|^k(1 + o(1)) \quad \text{as } t \to 0,$$

where $k > 0$, i.e., the obstacles ψ_1, ψ_2 are "weakly separated" near the set $\{\psi_1 = \psi_2\}$. The asymptotic behavior of $\zeta_i(t)$ will now depend on both p and k.

Remark 2.3. The reader will find in [5] references to several other problems in sequential analysis which lead to variational inequalities with singularities at $s = 0$ or $s = \infty$. The methods of this paper should be useful in studying the asymptotic behavior of the corresponding free boundaries.

3. Properties of the solution of the variational inequality

Set

$$L = \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}.$$

Throughout Sections 3 and 4 we impose the following conditions:

(3.1) $f(x, t)$ is a positive, bounded, continuously differentiable function in $\mathbf{R}^1 \times (0, T]$, and

$$t^p f(x, t) \to \alpha \text{ if } t \to 0, \quad \text{uniformly in } x \ (\alpha > 0,\ p > \tfrac{1}{2});$$

(3.2) $\psi_i(x, t)$ $(i = 1, 2)$ is a nonnegative, bounded, continuous function in $\mathbf{R}^1 \times [0, T]$, and its first three derivatives are bounded continuous functions in every strip $\mathbf{R}^1 \times [\varepsilon, T]$, $\varepsilon > 0$;

(3.3) $\psi_1(0, 0) = \psi_2(0, 0) = 0$;

(3.4) $(\psi_1 - \psi_2)_x = -1 + o(1)$ where $o(1) \to 0$ if $t \to 0$, uniformly in x;

(3.5) $(\psi_1 - \psi_2)_t \leq 0$, $|(\psi_1 - \psi_2)_t| \leq Ct^p$ $(C > 0)$.

The next set of conditions are imposed in a rectangle

$$S = \{(x, t);\ -\delta_* < x < \delta_*,\ 0 < t < \delta_*\}$$

with some $\delta_* > 0$:

$$t^p L\psi_2 \to 0 \text{ if } t \to 0, \quad \text{uniformly in } S. \tag{3.6}$$

$$(L\psi_1 - f)_x \leq 0 \quad \text{and} \quad (L\psi_2 - f)_x \geq 0 \text{ in } S. \tag{3.7}$$

$$(L\psi_2 - f)_t \geq 0 \text{ in } S; \tag{3.8}$$

$$(L\psi_1 - f)_t \geq \frac{\Omega(t)}{t^p} \text{ in } S, \quad \Omega(t) \to \infty \text{ if } t \to 0. \tag{3.9}$$

For any $\varepsilon > 0$ denote by u^ε the solution of the variational inequality

$$\left.\begin{aligned} &Lu^\varepsilon \leq f \\ &u^\varepsilon \leq \Psi \\ &(Lu^\varepsilon - f)(u^\varepsilon - \Psi) = 0 \end{aligned}\right\} \quad \text{a.e. in } x \in \mathbf{R}^1,\ 0 < t < T, \tag{3.10}$$

with the initial condition

$$u^\varepsilon(x, \varepsilon) = \Psi(x, \varepsilon), \tag{3.11}$$

where

$$\Psi = \min(\psi_1, \psi_2).$$

One can easily show that $u^\varepsilon \geq 0$. By general regularity results [3], u_x^ε, u_{xx}^ε, u_t^ε are locally bounded by a constant independent of ε. Hence we can take a sequence $\varepsilon \to 0$ such that $u^\varepsilon(x, t) \to u(x, t)$ uniformly in compact subsets of $\mathbf{R}^1 \times (0, T]$, and u is a solution of the variational inequality (1.1); (1.2) follows from Lemma 3.3 below; cf. (3.24).

The uniqueness of u in the class of bounded functions can be proved either by a standard variational inequality technique [12] or by the usual stochastic representation of the bounded solution of (1.1) [9].

From (3.3)–(3.5) it follows that the set

$$\{(x, t);\ \psi_1(x, t) = \psi_2(x, t)\} \cap \{t < t^*\}$$

is given by a curve

$$x = \gamma(t), \quad \gamma(0) = 0, \quad \gamma'(t) \le 0, \quad |\gamma'(t)| = O(t^p); \tag{3.12}$$

here, and in the sequel, t^* is a sufficiently small positive number.

The curve $x = \gamma(t)$ must lie in the non-coincidence set of both u^ε and any solution u of (1.1) (see [10]). It follows that the set $\{u = \psi_1\}$ lies to the right of $x = \gamma(t)$ whereas the set $\{u = \psi_2\}$ lies to the left of $x = \gamma(t)$ (for $0 < t < t^*$).

We denote by Γ_i^ε the free boundary of u^ε where $u^\varepsilon = \psi_i$.

LEMMA 3.1. *For any $\delta > 0$ there exists a small number t_δ independent of ε, such that, if $0 < t < t_\delta$, $u^\varepsilon(x, t) < \Psi(x, t)$, then*

$$x < \frac{(1 + \delta)t^p}{4\alpha}, \tag{3.13}$$

$$x > -\frac{(1 + \delta)t^p}{4\alpha}. \tag{3.14}$$

Proof. Suppose $u^\varepsilon(x_0, t_0) < \Psi(x_0, t_0)$ and set

$$U = v - \frac{\alpha'}{t_0^p}(x - x_0)^2$$

where $v = \psi_1 - u^\varepsilon$ and $0 < \alpha' < \alpha$. In the open set

$$G = \{(x, t);\ \varepsilon < t < t_0,\ u^\varepsilon(x, t) < \Psi(x, t)\},$$

the function U satisfies

$$U_t - \tfrac{1}{2}U_{xx} < 0$$

provided t_0 is sufficiently small (depending on $\alpha - \alpha'$). Hence U must take its maximum in $\bar{G}$ on the parabolic boundary of G; this maximum is positive since $U(x_0, t_0) > 0$. Notice now that $U(x, \varepsilon) < 0$ if $x > \gamma(\varepsilon)$, $U \le 0$ on Γ_1^ε, and $U(x, t) \to -\infty$ if $x \to \pm\infty$. Hence U must take positive values at some points $(\bar{x}, \bar{t})$ of ∂G which lie either on Γ_2^ε or on $t = \varepsilon$ (with $\bar{x} < \gamma(\varepsilon)$). In either case we have

$$v(\bar{x}, \bar{t}) > \frac{\alpha'(\bar{x} - x_0)^2}{t_0^p}$$

and

$$v(\bar{x}, \bar{t}) = \psi_1(\bar{x}, \bar{t}) - \psi_2(\bar{x}, \bar{t}).$$

Recalling (3.4), (3.12) we get

$$v(\bar{x}, \bar{t}) = -\bar{x}(1 + o(1)) + o(t^p).$$

Hence

$$-\bar{x}(1 + o(1)) \ge \frac{\alpha'}{t_0^p}(x_0 - \bar{x})^2 + o(t^p). \tag{3.15}$$

Setting

$$-\bar{x} = \frac{B^2\bar{t}^p}{4\alpha'}, \quad x_0 = \frac{A^2 t_0^p}{4\alpha'}, \quad \frac{\bar{t}}{t_0} = \lambda\ (\lambda \le 1),$$

we obtain

$$(1 + o(1))4B^2\lambda^p \ge (A^2 + B^2\lambda^p)^2 + o(1),$$

so that

$$A^2 + B^2\lambda^p \le 2B\lambda^{p/2}(1 + o(1)) + o(1).$$

It follows that

$$\begin{aligned} 0 &\le (B\lambda^{p/2} - (1 + o(1)))^2 = B^2\lambda^p - 2B\lambda^{p/2}(1 + o(1)) + (1 + o(1))^2 \\ &\le 1 - A^2 + o(1). \end{aligned}$$

Consequently

$$A^2 \le 1 + o(1),$$

that is

$$x_0 \le \frac{t_0^p}{4\alpha'}(1 + o(1)).$$

This gives the assertion (3.13). The proof of (3.14) is similar.

LEMMA 3.2. *The following inequalities hold for* $x \in \mathbf{R}^1$, $0 < t < t^*$:

$$(\psi_1 - u^\varepsilon)_x \le 0; \tag{3.16}$$

$$(\psi_2 - u^\varepsilon)_x \ge 0; \tag{3.17}$$

$$(\psi_2 - u^\varepsilon)_t \ge 0. \tag{3.18}$$

Proof. The function $w = \psi_2 - u^\varepsilon$ satisfies the variational inequality

$$\left.\begin{aligned} &w_t - \tfrac{1}{2}w_{xx} \ge L\psi_2 - f \\ &w \ge \tilde{\psi} \\ &(w_t - \tfrac{1}{2}w_{xx} - (L\psi_2 - f))(w - \tilde{\psi}) = 0 \end{aligned}\right\} \quad \text{a.e. in } x \in \mathbf{R}^1,\ \varepsilon < t < T, \tag{3.19}$$

and

$$w(x, \varepsilon) = \tilde{\psi}(x, \varepsilon), \tag{3.20}$$

where $\tilde{\psi} = \psi_2 - \Psi$. In view of (3.4), (3.5),

$$\tilde{\psi}_t \ge 0, \quad \tilde{\psi}_{xx} \ge -C_0\ (C_0 \ge 0),$$

where $\tilde{\psi}_{xx}$ is taken in the sense of distributions (since $\tilde{\psi}_x$ has a jump across $x = \gamma(t)$); C_0 may depend on ε.

Let $\tilde{\psi}_\delta$ be a smooth function in (x, t) (for each $\delta > 0$) satisfying

$$\tilde{\psi}_\delta \to \tilde{\psi} \text{ pointwise} \quad (\text{as } \delta \to 0),$$

$$\frac{\partial}{\partial t} \tilde{\psi}_\delta \geq 0, \quad \frac{\partial^2}{\partial x^2} \tilde{\psi}_\delta \geq -C_0.$$

Let $\beta_\delta(t)$ be C^∞ functions in t satisfying

$$\beta_\delta(t) \to -\infty \quad \text{if } t < 0, \ \delta \to 0,$$

$$\beta_\delta(t) \to 0 \quad \text{if } t > 0, \ \delta \to 0,$$

$$\beta'_\delta(t) \geq 0,$$

$$\beta_\delta(0) = A_0 - \tfrac{1}{2}C_0 \quad \left(A_0 = \inf_x (L\psi_2 - f)(x, \varepsilon)\right).$$

Notice (by (3.1) and (3.6)) that $A_0 \leq 0$ if ε is small enough, so that $\beta_\delta(0) \leq 0$.

Consider the penalized problem

$$W_t - \tfrac{1}{2}W_{xx} + \beta_\delta(W - \tilde{\psi}_\delta) = L\psi_2 - f \quad (x \in \mathbf{R}^1, 0 < \varepsilon < T), \tag{3.21}$$

$$W(x, \varepsilon) = \tilde{\psi}_\delta(x, \varepsilon) \quad (x \in \mathbf{R}^1). \tag{3.22}$$

Then $Z = \partial W/\partial t$ satisfies

$$Z_t - \tfrac{1}{2}Z_{xx} + \beta'_\delta(W - \tilde{\psi}_\delta)Z = \beta'_\delta(W - \tilde{\psi}_\delta) \frac{\partial \tilde{\psi}_\delta}{\partial t} + (L\psi_2 - f)_t \geq 0$$

where (3.8) was used. Also, by (3.21) and (3.22),

$$Z(t, \varepsilon) \geq \frac{1}{2} \frac{\partial^2}{\partial x^2} \tilde{\psi}_\delta + A_0 - \beta_\delta(0) \geq 0.$$

We can therefore apply the maximum principle to conclude that $Z(x, t) \geq 0$. Taking $\delta \to 0$ the assertion (3.18) follows.

The proofs of (3.16) and (3.17) is similar.

From Lemmas 3.1 and 3.2 it follows that there exist two curves

$$x = \zeta_{i\varepsilon}(t) \quad (\varepsilon < t < t^*; \ i = 1, 2)$$

such that $\zeta_{2\varepsilon}(t)$ is monotone decreasing, and

$$u^\varepsilon(x, t) = \psi_1(x, t) \quad \text{if and only if } x > \zeta_{1\varepsilon}(t),$$

$$u^\varepsilon(x, t) = \psi_2(x, t) \quad \text{if and only if } x < \zeta_{2\varepsilon}(t).$$

These curves lie in a parabolic region

$$|x| < \frac{\delta(t)t^p}{4\alpha} \quad (\delta(t) \to 1 \text{ if } t \to 0)$$

and $\zeta_{2\varepsilon}(t) < \gamma(t) < \zeta_{1\varepsilon}(t)$.

LEMMA 3.3. *If* $\zeta_{2\varepsilon}(t) < x < \zeta_{1\varepsilon}(t)$, $0 < t < t^*$, *then*

$$0 \le \psi_i - u_\varepsilon \le \frac{2t^p}{\alpha}. \tag{3.23}$$

Proof. Observe that

$$\begin{aligned}\tfrac{1}{2}(\psi_1 - u^\varepsilon)_{xx} &= (\psi_1 - u^\varepsilon)_t + f - L\psi_1 \\ &= (\psi_1 - \psi_2)_t + (\psi_2 - u^\varepsilon)_t + f - L\psi_1 \\ &> 0\end{aligned}$$

by (3.18) and (3.1), (3.5), (3.6). Also, since

$$(\psi_1 - u^\varepsilon)_x = 0 \quad \text{on } x = \zeta_{1\varepsilon}(t),$$
$$(\psi_1 - u^\varepsilon)_x = (\psi_1 - \psi_2)_x = -1 + o(1) \quad \text{on } x = \zeta_{2\varepsilon}(t),$$

we deduce that

$$|(\psi_1 - u^\varepsilon)_x| \le 1 + o(1).$$

Combining this with Lemma 3.1 and the fact that $\psi_1 - u^\varepsilon = 0$ on $x = \zeta_{1\varepsilon}(t)$, we obtain

$$\psi_1 - u^\varepsilon \le \frac{t^p}{2\alpha}(1 + o(1)),$$

Observing that (for $\zeta_{2\varepsilon}(t) < x < \zeta_1(t)$)

$$|\psi_2 - \psi_1| \le \frac{t^p}{2\alpha}(1 + o(1)),$$

we then also get

$$\psi_2 - u^\varepsilon \le \frac{t^p}{\alpha}(1 + o(1)),$$

and the proof of the lemma is complete.

We now let $\varepsilon \to 0$. Then we obtain for $u = \lim u^\varepsilon$ the inequalities (3.16)–(3.18). It follows that the free boundary in a strip $0 < t < t_*$ consists of two curves

$$x = \zeta_i(t),$$

$\zeta_2(t)$ is monotone decreasing and $\zeta_i(0) = 0$. Since (3.23) holds also for u, we conclude that

(3.24) $u(x, t)$ is continuous as $t \downarrow 0$; $u(x, 0) = 0$.

We can therefore duplicate the proof of Lemma 3.1 for u.

We summarize the results obtained so far for u.

THEOREM 3.4. *The free boundary for u, in $0 < t < t^*$, consists of two curves $x = \zeta_i(t)$, where $\zeta_2(t)$ is monotone decreasing, and the following inequalities hold for $\zeta_2(t) < x < \zeta_1(t), 0 < t < t^*$:*

$$(\psi_1 - u)_x < 0, \tag{3.25}$$

$$(\psi_2 - u)_x > 0, \tag{3.26}$$

$$(\psi_2 - u)_t > 0, \tag{3.27}$$

$$0 \le \psi_i - u \le \frac{2t^p}{\alpha}, \tag{3.28}$$

$$\zeta_1(t) \le \frac{t^p}{4\alpha}(1 + o(1)), \tag{3.29}$$

$$\zeta_2(t) \ge -\frac{t^p}{4\alpha}(1 + o(1)). \tag{3.30}$$

The strict inequalities in (3.25)–(3.27) follow by applying the strong maximum principle to each of the nonnegative functions $(u - \psi_1)_x$, $(\psi_2 - u)_x$, $(\psi_2 - u)_t$.

Using (3.25), (3.26) we conclude, by well-known results on variational inequalities (see, for instance, [4]), that the $\zeta_i(t)$ are $C^{k+2+\beta}$ curves (for $0 < t < t^*$) provided f is in $C^{k+\beta}$ and the ψ_i belong to $C^{k+2+\beta}$; here k is any nonnegative integer and $0 < \beta < 1$.

4. Monotonicity of $\zeta_1(t)$; lim $\zeta_i(t)/t^p$ exists.

THEOREM 4.1. *If $\gamma(t) < x < \zeta_1(t), 0 < t < t^*$, then*

$$(\psi_1 - u)_t > 0; \tag{4.1}$$

consequently $\zeta_1(t)$ is monotone increasing.

Set

$$\rho_1(t) = \zeta_1(t), \quad \rho_2(t) = -\zeta_2(t). \tag{4.2}$$

First we prove:

LEMMA 4.2. *For any $0 < \theta_0 < 1$,*

$$\rho_2(t) \ge \theta_0 \frac{t^p}{4\alpha} \tag{4.3}$$

if t is sufficiently small.

Proof. The function $w = \psi_2 - u$ satisfies

$$w_t - \tfrac{1}{2}w_{xx} = L\psi_2 - f$$

if $\zeta_2(t) < x < \zeta_1(t)$. Integrating this equation over (x, t) and recalling $L\psi_2 - f \sim \alpha t^{-p}$ and (3.29), (3.30), we get

$$\int_{\zeta_2(t)}^{\zeta_1(t)} w(x, t)\, dx - \frac{1}{2}\int_0^t w_x(\zeta_1(s), s)\, ds = -\int_0^t \frac{\alpha(\rho_1(s) + \rho_2(s))}{s^p}\, ds + o(t).$$

As $x \to \zeta_1(s)$,

$$w_x \to (\psi_2 - \psi_1)_x = 1 + o(1).$$

Also

$$w(x, s) \leq 2s^p/\alpha, \quad \zeta_1(t) - \zeta_2(t) \leq t^p/\alpha.$$

Using these facts we obtain

$$\begin{aligned}\int_0^t \frac{\alpha}{s^p}(\rho_1(s) + \rho_2(s))\, ds &= \frac{t}{2} + O(t^{2p}) + o(t)\\ &= \frac{t}{2} + o(t) \quad (\text{since } p > \tfrac{1}{2}).\end{aligned}$$

In view of (3.29) we also have

$$\int_0^t \frac{\alpha}{s^p}\rho_1(s)\, ds \leq \frac{t}{4} + o(t).$$

Hence

$$\int_0^t \frac{\alpha}{s^p}\rho_2(s) \geq \frac{t}{4} + o(t). \tag{4.4}$$

Next we write, for any $\theta_0 < \theta < 1$,

$$\begin{aligned}\int_0^t \frac{\alpha}{s^p}\rho_2(s)\, ds &= \int_0^{\theta t} \frac{\alpha}{s^p}\rho_2(s)\, ds + \int_{\theta t}^t \frac{\alpha}{s^p}\rho_2(s)\, ds\\ &\equiv I_1 + I_2.\end{aligned} \tag{4.5}$$

In view of (3.30),

$$I_1 \leq \int_0^{\theta t} \frac{\alpha s^p}{4\alpha s^p}\, ds + o(t) = \frac{\theta t}{4} + o(t).$$

Since $\rho_2(t)$ is monotone increasing,

$$I_2 \leq \rho_2(t)\int_{\theta t}^t \frac{\alpha\, ds}{s^p} \leq \alpha\rho_2(t)\,\frac{1-\theta}{\theta^p}\,\frac{1}{t^{p-1}}.$$

Substituting these estimates into (4.5) and comparing with (4.4) we find that

$$\rho_2(t) \geq \frac{\theta^p t^p}{4\alpha} + o(t^p),$$

and (4.3) follows.

Proof of Theorem 4.1. Consider the function

$$W = (\psi_1 - u)_t$$

in a domain G defined

$$\zeta_2(t) < x < \zeta_1(t), \quad 0 < t < t_0 \ (t_0 \text{ small}).$$

By (3.5) and (3.27) we have

$$W = (\psi_1 - \psi_2)_t + (\psi_2 - u)_t \geq (\psi_1 - \psi_2)_t \geq -Ct^p. \tag{4.6}$$

Also

$$W(\zeta_1(t), t) = 0, \qquad W_t - \tfrac{1}{2}W_{xx} = (L\psi_1 - f)_t.$$

Consider the function

$$Z = -Mt^{-p}(x - \gamma(t))^2$$

in the same domain G. On $x = \zeta_2(t)$,

$$Z = -Mt^{-p}(\zeta_2(t) - \gamma(t))^2 \leq -cMt^p \quad (c > 0)$$

by Lemma 4.2 (where c is independent of M), so that $Z < W$ on $x = \zeta_2(t)$ if M is large enough (we use here (4.6)).

Also

$$Z < 0 = W \quad \text{on } x = \zeta_1(t),$$

$$\liminf_{t \to 0} (W - Z) \geq 0.$$

Finally, with M fixed, we have

$$\begin{aligned} Z_t - \tfrac{1}{2}Z_{xx} &= M\{pt^{-p-1}(x - \gamma(t))^2 + 2t^{-p}(x - \gamma(t))\gamma'(t) + t^{-p}\} \\ &\leq M\{Ct^{-p-1}t^{2p} + t^{-p}\} \\ &\leq 2Mt^{-p} \quad (C > 0). \end{aligned}$$

It follows, by (3.9), that

$$Z_t - \tfrac{1}{2}Z_{xx} < (L\psi_1 - f)_t$$

if t is small enough.

We can therefore apply the maximum principle to conclude that $Z < W$ in G and, in particular,

$$W(\gamma(t), t) > 0.$$

In the domain $G_1 = \{(x, t); \gamma(t) < x < \zeta_1(t), 0 < t < t_0\}$ we then have

$$W_t - \tfrac{1}{2}W_{xx} > 0;$$

also

$$W > 0 \quad \text{on } x = \gamma(t),$$
$$W = 0 \quad \text{on } x = \zeta_1(t),$$
$$\liminf_{t \to 0} W \geq 0.$$

Hence, by the maximum principle, $W > 0$ in G_1, and (4.1) follows.

Having proved that $\zeta_1(t)$ is monotone increasing for $0 < t < t_0$, we can apply the proof of Lemma 4.2 to $\rho_1(t)$, instead of $\rho_2(t)$, and deduce that, for any $0 < \theta_0 < 1$,

$$\rho_1(t) \geq \frac{\theta_0 t^p}{4\alpha} \quad \text{if } t \text{ is small enough.} \tag{4.7}$$

Combining (4.7) with (4.3) and (3.29), (3.30), we obtain:

THEOREM 4.3.

$$\lim_{t \to 0} \frac{\zeta_1(t)}{t^p} = \frac{1}{4\alpha}, \quad \lim_{t \to 0} \frac{\zeta_2(t)}{t^p} = -\frac{1}{4\alpha}. \tag{4.8}$$

The assumption $p > 1/2$ is essential for the validity of (4.8).

To show this we first prove the following result, which is of intrinsic interest.

THEOREM 4.4. *Let w be a solution of the variational inequality*

$$w_t - \tfrac{1}{2} w_{xx} \geq f,$$
$$w \geq 0,$$
$$(w_t - \tfrac{1}{2} w_{xx} - f) w = 0$$

a.e. in $x > 0$, $0 < t < T$, and

$$w_x(0, t) = g(t) \quad \textit{for } 0 < t < T,$$
$$w(x, 0) = 0 \quad \textit{for } x > 0;$$

w is continuous for $x \geq 0$, $t \geq 0$. If

$$-C_1 \leq f(x, t) \leq -c_1, \quad -C_2 \leq g(t) \leq -c_2$$

where C_i, c_i are positive constants, then the free boundary lies in a region

$$(t \log 1/t)^{1/2}(1 + o(1)) < x < (t \log 1/t)^{1/2}(1 + o(1)) \tag{4.9}$$

for all t sufficiently small; $o(t) \to 0$ if $t \to 0$.

Estimates on the free boundary for the Cauchy problem for parabolic variational inequalities in n-dimensions were obtained by Brezis and Friedman [2]. Simpler proofs were given by Evans and Knerr [6]; we shall adapt their method to the present case.

Proof. Let v be the solution of

$$v_t - \tfrac{1}{2}v_{xx} = 2C_1 \quad \text{if } x > 0,\ t > 0,$$
$$v_x(0, t) = -C_2 \quad \text{if } t > 0,$$
$$v(x, 0) = 0 \quad \text{if } x > 0.$$

It is well known that

$$|w_t - \tfrac{1}{2}w_{xx}| \le 2|f| \tag{4.10}$$

(this can be shown using a penalized problem

$$w_t - \tfrac{1}{2}w_{xx} + \beta_\delta(w) = f$$

and proving for the solution $w = w_\delta$, that $|\beta_\delta(w)| \le |f|$ (by the maximum principle).)

It follows, by the maximum principle, that $v \ge w$ if $x > 0$, $t > 0$. Hence

$$w(x, t) \le v(x, t) = \frac{2C_2}{\sqrt{2\pi}} \int_0^t \frac{e^{-x^2/2s}}{\sqrt{s}}\, ds + 2C_1 t. \tag{4.11}$$

Suppose now that

$$w(x^0, t^0) > 0$$

and consider the function

$$W = w - c_1(x - x^0)^2$$

in

$$G = \{\varepsilon < x < \infty, 0 < t < t^0\} \cap \{w > 0\}.$$

Since

$$W_t - \tfrac{1}{2}W_{xx} = f + c_1 < 0 \quad \text{in } G,$$

W cannot attain local maximum at any point in G. Further, since $W(x, t) \to -\infty$ if $x \to \infty$, and since

$$W(x^0, t^0) > 0,\ W(x, 0) \le 0,\ W \le 0 \quad \text{on } \partial\{w > 0\},$$

W must attain a positive maximum in $\bar{G}$ at a point on $\partial G \cap \{x = \varepsilon\}$, say at $(\varepsilon, \bar{t})$ where $0 < \bar{t} \le t^0$. Thus

$$|\varepsilon - x^0| \le [\bar{c}w(\varepsilon, \bar{t})]^{1/2}, \quad \bar{c} = 1/c_1.$$

Recalling (4.11) and choosing $\varepsilon = x^0/2$, we obtain

$$\tfrac{1}{2}x^0 \le \left[2C_1 t^0 + \frac{4C_2}{\sqrt{2\pi}} \int_{x^0/(t^0)^{1/2}}^{\infty} \frac{e^{-\lambda^2/2}}{\lambda^2}\, d\lambda\right]^{1/2} \bar{c}^{1/2} \tag{4.12}$$

where the substitution $\lambda = x/\sqrt{s}$ has been used.

Suppose now that $x^0 = M\sqrt{t^0}$, for a sufficiently large constant M. Then (4.12) yields

$$M\sqrt{t^0} \le \frac{C}{M} e^{-M^2/2} \quad (C > 0),$$

or

$$M \le (\log 1/t^0)^{1/2}(1 + o(1)) \quad \text{as } t^0 \downarrow 0.$$

It follows that the free boundary of w lies in the region

$$x < (t \log 1/t)^{1/2}(1 + o(1)).$$

To prove the reverse inequality (with a suitable $o(1)$) we compare w with the solution $\bar{v}$ of

$$\begin{aligned} \bar{v}_t - \tfrac{1}{2}\bar{v}_{xx} &= -C_1 && \text{if } x > 0,\ t > 0, \\ \bar{v}_x(0, t) &= -c_2 && \text{if } t > 0, \\ \bar{v}(x, 0) &= 0 && \text{if } x > 0. \end{aligned}$$

By the maximum principle, $w \ge \bar{v}$ if $x > 0$, $t > 0$, i.e.,

$$w(x, t) \ge \bar{v}(x, t) = \frac{2c_2}{\sqrt{2\pi}} \int_0^t \frac{e^{-x^2/2s}}{\sqrt{s}} - C_1 t.$$

Taking $x = m\sqrt{t}$, t small, we get

$$w(m\sqrt{t}, t) > 0$$

if

$$m\sqrt{t} \int_m^\infty \frac{e^{-\lambda^2/2}}{\lambda^2}\, d\lambda - Ct \ge 0 \quad (C > 0),$$

i.e., if

$$m \le (\log 1/t)^{1/2}(1 - |o(1)|).$$

Hence the free boundary for w lies in the region

$$x > (t \log 1/t)^{1/2}(1 - |o(1)|).$$

Consider now the case where the conditions (3.1)–(3.9) are satisfied with

$$\psi_1(x, t) = \psi_2(-x, t), \quad f = \alpha/t^p.$$

In this case, $u(x, t) = u(-x, t)$ so that $u_x(0, t) = 0$. Consequently, the function $w = \psi_1 - u$ satisfies

$$w_x(0 + 0, t) = \psi_{1x}(0, t) \equiv g(t)$$

where $g(t)$ is a bounded and strictly negative function for $0 \le t \le T$. The free boundaries $\zeta_i(t)$ satisfy $\zeta_1(t) = -\zeta_2(t)$; we designate $\zeta_1(t)$ by $\zeta_1(t; p)$ in order to indicate its dependence on p. By a standard comparison theorem for variational inequalities (see, for instance, [10]), if p decreases then w increases. Consequently

$$\zeta_1(t, 1/2) \le \zeta_1(t, p) \le \zeta_1(t, 0) \quad \text{if } 0 \le p \le \tfrac{1}{2}. \tag{4.13}$$

Next, if we take $\rho_1 = \rho_2$, $p = 1/2$ in the proof of Lemma 4.2, we get

$$-2\alpha \int_0^t \frac{\rho_2(s)}{s^{1/2}}\, ds = \int_{\zeta_1(t)}^{\zeta_2(t)} w(x, t)dx - \frac{t}{2} + o(t) \ge -\frac{t}{2} + o(t).$$

Since $\rho_2(s)$ is monotone increasing, we obtain as before,

$$\rho_2(t) \ge \frac{\theta_0^{1/2} t^{1/2}}{8\alpha} \quad (\theta_0 < \theta < 1).$$

Using this and Theorem 4.4 in (4.13), we obtain:

COROLLARY 4.5. *If $\psi_1(x, t) = \psi_2(-x, t)$ and $f = \alpha/t^p$ then, for $0 \le p \le 1/2$,*

$$\frac{t^{1/2}}{8\alpha}(1 + o(1)) \le \zeta_1(t, p) \le (t \log 1/t)^{1/2}(1 + o(1)). \tag{4.14}$$

This complements Theorem 4.3 for $0 < p \le 1/2$, for the symmetric case $\psi_1(x, t) = \psi_2(-x, t)$. The non-symmetric case can be treated in a similar manner; at least for $\zeta_2(t, p)$.

5. The functions z_i

In Sections 5–8 we require additional local conditions in S, on the functions ψ_i. These conditions are:

$$(\psi_1 - \psi_2)_x = -1 + O(t^r), \quad r > 0, \tag{5.1}$$

$$(\psi_1 - \psi_2)_t = O(t^q), \tag{5.2}$$

$$(\psi_1 - \psi_2)_{tx} = O(t^\sigma), \tag{5.3}$$

$$(\psi_1 - \psi_2)_{xx} = O(t^{-p}), \tag{5.4}$$

$$(\psi_1 - \psi_2)_{tt} = O(t^\nu), \tag{5.5}$$

with suitable powers of r, q, σ and ν; here $|O(t^s)| \le Ct^s$ where C is a constant independent of x. Notice that (5.2) with $q = p$ is included in the condition (3.5) already assumed.

As for $f(x, t)$ we shall henceforth take, for simplicity,

$$f = \alpha/t^p \quad (p > 1/2,\ \alpha > 0). \tag{5.6}$$

Notice that (5.2) implies

$$\gamma(t) = O(t^{q+1}). \tag{5.7}$$

LEMMA 5.1. *If (5.1)–(5.3) hold, then*

$$(\psi_1 - \psi_2)(\zeta_i(t), t) = -\zeta_i(t) + O(t^{r+p} + t^{q+1}), \tag{5.8}$$

$$(-1)^i \frac{\partial}{\partial t} [t^p(\psi_1 - \psi_2)](\zeta_i(t), t) = -pt^{p-1}\zeta_i(t) + O(t^{r+2p-1} + t^{p+q}), \tag{5.9}$$

$$\frac{\partial^2}{\partial x\, \partial t} [t^p(\psi_1 - \psi_2)](\zeta_i(t), t) = -pt^{p-1} + O(t^{r+p-1} + t^{p+\sigma}); \tag{5.10}$$

if (5.5) also holds, then

$$\frac{d}{dt}\left[\frac{\partial}{\partial t} (t^p(\psi_1 - \psi_2))(\zeta_i(t), t)\right] \le O(t^{2p-2} + t^{p-1+q} + t^{p+\nu}) + \zeta_i'(t)\,|\,O(t^{\sigma+p} + t^{p-1})\,|. \tag{5.11}$$

Proof. The assertion (5.8) follows by (5.1) and (5.7); (5.9) follows from (5.2) and (5.8); and (5.10) follows from (5.1) and (5.3). From (5.8) we have

$$(\psi_1 - \psi_2)(\zeta_i(t), t) = O(t^p). \tag{5.12}$$

Using this and (5.3), (5.5), the assertion (5.11) easily follows.

In this section we require:

Condition (a). The conditions (5.1)–(5.4) and (5.6) hold with

$$p > 1/2 \text{ and } r \ge p + 1/2, \quad q \ge 2p - 3/2, \quad \sigma \ge p - 1/2.$$

We now introduce the functions

$$w_i = \psi_i - u, \tag{5.13}$$

$$z_i = \frac{\partial}{\partial t} (t^p w_i) \tag{5.14}$$

in a rectangle $S_0 = \{(x, t);\ |x| < \delta_0,\ t < t_0\}$ with δ_0, t_0 small. In the subset where $\zeta_2(t) < x < \zeta_1(t)$,

$$\frac{\partial}{\partial t} z_i - \frac{1}{2} \frac{\partial^2}{\partial x^2} z_i = f_i \tag{5.15}$$

where

$$f_i = \frac{\partial}{\partial t} (pt^{p-1}w_i) = \frac{pz_i}{t} - pt^{p-2}w_i. \tag{5.16}$$

One can easily compute that

$$z_i(\zeta_i(t), t) = 0, \tag{5.17}$$

and

$$\frac{\partial}{\partial x} z_i(\zeta_i(t), t) = -2\alpha\zeta_i'(t). \tag{5.18}$$

We also have

$$z_1(\zeta_2(t), t) = \frac{\partial}{\partial t}(t^p(\psi_1 - \psi_2))(\zeta_2(t), t) \tag{5.19}$$
$$= -pt^{p-1}\zeta_2(t) + O(t^{r+2p-1} + t^{p+q}),$$

and

$$z_2(\zeta_1(t), t) = \frac{\partial}{\partial t}(t^p(\psi_2 - \psi_1))(\zeta_1(t), t) \tag{5.20}$$
$$= pt^{p-1}\zeta_1(t) + O(t^{r+2p-1} + t^{p+q})$$

where (5.9) was used. Next,

$$\frac{\partial}{\partial x} z_1(\zeta_2(t), t) = \frac{\partial^2}{\partial x\, \partial t}(t^p(\psi_1 - \psi_2))(\zeta_2(t), t) - 2\alpha\zeta_2'(t),$$

so that, by (5.10),

$$\frac{\partial}{\partial x} z_1(\zeta_2(t), t) = -pt^{p-1} - 2\alpha\zeta_2'(t) + O(t^{r+p-1} + t^{p+\sigma}). \tag{5.21}$$

Similarly,

$$\frac{\partial}{\partial x} z_2(\zeta_1(t), t) = pt^{p-1} - 2\alpha\zeta_1'(t) + O(t^{r+p-1} + t^{p+\sigma}). \tag{5.22}$$

Since

$$z_2 = pt^{p-1}w_2 + t^p \frac{\partial w_2}{\partial t}, \tag{5.23}$$

it follows from (3.27) that

$$z_2 > 0 \quad \text{if } \zeta_2(t) < x < \zeta_1(t),\ t < t^*. \tag{5.24}$$

Similarly, it follows from Theorem 4.1 that

$$z_1 > 0 \quad \text{if } \gamma(t) < x < \zeta_1(t),\ t < t^*. \tag{5.25}$$

Orientation 5.1. The formulas derived for the z_i show that the functions $(-1)^i z_i$ may roughly be viewed as the temperatures of ice and water in the classical Stefan problem. Our present problem however is substantially more complicated because the "ice" and "water" occupy the same space and because there are two free boundary curves. Nevertheless the analogy with the Stefan problem will be instructive and a guiding line throughout the estimates which follow in the rest of this paper.

Orientation 5.2. It is well known that the one phase Stefan problem is equivalent to a variational inequality. Each of the two formulations has its own advantages. Until now we have worked only with the variational inequality approach (for u). From now on we shall work mainly with the "Stefan problem" formulation, that is, with the z_i.

Orientation 5.3. For the one phase Stefan problem one can obtain asymptotic estimates on the free boundary by using "conservation (of energy) laws". By this we mean that one multiplies the parabolic equation by 1 or by x and then integrates over the entire domain in x and over t, for $0 \le t \le \sigma$. The resulting equation gives an expression for the free boundary $\zeta(\sigma)$ in terms of quantities which are then estimated up to some error terms. We shall adopt this procedure here. Since however z_i may have a singularity at (0, 0) we must first proceed carefully to estimate z_i. This will be done in Lemmas 5.2–5.4. Then, in Theorem 5.5, we shall use the "conservation of energy" procedure and establish a preliminary estimate on $\zeta_i(t)$.

LEMMA 5.2. *If R and T_0 are sufficiently small,*

$$\int_0^{T_0}\int_{-R}^{R} |z_i|^\lambda \, dx \, dt < \infty \quad \textit{for any } 1 < \lambda < \infty. \tag{5.26}$$

Proof. Consider the function

$$W = t^p w_2.$$

It satisfies

$$W_t - \tfrac{1}{2}W_{xx} = -\alpha + pt^{p-1}w_2 \quad \text{if } \zeta_2(t) < x < \zeta_1(t),$$

and the right hand side is a bounded function. If $x > \zeta_1(t)$, then

$$W_t - \tfrac{1}{2}W_{xx} = [t^p(\psi_2 - \psi_1)]_t - \tfrac{1}{2}t^p(\psi_2 - \psi_1)_{xx},$$

and the right hand side is again a bounded function (by (5.2) and (5.4)). If $x < \zeta_2(t)$ then $W = 0$.

Notice next that W_x is continuous across the curves $x = \zeta_i(t)$. Hence, setting

$$W_t - \tfrac{1}{2}W_{xx} = g \quad (x \ne \zeta_i(t),\ g \text{ bounded})$$

we can represent W in the form

$$W(x, t) = \int_\varepsilon^t \int_{-R_0}^{R_0} K(x, t; y, \tau) g(y, \tau) \, dy \, d\tau + \int_{-R_0}^{R_0} K(x, t; y, \varepsilon) W(y, \varepsilon) \, dy + J_\varepsilon$$

where

$$K(x, t; y, \tau) = \frac{1}{\sqrt{2\pi(t - \tau)}} \exp\left\{-\frac{(x - y)^2}{2(t - \tau)}\right\}, \tag{5.27}$$

R_0 is any small positive number, and J_ε consists of boundary integrals, on $y = \pm R_0$, $\varepsilon \le \tau \le t$, of $KW_y - K_yW$. Since $W(y, \varepsilon) \to 0$ if $\varepsilon \to 0$ (by (3.28)), we obtain

$$W(x, t) = \int_0^t \int_{-R_0}^{R} K(x, t; y, \tau) g(y, \tau)\, dy\, d\tau + J_0$$

and J_0 is a bounded function in $|x| < R_0/2$, $0 \le t \le T_0$. We now use the L^p estimates for the parabolic equations (in fact, all we need is a special case of [11]) to deduce that

$$\int_0^t \int_{-R}^{R} |W_t|^\lambda\, dr\, dt \le C_\lambda \quad \text{for any } 1 < \lambda < \infty,$$

provided $R = R_0/3$ (we needed here just the fact that $g \in L^\lambda$). We have thus established (5.26) for $i = 2$; the proof for $i = 1$ is similar.

LEMMA 5.3. *There is a sequence $h_m \downarrow 0$ such that*

$$\int_{\zeta_2(h)}^{\zeta_1(h)} |z_i(y, h)|\, dy \to 0 \quad \textit{if } h = h_m \downarrow 0. \tag{5.28}$$

Proof. Consider the integral

$$I_h \equiv \frac{1}{h} \int_0^h \int_{\zeta_2(t)}^{\zeta_1(t)} |z_i(y, t)|\, dy\, dt$$
$$\le \frac{1}{h} \left(\iint |z_i|^\lambda \right)^{1/\lambda} \left(\iint 1 \right)^{1/\mu} \quad \left(\frac{1}{\lambda} + \frac{1}{\mu} = 1 \right).$$

The last integral is bounded by

$$\int_0^h (\zeta_1(t) - \zeta_2(t))\, dt \le Ch^{p+1}.$$

It follows, after using Lemma 5.2 with $\lambda > (p + 1)/p$, that

$$I_h \le \frac{C}{h} h^{(p+1)/\mu} \to 0 \quad \text{if } h \to 0.$$

The assertion of the lemma now follows immediately.

The next lemma gives a sharper estimate on z_i.

LEMMA 5.4. *If $\zeta_2(t) < x < \zeta_1(t)$, $0 < t < t^*$, then*

$$|z_i(x, t)| \le Ct^{p-1/2}. \tag{5.29}$$

Proof. We can represent $z_2(x, t)$ in the form

$$
\begin{aligned}
2z_2(x, t) = & \int_h^t K(x, t; \zeta_1(\tau), \tau) z_{2x}(\zeta_1(\tau), \tau)\, d\tau \\
& + \left\{ -\int_h^t K(x, t; \zeta_2(\tau), \tau) z_{2x}(\zeta_2(\tau), \tau)\, d\tau \right\} \\
& + \int_h^t K_x(x, t; \zeta_1(\tau), \tau) z_2(\zeta_1(\tau), \tau)\, d\tau \\
& + 2 \int_h^t \int_{\zeta_2(\tau)}^{\zeta_1(\tau)} K(x, t; y, \tau) f_2(y, \tau)\, dy\, d\tau \\
& + 2 \int_{\zeta_1(h)}^{\zeta_1(h)} K(x, t; y, h) z_2(y, h)\, dy \\
\equiv & I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{5.30}
$$

Set

$$\phi_{ih}(t) = \sup_{\substack{\zeta_2(s) < x < \zeta_1(s) \\ h < s < t}} |z_i(x, s)|.$$

If $\zeta_2(t) < x < \gamma(t)$, then, by Theorem 4.3,

$$x - \zeta_1(\tau) \geq c_0 t^p \text{ if } t/2 < \tau < t \quad (c_0 > 0).$$

Using (5.20) also, we get

$$
\begin{aligned}
I_3 & \leq C t^{2p-1} \left\{ \int_0^{t/2} \frac{d\tau}{t - \tau} + \int_{t/2}^t \frac{e^{-ct^p/(t-\tau)}}{t - \tau}\, d\tau \right\} \quad (c > 0) \\
& \leq C t^{2p-1} \left\{ 1 + \int_{c_0 t^{p-1}}^{\infty} \frac{e^{-y}}{y}\, dy \right\} \quad (c_0 > 0) \\
& \leq C_\varepsilon t^{2p-1-\varepsilon} \quad \text{for any } \varepsilon > 0.
\end{aligned}
\tag{5.31}
$$

Next,

$$I_2 \leq 0 \tag{5.32}$$

since $z_{2x}(\zeta_2(\tau), \tau) = -2\alpha\zeta_2'(\tau) \geq 0$. Also

$$
\begin{aligned}
I_1 = & \int_h^t K(x, t; \zeta_1(\tau), \tau) z_{1x}(\zeta_1(\tau), \tau)\, d\tau \\
& + \int_h^t K(x, t; \zeta_1(\tau), \tau)(z_{2x} - z_{1x})(\zeta_1(\tau), \tau)\, d\tau \\
& + \int_h^t K(x, t; \zeta_1(\tau), \tau)(z_{2x} - z_{1x})(\zeta_1(\tau), \tau)\, d\tau \\
\equiv & I_{11} + I_{12}.
\end{aligned}
$$

Since $z_{1x}(\zeta_1(\tau), \tau) = -2\alpha\zeta_1'(\tau) \le 0$ (by Theorem 4.1) we have

$$I_{11} \le 0.$$

Noting, by (5.10), that

$$(5.33) \qquad (z_{2x} - z_{1x})(\zeta_i(t), t) = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\, t^p(\psi_2 - \psi_1)\right)(\zeta_i(t), t) = O(t^{p-1}),$$

we get $I_{12} \le Ct^{p-1/2}$, so that

$$(5.34) \qquad I_1 \le Ct^{p-1/2}.$$

Next, by (5.16) and (3.28),

$$\begin{aligned} I_4 &\le p\phi_{2h}(t) \iint K(x, t; y, \tau)\, dy\, \frac{d\tau}{\tau} + Ct^{2p-2} \iint K(x, t; y, \tau)\, dy\, d\tau \\ &\le C\phi_{2h}(t) \int_h^t \frac{\tau^p}{\sqrt{t-\tau}\,\tau}\, d\tau + Ct^{2p-2} \int_h^t \frac{\tau^p}{\sqrt{t-\tau}}\, d\tau \end{aligned}$$

and thus

$$(5.35) \qquad I_4 \le Ct^{p-1/2}\phi_{2h}(t) + Ct^{3(p-1/2)}.$$

Substituting (5.35), (5.34), (5.32) and (5.31) into (5.30), we obtain (recalling (5.24))

$$(5.36) \qquad 0 \le z_2(x, t) \le Ct^{p-1/2}\phi_{2h}(t) + Ct^{p-1/2} + C \int_{\zeta_2(h)}^{\zeta_1(h)} z_2(x, h)\, dx$$

provided $\zeta_2(t) \le x \le \gamma(t)$.

Similarly (using (5.25)),

$$(5.37) \qquad 0 \le z_1(x, t) \le Ct^{p-1/2}\phi_{1h}(t) + Ct^{p-1/2} + C \int_{\zeta_2(h)}^{\zeta_1(h)} |z_1(x, h)|\, dx$$

provided $\gamma(t) \le x \le \zeta_1(t)$.

By the proof of (5.9) we also have

$$(5.38) \qquad z_1(x, t) - z_2(x, t) = \frac{\partial}{\partial t}\,[t^p(\psi_1(x, t) - \psi_2(x, t))] = O(t^{2p-1}).$$

From this inequality and from (5.37) we deduce that (5.36) is valid also if $\gamma(t) \le x \le \zeta_1(t)$. We now take $h = h_m \to 0$ in (5.36), h_m as in Lemma 5.3, and obtain the inequality

$$\phi_2(t) \le Ct^{p-1/2}\phi_2(t) + Ct^{p-1/2}$$

where

$$\phi_2(t) = \lim_{h\to 0} \phi_{2h}(t) = \sup_{\zeta_2(\tau) < x < \zeta_1(\tau),\ 0 < \tau \le t} z_2(x, \tau).$$

This gives (5.29) for $i = 2$ (and t^* small enough). The assertion (5.29) for $i = 1$ then follows by (5.38).

As an application of Lemma 5.4 we prove:

THEOREM 5.5. *If* (a) *holds then*

$$\rho_i(t) = \frac{t^p}{4\alpha} + O(t^{2p-1/2}). \tag{5.39}$$

Proof. Integrating (5.15) for $i = 2$, over (x, t) and using (5.17), (5.18) and (5.22), we obtain

$$\begin{aligned} &\int_{\zeta_2(t)}^{\zeta_1(t)} z_2(x, t)\, dx - \int_0^{\zeta_1(t)} z_2(x, \zeta_1^{-1}(x))\, dx \\ &\qquad + \alpha(\rho_1(t) + \rho_2(t)) - \tfrac{1}{2}t^p + O(t^{p+r-1} + t^{p+\sigma}) \\ &\qquad = \int_{\zeta_2(t)}^{\zeta_1(t)} pt^{p-1} w_2(x, t)\, dx - \int_0^{\zeta_1(t)} p(\zeta_1^{-1}(x))^{p-1} w_2(x, \zeta_1^{-1}(x))\, dx \end{aligned} \tag{5.40}$$

where

$$\zeta_i^{-1}(x) = t \quad \text{if } x = \zeta_i(t).$$

Using Lemma 5.4 and condition (a) (which implies that $p + r - 1 \geq 2p - \frac{1}{2}$, $p + \sigma \geq 2p - \frac{1}{2}$) we get

$$\rho_1(t) + \rho_2(t) = \frac{t^p}{2\alpha} + O(t^{2p-1/2}). \tag{5.41}$$

To derive another relation between ρ_1, ρ_2 we multiply both sides of (5.15), for $i = 2$, by x and then integrate over (x, t). We obtain

$$\begin{aligned} &\int_{\zeta_2(t)}^{\zeta_1(t)} xz_2(x, t)\, dx - \int_0^{\zeta_1(t)} xz_2(x, \zeta_1^{-1}(x))\, dx \\ &\quad - \frac{1}{2}\int_0^t \zeta_1(s) z_{2x}(\zeta_1(s), s)\, ds + \frac{1}{2}\int_0^t \zeta_2(s) z_{2x}(\zeta_2(s), s)\, ds + \frac{1}{2}\int_0^t z_2(\zeta_1(s), s)\, ds \\ &\quad = \int_{\zeta_2(t)}^{\zeta_1(t)} pxt^{p-1} w_2(x, t)\, dx - \int_0^{\zeta_1(t)} px(\zeta_1^{-1}(x))^{p-1} w_2(x, \zeta_1^{-1}(x))\, dx. \end{aligned} \tag{5.42}$$

Using (5.17), (5.18) and (5.22), we get

$$\begin{aligned} &-\frac{1}{2}\int_0^t \zeta_1(s) z_{2x}(\zeta_1(s), s)\, ds + \frac{1}{2}\int_0^t \zeta_2(s) z_{2x}(\zeta_2(s), s)\, ds + \frac{1}{2}\int_0^t z_2(\zeta_1(s), s)\, ds \\ &\quad = \alpha \int_0^t (\zeta_1(s)\zeta_1'(s) - \zeta_2(s)\zeta_2'(s))\, ds + O(t^{r+2p} + t^{\sigma+1+2p} + t^{p+q+1}). \end{aligned} \tag{5.43}$$

By (3.28) and Lemma 5.4, each of the remaining integrals in (5.42) is bounded by

$$O(t^{3p-1/2}).$$

It follows that

$$\zeta_1^2(t) - \zeta_2^2(t) = O(t^{3p-1/2}), \tag{5.44}$$

where the inequalities (ensured by condition (a))

$$r + 2p \geq 3p - \tfrac{1}{2}, \quad \sigma + 1 + 2p \geq 3p - \tfrac{1}{2}, \quad p + q + 1 \geq 3p - \tfrac{1}{2}$$

have been used.

Dividing both sides of (5.44) by $\zeta_1 - \zeta_2$ and using (5.41) we get

$$\rho_1(t) - \rho_2(t) = O(t^{2p-1/2}). \tag{5.45}$$

Finally, comparing (5.45) with (5.41), the assertion (5.39) follows.

6. Additional estimates on z_i

In this section and in Sections 7 and 8 we replace the condition (a) by the stronger condition:

Condition (b). The conditions (5.1)–(5.6) hold with $p > 1$ and $r \geq 2p$, $q \geq 3p - 2$, $\sigma \geq 2p - 1$, $v \geq p - 2$.

Orientation 6.1. In order to obtain sharper estimates on $\zeta_i(t)$ and in order to estimate $\zeta_i'(t)$ we need to obtain better estimates on z_i near (0, 0). This we can do using results derived in Section 5. In Lemma 6.1 we give a sharp lower bound on $\partial z_i/\partial t$ which leads (in Lemma 6.2) to an upper bound on $\rho_1'(t) + \rho_2'(t)$. This in turn enables us to estimate (in Lemma 6.3) $\partial z_i/\partial x$, using the integral representation for z_i (a representation which is very useful also for the classical Stefan problem). Finally, the "conservation of energy" method coupled with the new estimates on z_i gives an improved version of the estimates on $\rho_i(t)$, $\rho_i'(t)$.

LEMMA 6.1. *There exists a positive constant C such that if* $\zeta_2(t) < x < \zeta_1(t)$, $0 < t < t^*$, *then*

$$\frac{\partial}{\partial t} z_i(x, t) \geq -Ct^{2p-2}. \tag{6.1}$$

Proof. For any $h > 0$, introduce the finite difference quotients

$$Z_h(x, t) = \frac{z(x, t+h) - z(x, t)}{h}, \qquad z = z_2,$$

$$W_h(x, t) = \frac{(t^{p-2}w)(x, t+h) - (t^{p-2}w)(x, t)}{h}, \quad w = w_2.$$

Since

$$z = (t^p w)_t = pt^{p-1}w + t^p w_t, \tag{6.2}$$

it follows from (3.27) and Lemma 5.4 that

$$0 \le w_t \le \frac{z}{t^p} \le \frac{C}{\sqrt{t}} \tag{6.3}$$

provided $\zeta_2(t) < x < \zeta_1(t)$. The estimate $w_t \le C/\sqrt{t}$ remains valid also if $x \ge \zeta_1(t)$ ($w = \psi_2 - \psi_1$) and if $x < \zeta_2(t)$ ($w = 0$). From (6.3) we find that

$$(t^{p-2}w)_t \le Ct^{p-5/2}.$$

Hence,

$$W_h = \frac{1}{h}\int_t^{t+h} (s^{p-2}w(x, s))_s \, ds \le C(t^{p-5/2} + (t+h)^{p-5/2}). \tag{6.4}$$

By Lemma 5.4 we also get

$$\frac{pz}{t(t+h)} \le Ct^{p-5/2}. \tag{6.5}$$

Taking the finite difference quotients with respect to t in the parabolic equation (5.15) for $z = z_2$, we obtain

$$\frac{\partial}{\partial t} Z_h - \frac{1}{2}\frac{\partial^2}{\partial x^2} Z_h - \frac{p}{t+h} Z_h = \Phi_h \tag{6.6}$$

where

$$\Phi_h = -\frac{pz}{t(t+h)} - pW_h.$$

By (6.4) and (6.5),

$$\Phi_h \ge -C(t^{p-5/2} + (t+h)^{p-5/2}). \tag{6.7}$$

Since $\zeta_2(t)$ is monotone decreasing it follows from (5.24) that

$$Z_h > 0 \quad \text{on } x = \zeta_2(t). \tag{6.8}$$

Next, on $x = \zeta_1(t)$,

$$Z_h = [z_1]_h + \left[\frac{\partial}{\partial t}(t^p(\psi_2 - \psi_1))\right]_h \tag{6.9}$$

where $[v]_h$ designates the finite difference quotient of v with respect to t.

Using (5.1), (5.2), (5.5) we find that

$$\left|\frac{\partial^2}{\partial t^2}(t^p(\psi_2 - \psi_1)(x, t))\right| = O(t^{2p-2} + t^{p+q-1} + t^{p+\nu}) = O(t^{2p-2}). \tag{6.10}$$

Recalling that $p > 1$, we obtain

$$\left| \frac{\partial}{\partial t} (t^p(\psi_2 - \psi_1)) \right|_h = O((t + h)^{2p-2}).$$

Since $\zeta_1(t)$ is monotone increasing and (5.25) holds, we also have

$$[z_1]_h > 0 \quad \text{on } x = \zeta_1(t).$$

We thus conclude from (6.9) that

$$Z_h \geq -C_0(t + h)^{2p-2} \quad \text{on } x = \zeta_1(t) \quad (C_0 \text{ positive constant}) \tag{6.11}$$

We shall compare Z_h with the function

$$U = Cx^2(t^{p-5/2} + (t + h)^{p-5/2}) - \tilde{C}(t + h)^{2p-2} \tag{6.12}$$

where $\tilde{C} > 2C_0$ and C is as in (6.7). Clearly

$$\begin{aligned} U_t - \tfrac{1}{2}U_{xx} &= -C(t^{p-5/2} + (t + h)^{p-5/2}) + (p - \tfrac{5}{2})Cx^2(t^{p-7/2} + (t + h)^{p-7/2}) \\ &\quad - (2p - 2)\tilde{C}(t + h)^{2p-3} \\ &< -C(t^{p-5/2} + (t + h)^{p-5/2}) \end{aligned}$$

since $x^2 \leq t^{2p}/(3\alpha)^2$ and $p > 1$. Hence

$$U_t - \tfrac{1}{2}U_{xx} < \Phi_h.$$

Next, on $x = \zeta_i(t)$, $x^2 \sim t^{2p}/(4\alpha)^2$ so that

$$U \sim \frac{C}{(4\alpha)^2} (t^{3p-5/2} + t^{2p}(t + h)^{p-5/2}) - \tilde{C}(t + h)^{2p-2} < -\tfrac{1}{2}\tilde{C}(t + h)^{2p-2}.$$

Comparing with (6.8) and (6.11) we see that

$$U < Z_h \quad \text{on } x = \zeta_i(t).$$

Noting finally that

$$\liminf_{(x,t)\to(0,0)} [Z_h(x, t) - U(x, t)] > 0,$$

we can apply the maximum principle to $Z_h - U$ and conclude that

$$Z_h(x, t) \geq -\tilde{C}(t + h)^{2p-2}$$

if $\zeta_2(t) < x < \zeta_1(t)$, $0 < t < t^*$. Taking $h \to 0$, the assertion of the lemma follows for $i = 2$; the proof for $i = 1$ is similar.

LEMMA 6.2. *There exists a positive constant C such that, if $0 < t < t^*$,*

$$\rho_1'(t) + \rho_2'(t) \leq \frac{pt^{p-1}}{2\alpha} + Ct^{2p-3/2}. \tag{6.13}$$

Proof. Integrating the equation (5.15) for z_2 with respect to x, $\zeta_2(t) < x < \zeta_1(t)$, and setting $z = z_2$, $w = w_2$, we get, after using (5.22),

$$\alpha(\rho_1'(t) + \rho_2'(t)) - \frac{pt^{p-1}}{2} + O(t^{p+r-1} + t^{p+\sigma}) = \int_{\zeta_2(t)}^{\zeta_1(t)} [(pt^{p-1}w)_t - z_t]\, dx. \tag{6.14}$$

Now,

$$(pt^{p-1}w)_t = \left(\frac{p}{t}\, t^p w\right)_t = \frac{p}{t}\, z - \frac{p}{t^2}\, t^p w = O(t^{p-3/2})$$

by Lemma 5.4. Also $-z_t \leq Ct^{2p-2}$, by Lemma 6.1. It follows that the right hand side of (6.14) is bounded above by $Ct^{2p-3/2}$, and (6.13) follows.

LEMMA 6.3.

$$\left|\frac{\partial}{\partial x}\, z_i(x, t)\right| = O(t^{p-1}). \tag{6.15}$$

Proof. For $x > \zeta_1(t)$ or $x < \zeta_2(t)$ the estimate (6.15) is a consequence of our assumptions on the ψ_j. Let $\zeta_2(t) < x < \zeta_1(t)$ and take $h \to 0$ in (5.30), noting that $I_5 \to 0$. After differentiating the resulting relation with respect to x, we get

$$\begin{aligned} 2z_{2x}(x, t) &= \int_0^t K_x(x, t; \zeta_1(\tau), \tau) z_{2x}(\zeta_1(\tau), \tau)\, d\tau \\ &\quad + \left\{- \int_0^t K_x(x, t; \zeta_2(\tau), \tau) z_{2x}(\zeta_2(\tau), \tau)\, d\tau\right\} \\ &\quad + \int_0^t K_{xx}(x, t; \zeta_1(\tau), \tau) z_2(\zeta_1(\tau), \tau)\, d\tau \\ &\quad + 2 \int_0^t \int_{\zeta_2(\tau)}^{\zeta_1(\tau)} K_x(x, t; y, \tau) f_2(y, \tau)\, dy\, d\tau \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{6.16}$$

From (5.18) and Lemma 6.2 we have

$$|z_{2x}(\zeta_2(\tau), \tau)| \leq Ct^{p-1}.$$

Using the inequalities $0 \leq \rho_2'(\tau) \leq C\tau^{p-1}$, we can deduce as in [8; p. 219, formula (1.16)] that

$$\int_0^t |K_x(x, t; \zeta_2(\tau), \tau)|\, d\tau \leq C. \tag{6.17}$$

Hence

$$|J_2| \leq Ct^{p-1}.$$

Similarly, $|z_{2x}(\zeta_1(\tau), \tau)| \le Ct^{p-1}$ and

$$|J_1| \le Ct^{p-1}.$$

Next, by Lemma 5.4. and (3.28),

$$|f_2| \le Ct^{p-3/2}.$$

Therefore

$$|J_4| \le Ct^{p-3/2} \int |K_x| \, dy \, d\tau \le Ct^{p-3/2} \int \frac{d\tau}{\sqrt{t-\tau}} = Ct^{p-1}.$$

We now restrict x to lie in the interval $\zeta_2(t) < x < \gamma(t)$. Then, by (5.20),

$$\begin{aligned} |J_3| &\le Ct^{2p-1} \int_0^t |K_{xx}| \, d\tau \\ &= Ct^{2p-1} \left\{ \int_0^{t/2} \frac{d\tau}{(t-\tau)^{3/2}} + \int_{t/2}^t \frac{e^{-ct^{2p}/(t-\tau)}}{(t-\tau)^{3/2}} \, d\tau \right\} \end{aligned}$$

for some $c > 0$; in obtaining the exponent in the last integral we have used the relation $\rho_1(t) \sim t^p/(4\alpha)$. Thus

$$|J_3| \le Ct^{2p-1} \left\{ \frac{1}{\sqrt{t}} + \frac{1}{t^p} \int_0^\infty \sqrt{y}\, e^{-cy} \, dy \right\} \le Ct^{p-1}.$$

Substituting the estimates on the J_i into (6.16), the assertion (6.15) follows for $i = 2$ and $\zeta_2(t) \le x \le \gamma(t)$. The proof for $i = 1$ and $\gamma(t) < x < \zeta_1(t)$ is similar. Finally, since

$$(z_1 - z_2)_x = \frac{\partial}{\partial x} \frac{\partial}{\partial t} (t^p(\psi_1 - \psi_2)) = O(t^{p-1}),$$

the proof of the lemma is complete.

We can now use Lemma 6.3 in order to bootstrap previous inequalities for z_i and ρ_i. Indeed, writing

$$z_1(x, t) = \int_{\zeta_1(t)}^x z_{1x}(y, t) \, dy$$

and using Lemma 6.3, we get

$$|z_1(x, t)| \le Ct^{p-1} |x - \zeta_1(t)| \le Ct^{2p-1}.$$

A similar inequality holds for z_2. Using these estimates in the proof of Theorem 5.5 and noting that

$$p + r - 1 \ge 3p - 1, \quad p + \sigma \ge 3p - 1,$$

we obtain

$$\rho_1(t) + \rho_2(t) = \frac{t^p}{2\alpha} + O(t^{3p-1}) \tag{6.18}$$

instead of (5.41); similarly, since

$$r + 2p \geq 4p - 1, \quad \sigma + 1 + 2p \geq 4p - 1, \quad p + q + 1 \geq 4p - 1,$$

we obtain

$$\zeta_1^2(t) - \zeta_2^2(t) = O(t^{4p-1}). \tag{6.19}$$

We can therefore state:

THEOREM 6.4. *If the condition* (b) *holds then*

$$|z_i(x, t)| \leq Ct^{2p-1} \quad (C \text{ constant}), \tag{6.20}$$

and

$$\rho_i(t) = \frac{t^p}{4\alpha} + O(t^{3p-1}) \tag{6.21}$$

for $\zeta_2(t) < x < \zeta_1(t)$, $0 < t < t^*$ (t^* *small enough*).

We conclude this section with an improvement of Lemma 6.2, which will be needed in Section 7:

LEMMA 6.6. *For all* $0 < t < t^*$,

$$\rho_i'(t) \leq \frac{pt^{p-1}}{4\alpha} + Ct^{3p-2} \tag{6.22}$$

where C is a positive constant.

Proof. We shall prove (6.22) for $i = 1$; the proof for $i = 2$ is similar. Multiplying the equation (5.15) for $i = 1$ by $x - \zeta_2(t)$ and integrating over x, $\zeta_2(t) < x < \zeta_1(t)$, we obtain (cf. (6.14))

$$\begin{aligned} &\alpha(\rho_1(t) + \rho_2(t))\rho_1'(t) - \tfrac{1}{2}z_1(\zeta_2(t), t) \\ &\quad = \int_{\zeta_2(t)}^{\zeta_1(t)} (x - \zeta_2(t)) \left[\frac{\partial}{\partial t}(pt^{p-1}w_1)_t - \frac{\partial}{\partial t} z_1 \right] dx. \end{aligned} \tag{6.23}$$

By (5.9),

$$z_1(\zeta_2(t)) = pt^{p-1}\rho_2(t) + O(t^{r+2p-1} + t^{p+q}). \tag{6.24}$$

From (6.2) with $w = w_i$, $z = z_i$ and (6.20),

$$\left| \frac{\partial}{\partial t} w_i \right| \leq Ct^{p-1}. \tag{6.25}$$

Using (6.24), (6.25) and (6.1) in (6.23), we obtain

$$\alpha(\rho_1(t) + \rho_2(t))\rho_1'(t) \le pt^{p-1}\rho_2(t) + Ct^{4p-2}. \tag{6.26}$$

Substituting $\rho_i(t)$ from (6.21) and using the inequality $\rho_1'(1) = O(t^{p-1})$ (Lemma 6.2), the assertion (6.22) follows (for $i = 1$).

7. The integral equation for ρ_i'

In this section we continue to assume the condition (B).

Orientation 7.1. In order to improve our estimates on $\rho_i(t)$ we must work with $\rho_i(t)$, $\rho_i'(t)$ together. In the classical Stefan problem this can be done by working with a nonlinear Volterra type integral equation satisfied by $\zeta'(t)$. Here we shall have a system of two equations. The nice thing about working with such integral equations is the possibility of bootstrapping the estimates step-by-step. The initial estimates needed to begin the process were already derived in Section 6. We shall now start with a preliminary estimate on $\rho_i'(t)$ (Lemma 7.1) and a corresponding estimate on $\partial z_i(\zeta_j(t), t)/\partial x$ (Corollary 7.2). Next we use again the integral equation in order to obtain better estimates on $\rho_i'(t)$ (Lemma 7.3, Theorem 7.4). The procedure can probably be continued step-by-step, but because the calculations are tedious and increasingly lengthy, we shall not pursue this further.

We let $x \to \zeta_2(t) + 0$ in (6.16) and use a standard jump relation [7] [8]. We get

$$\begin{aligned} z_{2x}(\zeta_2(t), t) &= \int_0^t K_x(\zeta_2(t), t; \zeta_1(\tau), \tau) z_{2x}(\zeta_1(\tau), \tau)\, d\tau \\ &\quad + \left\{ -\int_0^t K_x(\zeta_2(t), t; \zeta_2(\tau), \tau) z_{2x}(\zeta_2(\tau), \tau)\, d\tau \right\} \\ &\quad + \int_0^t K_{xx}(\zeta_2(t), t; \zeta_1(\tau), \tau) z_2(\zeta_1(\tau), \tau)\, d\tau \\ &\quad + 2\int_0^t \int_{\zeta_2(\tau)}^{\zeta_1(\tau)} K_x(\zeta_2(t), t; y, \tau) f_2(y, \tau)\, dy\, d\tau \\ &\equiv \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4. \end{aligned} \tag{7.1}$$

Set

$$\phi_i(t) = z_{ix}(\zeta_i(t), t), \quad \phi(t) = \phi_2(t) - \phi_1(t), \quad \rho(t) = \rho_1(t) + \rho_2(t).$$

Then

$$\phi_i(t) = -2\alpha\zeta_i'(t), \; \phi(t) = 2\alpha\rho'(t).$$

LEMMA 7.1.

$$\phi(t) - pt^{p-1} = O(t^{2p-3/2}). \tag{7.2}$$

Proof. Since, by Lemma 6.6,

$$\phi(t) - pt^{p-1} \le Ct^{3p-2} \quad (C > 0), \tag{7.3}$$

it remains to prove that

$$\phi(t) - pt^{p-1} \ge -Ct^{2p-3/2}. \tag{7.4}$$

Using (6.20) we find that

$$|f_2(x, t)| \le Ct^{2p-2}.$$

Hence

$$|\tilde{J}_4| \le Ct^{2p-2}\left\{\int_0^{t-t^{2p}} \int_{\zeta_2(\tau)}^{\zeta_1(\tau)} |K_x(x, t; y, \tau)|\, dy\, d\tau + \int_{t-t^{2p}}^{0} \int_{\zeta_2(\tau)}^{\zeta_1(\tau)} |K_x(x, t; y, \tau)|\, dy\, d\tau\right\}.$$

The first integral on the right hand side is bounded by

$$C\int_0^{t-t^{2p}} \frac{d\tau}{(t-\tau)^{3/2}} \int_{\zeta_2(\tau)}^{\zeta_1(\tau)} \rho(\tau)\, d\tau \le C\,\frac{t^{2p}}{t^p} = Ct^p,$$

and the second integral is bounded by

$$\int_{t-t^{2p}}^{t} \frac{d\tau}{\sqrt{t-\tau}} \int_0^{\infty} ze^{-z^2/2}\, dz \le Ct^p.$$

It follows that

$$|\tilde{J}_4| \le Ct^{3p-2}. \tag{7.5}$$

Next

$$\begin{aligned}\tilde{J}_1 &= -\int_0^t \frac{\zeta_2(t) - \zeta_1(\tau)}{t-\tau} K(\zeta_2(t), t; \zeta_1(\tau), \tau) z_{2x}(\zeta_1(\tau), \tau)\, d\tau \\ &= -\int_0^t \frac{\zeta_2(t) - \zeta_1(\tau)}{t-\tau} K(\zeta_2(t), t; \zeta_1(t), \tau) z_{2x}(\zeta_1(t), \tau)\, d\tau + O(t^{2p-3/2});\end{aligned}$$

the proof uses (6.15) and the same argument as in [8; p. 218, formulas (1.11) and (1.12)]. We can then write

$$\tilde{J}_1 = -\int_0^t \frac{\zeta_2(t)-\zeta_1(t)}{t-\tau} K(\zeta_2(t), t; \zeta_1(t), \tau) z_{2x}(\zeta_1(\tau), \tau)) \, d\tau$$
$$+ \tilde{J}_{11} + O(t^{2p-3/2}), \tag{7.6}$$
$$\tilde{J}_{11} = -\int_0^t \frac{\zeta_1(t)-\zeta_1(\tau)}{t-\tau} K(\zeta_2(t), t; \zeta_1(t), \tau) z_{2x}(\zeta_1(\tau), \tau)) \, d\tau.$$

By Lemmas 6.2, 6.3,

$$|\tilde{J}_{11}| \le Ct^{2p-3/2}. \tag{7.7}$$

Since

$$\tilde{J}_2 = \int_0^t \frac{\zeta_2(t)-\zeta_2(\tau)}{t-\tau} K(\zeta_2(t), t; \zeta_2(\tau), \tau) z_{2x}(\zeta_2(\tau), \tau) \, d\tau,$$

we also have

$$|\tilde{J}_2| \le Ct^{2p-3/2}. \tag{7.8}$$

As for $\tilde{J}_3$, we can use Lemma 6.2 to write

$$\tilde{J}_3 = \int_0^t \left[\frac{(\zeta_2(t)-\zeta_1(\tau))^2}{(t-\tau)^2} - \frac{1}{t-\tau}\right] K(\zeta_2(t), t; \zeta_1(\tau), \tau) z_2(\zeta_1(\tau), \tau) \, d\tau$$
$$= \int_0^t \left[\frac{(\zeta_2(t)-\zeta_1(t))^2}{(t-\tau)^2} - \frac{1}{t-\tau}\right] K(\zeta_2(t), t; \zeta_1(\tau), \tau) z_2(\zeta_1(\tau), \tau) \, d\tau \tag{7.9}$$
$$+ O(t^{2p-1}) \int \frac{1}{t-\tau} K(\zeta_2(t), t; \zeta_1(\tau), \tau) z_2(\zeta_1(\tau), \tau) \, d\tau.$$

The last integral can be estimated by $O(t^{p-1})$, using the same calculations as in the estimate of J_3 (in the proof of Lemma 6.3).

If in the middle integral in (7.9) we replace

$$K(\zeta_2(t), t; \zeta_1(\tau), \tau)$$

by

$$K(\zeta_2(t), t; \zeta_1(t), \tau),$$

then the resulting error (cf. (7.6)) is bounded by the last term on the right hand side of (7.9). Thus

$$\tilde{J}_3 = \int_0^t \left[\frac{(\zeta_2(t)-\zeta_1(t))^2}{t-\tau} - \frac{1}{t-\tau}\right] \tag{7.10}$$
$$\times K(\zeta_2(t), t; \zeta_1(t), \tau) z_2(\zeta_1(\tau), \tau) \, d\tau + O(t^{3p-2}).$$

We now replace $z_2(\zeta_1(\tau), \tau)$ by $z_2(\zeta_1(t), t)$ in (7.10); by (5.19) and (5.11), the resulting error is

$$O(t^{2p-2} + t^{p+\nu} + t^{p-1+q} + t^{2p-1+\sigma}) \int_0^t \frac{d\tau}{\sqrt{t-\tau}} = O(t^{2p-3/2}).$$

Thus

$$\tilde{J}_3 = \int_0^t \left[\frac{(\zeta_1(t) - \zeta_2(t))^2}{(t-\tau)^2} - \frac{1}{t-\tau}\right] \times K(\zeta_2(t), t; \zeta_1(\tau), \tau) z_2(\zeta_1(t))\, d\tau + O(t^{2p-3/2}). \tag{7.11}$$

We now substitute the expressions in (7.11), (7.8), (7.5) and (7.6), (7.7) into (7.1) and obtain

$$\begin{aligned} \phi_2(t) = {} & \int_0^t \frac{\rho(t)}{t-\tau} K(\rho(t), t; 0, \tau)(p\tau^{p-1} - 2\alpha\rho_1'(\tau))\, d\tau \\ & + \int_0^t \left[\frac{\rho^2(t)}{(t-\tau)^2} - \frac{1}{t-\tau}\right] K(\rho(t), t; 0, \tau) p t^{p-1} \rho_1(t)\, d\tau \\ & + O(t^{2p-3/2}). \end{aligned} \tag{7.12}$$

Similarly, we get

$$\begin{aligned} \phi_1(t) = {} & - \int_0^t \frac{\rho(t)}{t-\tau} K(\rho(t), t; 0, \tau)(p\tau^{p-1} + 2\alpha\rho_2'(\tau)\, d\tau \\ & - \int_0^t \left[\frac{\rho^2(t)}{(t-\tau)^2} - \frac{1}{t-\tau}\right] K(\rho(t), t; 0, \tau) p t^{p-1} \rho_2(t)\, d\tau \\ & + O(t^{2p-3/2}). \end{aligned} \tag{7.13}$$

Consequently

$$\begin{aligned} \phi(t) = {} & - \int_0^t \frac{\rho(t)}{t-\tau} K(\rho(t)t; 0, \tau)(\phi(\tau) - p\tau^{p-1})\, d\tau \\ & + \int_0^t \frac{\rho(t)}{t-\tau} K(\rho(t), t; 0, \tau) p\tau^{p-1}\, d\tau \\ & + \int_0^t \left[\frac{\rho^2(t)}{(t-\tau)^2} - \frac{1}{t-\tau}\right] K(\rho(t), t; 0, \tau) p t^{p-1} \rho(t)\, d\tau \\ & + O(t^{2p-3/2}) \\ \equiv {} & H_1 + H_2 + H_3 + O(t^{2p-3/2}). \end{aligned} \tag{7.14}$$

Substituting

$$y = \frac{\rho(t)}{(t-\tau)^{1/2}}$$

we derive the formulas

$$\int_0^t \frac{\rho(t)}{t-\tau} K(\rho(t), t; 0, \tau)\, d\tau = 1 + O(t^{p-1/2}), \tag{7.15}$$

$$\int_0^t \frac{\rho^3(t)}{(t-\tau)^2} K(\rho(t), t; 0, \tau)\, d\tau = 1 + t^{3p-3/2}). \tag{7.16}$$

Now, in view of (7.3) and (7.15),

$$H_1 \geq -Ct^{3p-2}.$$

Next, by (7.15) and (7.16),

$$|H_3| \leq C^{2p-3/2}.$$

Finally, since

$$H_2 = \int_0^{t/2} + \int_{t/2}^t, \quad \int_0^{t/2} \leq Ct^{2p-1},$$

we can easily verify that

$$H_2 = \int_0^t \frac{\rho(t)}{t-\tau} K(\rho(t), t; 0, \tau) pt^{p-1}\, d\tau + O(t^{2p-3/2}) = pt^{p-1} + O(t^{2p-3/2}).$$

Substituting the estimates for the H_j into (7.14) we obtain the assertion (7.4). From Lemmas 6.6, 7.1 we deduce:

COROLLARY 7.2. *We have*

$$\rho_i'(t) = \frac{pt^{p-1}}{4\alpha} + O(t^{2p-3/2}) \tag{7.17}$$

and, consequently (*by* (5.21) *and* (5.22)),

$$\frac{\partial}{\partial x} z_2(\zeta_i(t), t) = \frac{pt^{p-1}}{2} + O(t^{2p-3/2}) \tag{7.18}$$

and

$$\frac{\partial}{\partial x} z_1(\zeta_i(t), t) = -\frac{pt^{p-1}}{2} + O(t^{2p-3/2}). \tag{7.19}$$

Using these results we shall bootstrap the proof of Lemma 7.1., thereby establishing:

LEMMA 7.3.

$$\phi(t) - pt^{p-1} = O(t^{3p-2}). \tag{7.20}$$

Proof. In view of (7.3), it suffices to show that

$$\phi(t) - pt^{p-1} \geq -Ct^{3p-2}. \tag{7.21}$$

We re-evaluate $\tilde{J}_1$, writing

$$\tilde{J}_1 = -\int_0^t \frac{\zeta_2(t) - \zeta_1(t)}{t - \tau} K(\rho(t), t; 0, \tau) z_{2x}(\zeta_1(\tau), \tau)\, d\tau$$
$$- \int_0^t \frac{\zeta_1(t) - \zeta_1(\tau)}{t - \tau} K(\rho(t), t; 0, \tau) z_{2x}(\zeta_1(\tau), \tau)\, d\tau,$$

and replacing (by (7.18)) z_{2x} by $p\tau^{p-1}/2 + O(\tau^{2p-3/2})$ in the last integral, we obtain

$$\tilde{J}_1 = \int_0^t \frac{\rho(t)}{t - \tau} K(\rho(t), t; 0, \tau) z_{2x}(\zeta_1(\tau), \tau)\, d\tau$$
$$- \int_0^t \frac{\zeta_1(t) - \zeta_1(\tau)}{t - \tau} K(\rho(t), t; 0, \tau) \frac{p}{2} \tau^{p-1}\, d\tau + O(t^{3p-2}).$$

Next, by using again (7.18),

$$\tilde{J}_2 = \int_0^t \frac{\zeta_2(t) - \zeta_2(\tau)}{t - \tau} K(\rho(t), t; 0, \tau) \frac{p}{2} \tau^{p-1}\, d\tau + O(t^{3p-2}).$$

For $\phi_1(t)$ we obtain similar expressions. If we denote by $\hat{J}_i$ the expressions analogous to $\tilde{J}_i$, then

$$\hat{J}_1 + \hat{J}_2 = \int_0^t \frac{\zeta_1(t) - \zeta_1(\tau)}{t - \tau} K(\rho(t), t; 0, \tau) \frac{p}{2} \tau^{p-1}\, d\tau$$
$$+ \int_0^t \frac{\rho(t)}{t - \tau} K(\rho(t), t; 0, \tau) z_{1x}(\zeta_2(\tau), \tau)\, d\tau$$
$$- \int_0^t \frac{\zeta_2(t) - \zeta_2(\tau)}{t - \tau} K(\rho(t), t; 0, \tau) \frac{p}{2} \tau^{p-1}\, d\tau$$
$$+ O(t^{3p-2}).$$

Setting $K = K(\rho(t), t; 0, \tau)$, and using (5.21) and (5.22), we conclude that

$$\phi(t) = \int_0^t \frac{(\zeta_2(t) - \zeta_2(\tau)) - (\zeta_1(t) - \zeta_1(\tau))}{t - \tau} Kp\tau^{p-1}\, d\tau$$
$$+ \int_0^t \frac{\rho(t)}{t - \tau} K[p\tau^{p-1} + (p\tau^{p-1} - \phi(\tau))]\, d\tau$$
$$+ \tilde{J}_3 - \hat{J}_3 + O(t^{3p-2}).$$

If we now use (7.3) and (7.15), we obtain

$$\begin{aligned}\phi(t) &\geq -p \int_0^t \frac{\rho(t)-\rho(\tau)}{t-\tau} K\tau^{p-1}\,d\tau + p\int_0^t \frac{\rho(t)}{t-\tau} K\tau^{p-1}\,d\tau \\ &\quad + \tilde{J}_3 - \hat{J}_3 + O(t^{3p-2}) \\ &= p\int_0^t \frac{\rho(t)}{t-\tau} K\tau^{p-1}\,d\tau + \tilde{J}_3 - \hat{J}_3 + O(t^{3p-2}).\end{aligned}$$

We now recall (7.11) and note that $\hat{J}_3$ is obtained from $\tilde{J}_3$ by replacing $z_2(\zeta_1(\tau), \tau)$ by $-z_1(\zeta_2(\tau), \tau)$. After making use of the relation

$$z_2(\zeta_1(\tau), \tau) + z_1(\zeta_2(\tau), \tau) = p\tau^{p-1}\rho(\tau) + O(t^{r+2p-1} + t^{p+q})$$

(which follows from (5.19) and (5.20)), we obtain

$$\begin{aligned}\phi(t) &\geq p\int_0^t \frac{\rho(t)}{t-\tau} K\tau^{p-1}\,d\tau \\ &\quad + \int_0^t \left[\frac{\rho^2(t)}{(t-\tau)^2} - \frac{1}{t-\tau}\right] Kp\tau^{p-1}\rho(\tau)\,d\tau + O(t^{3p-2}) \\ &= \int_0^t \frac{\rho^2(t)}{(t-\tau)^2} Kp\tau^{p-1}\rho(\tau)\,d\tau + O(t^{3p-2}).\end{aligned}$$

Substituting $\rho(t)$ from formula (6.21) and using (7.16), we finally get

$$\begin{aligned}\phi(t) &\geq \frac{p}{2\alpha}\int_0^t \frac{\rho^2(t)}{(t-\tau)^2} K(\rho(t), t; 0, \tau)\tau^{2p-1}\,d\tau + O(t^{3p-2}) \\ &\equiv J + O(t^{3p-2}).\end{aligned} \tag{7.22}$$

Substituting $z = \rho(t)/\sqrt{t-\tau}$ we obtain

$$J = \frac{p}{\sqrt{2\pi}\,\alpha\rho(t)} \int_{\rho(t)/\sqrt{t}}^{\infty} z^2\left(t - \frac{\rho^2(t)}{z^2}\right)^{2p-1} e^{-z^2/2}\,dz.$$

Since

$$\left|\left(t - \frac{\rho^2(t)}{z^2}\right)^{2p-1} - t^{2p-1}\right| \leq Ct^{2p-2}\,\frac{\rho^2(t)}{z^2},$$

we find that

$$\begin{aligned}J &= \frac{p}{\sqrt{2\pi}\,\alpha\rho(t)} \int_{\rho(t)/\sqrt{t}}^{\infty} z^2 t^{2p-1} e^{-z^2/2}\,dz + O(t^{3p-2}) \\ &= \frac{pt^{2p-1}}{\sqrt{2\pi}\,\alpha\rho(t)} \int_0^{\infty} z^2 e^{-z^2/2}\,dz + O(t^{3p-2}) \\ &= \frac{pt^{2p-1}}{2\alpha\rho(t)} + O(t^{3p-2}) \\ &= pt^{p-1} + O(t^{3p-2}).\end{aligned}$$

Substituing this into (7.22), the assertion (7.21) follows.

Combining Lemmas 7.3 and 6.6 we obtain:

THEOREM 7.4. *If the condition* (b) *holds then*

$$\rho_i'(t) = \frac{pt^{p-1}}{4\alpha} + O(t^{3p-2}). \tag{7.23}$$

It now follows from (5.21), (5.22) and (5.18) that

$$\frac{\partial}{\partial x} z_2(\zeta_i(t), t) = \frac{p}{2} t^{p-1} + O(t^{3p-2}) \tag{7.24}$$

and

$$\frac{\partial}{\partial x} z_1(\zeta_i(t), t) = -\frac{p}{2} t^{p-1} + O(t^{3p-2}); \tag{7.25}$$

this is an improvement over (7.18), (7.19).

Orientation 7.2. With these improved estimates we shall now estimate $\partial z_i(x, t)/\partial x$ for $\zeta_2(t) < x < \zeta_1(t)$ by comparing with special super- and sub-solutions via the maximum principle. The new estimates derived in Theorem 7.5 will enable us (in Section 8) to obtain sharp estimates on $z_i(x, t)$ which in turn, yield improved estimates on $\zeta_i(t)$ (by using the "conservation law" method). These improved estimates (stated in Theorem 8.1) together with Theorem 7.4 constitute the main results of this paper.

THEOREM 7.5. *If* (b) *holds then, for* $\zeta_2(t) < x < \zeta_1(t)$,

$$\frac{\partial}{\partial x} z_2(x, t) = \frac{p}{2} t^{p-1} + O(t^{3p-2}) \tag{7.26}$$

and

$$\frac{\partial}{\partial x} z_1(x, t) = -\frac{p}{2} t^{p-1} + O(t^{3p-2}) \tag{7.27}$$

where $|O(t^{3p-1})| \le Ct^{3p-2}$, *C independent of x.*

Proof. By Lemma 6.3,

$$|z_{ix}| \le Ct^{p-1}. \tag{7.28}$$

Hence, we also have

$$|(t^p w_i)_{tx}| \le Ct^{p-1}.$$

Since $(t^p w_i)_x = 0$ on $x = \zeta_i(t)$, we get

$$|(t^p w_1)_x| \le Ctt^{p-1} \quad \text{if } 0 < x < \zeta_1(t),$$

$$|(t^p w_2)_x| \le Ctt^{p-1} \quad \text{if } \zeta_2(t) < x < 0.$$

Recalling that $(t^p(w_1 - w_2))_x = t^p(\psi_1 - \psi_2)_x$, we find that

$$|(t^p w_i)_x| \le Ct^p \quad \text{for } \zeta_2(t) < x < \zeta_1(t). \tag{7.29}$$

Differentiating (5.15) (for $i = 2$) with respect to x and using (5.16) and (7.28), (7.29), we see that

$$\left| \frac{\partial}{\partial t}(z_{2x}) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(z_{2x}) \right| \le Ct^{p-2}. \tag{7.30}$$

Consider the function

$$W(x, t) = z_{2x} - \frac{p}{2} t^{p-1} - Ax^2 t^{p-2} + Bt^{3p-2} \tag{7.31}$$

in the region $\zeta_2(t) < x < \zeta_1(t)$, $0 < t < t^*$, where $A > 0$, $B > 0$. Then, by (7.30),

$$\begin{aligned} W_t - \tfrac{1}{2}W_{xx} &\ge -Ct^{p-2} - \frac{p(p-1)}{2} t^{p-2} + At^{p-2} - A(p-2)x^2 t^{p-3} + B(3p-2)t^{3p-3} \\ &\ge \frac{A}{2} t^{p-2} \\ &> 0 \end{aligned}$$

provided A is large enough independently of B (and t^* will depend on B).

Next, in view of (7.24),

$$W(\zeta_i(t), t) > 0$$

if B is large enough, say,

$$B > \frac{2A}{(4\alpha)^2};$$

here we use the relation $\rho_i(t) \sim t^p/(4\alpha)$.

Applying the maximum principle we deduce that $W(x, t) > 0$, that is,

$$z_{2x} - \frac{p}{2} t^{p-1} \ge C_0 t^{3p-2} \quad (C_0 > 0).$$

Similarly one obtains an upper bound, which completes the proof of (7.26). The proof of (7.27) is similar.

8. The final asymptotic formulas

THEOREM 8.1. *If the condition* (b) *holds with* $r \ge 3p$, $q \ge 4p - 1$, $\sigma \ge 3p - 1$, *then*

$$\zeta_1(t) = \frac{t^p}{4\alpha} - \frac{pt^{3p-1}}{96\alpha^2} + O(t^{4p}), \tag{8.1}$$

and

$$\zeta_2(t) = \frac{t^p}{4\alpha} + \frac{pt^{3p-1}}{96\alpha^2} + O(t^{4p}), \tag{8.2}$$

Recall that under just the condition (b) we have already proved that

$$(-1)^{i-1}\zeta_i'(t) = \frac{pt^{p-1}}{4\alpha} + O(t^{3p-2}) \quad (i = 1, 2) \tag{8.3}$$

and

$$\left| \frac{\partial^2}{\partial x \partial t} (t^2(\psi_i - u)) - (-1)^i \frac{p}{2} t^{p-1} \right| \le Ct^{3p-2} \quad (C > 0) \tag{8.4}$$

if $\zeta_2(t) < x < \zeta_1(t)$, $0 < t < t^*$ (t^* small enough).

Proof. From Theorem 7.5 and (6.21) we get

$$\begin{aligned} z_2(x, t) &= \frac{p}{2} t^{p-1}(x - \zeta_2(t)) + O(t^{4p-2}) \\ &= \frac{p}{2} t^{p-1}\left(x + \frac{t^p}{4\alpha}\right) + O(t^{4p-2}). \end{aligned} \tag{8.5}$$

From (6.21) we see that if $\zeta_2(\tau) = x$ then

$$\begin{aligned} \tau = \zeta_2^{-1}(x) &= \{4\alpha|x|[1 + O(|x|^{(2p-1)/p})\}^{1/p} \\ &= [4\alpha|x|]^{1/p} + O(x^2). \end{aligned}$$

Thus, if $\zeta_2(t) < x < 0$,

$$\begin{aligned} t^p w_2(x, t) &= \int_{\zeta_2^{-1}(x)}^t z_2(x, s)\, ds \\ &= \int_{[4\alpha|s|]^{1/p}}^t \frac{p}{2} s^{p-1}\left(x + \frac{s^p}{4\alpha}\right) ds + O(t^{4p-1}) \\ &= \frac{t^{2p}}{16\alpha} + \frac{xt^p}{2} + ax^2 + O(t^{4p-1}). \end{aligned}$$

Hence,

$$pt^{p-1}w_2(x, t) - z_2(x, t) = -\frac{pt^{2p-1}}{16\alpha} + \frac{p\alpha x^2}{t} + O(t^{4p-2}). \tag{8.6}$$

Next, if $x > 0$,

$$\begin{aligned} t^p w_2(x, t) - (\zeta_1^{-1}(x))^p w_2(x, \zeta_1^{-1}(x)) &= \int_{\zeta_1^{-1}(x)}^t z_2(x, s)\, ds \\ &= \int_{[4\alpha x]^{1/p}}^t \left(\frac{p}{2} s^{p-1}x + \frac{s^p}{4\alpha}\right) ds + O(t^{4p-1}) \\ &= \frac{t^{2p}}{16\alpha} + \frac{xt^p}{2} - 3\alpha x^2 + O(t^{4p-1}). \end{aligned}$$

Also, if $x = \zeta_1(t)$,

$$(\zeta_1^{-1}(x))^p w_2(x, \zeta_1^{-1}(x)) = (\zeta_1^{-1}(x))^p(\psi_2 - \psi_1) = 4\alpha x^2 + O(t^{4p-1}).$$

Hence, if $x < 0$,

$$pt^{p-1}w_2(x, t) - z_2(x, t) = -\frac{pt^{2p-1}}{16\alpha} + \frac{p\alpha x^2}{t} + O(t^{4p-2}). \tag{8.7}$$

With the estimates (8.6) and (8.7) at hand, we can now bootstrap the proofs of Theorem 5.5 and (6.21). We begin by computing

(8.8)

$$\begin{aligned}\int_{\zeta_2(t)}^{\zeta_1(t)} [pt^{p-1}w_2(x, t) - z_2(x, t)]\, dx &= \int_{\zeta_2(t)}^{\zeta_1(t)} \left(-\frac{pt^{2p-1}}{16\alpha} + \frac{p\alpha x^2}{t}\right) dx + O(t^{4p}) \\ &= -\frac{t^{2p-1}}{16\alpha}(\zeta_1(t) - \zeta_2(t)) \\ &\quad + \frac{p\alpha}{3t}[(\zeta_1(t))^3 - (\zeta_2(t))^3] + O(t^{4p}) \\ &= -\frac{pt^{3p-1}}{32\alpha} + \frac{p\alpha}{3t}\frac{2t^{3p}}{(4\alpha)^3} + O(t^{4p}) \\ &= -\frac{pt^{3p-1}}{48\alpha} + O(t^{4p}).\end{aligned}$$

Next,

$$\begin{aligned}&\int_a^{\zeta_1(t)} [p(\zeta_1^{-1}(x))^{p-1}w_2(x, \zeta_1^{-1}(x)) - z_2(x, \zeta_1^{-1}(x))]\, dx \\ &\quad = \int_0^t [ps^{p-1}w_2(\zeta_1(s), s) - z_2(\zeta_1(s), s)]\zeta_1'(s)\, ds \\ &\quad = \int_0^t \left[-\frac{ps^{2p-1}}{16\alpha} + \frac{p\alpha(\zeta_1(s))^2}{s}\right]\zeta_1'(s)\, ds + O(t^{4p}) \\ &\quad = \int_0^t \left[-\frac{ps^{2p-1}}{16\alpha} + \frac{p\alpha}{s}\frac{s^{2p}}{16\alpha^2}\right]\zeta_1'(s)\, ds + O(t^{4p}) \\ &\quad = O(t^{4p})\end{aligned} \tag{8.9}$$

since the last integrand vanishes.

Using (8.8) and (8.9) we conclude from (5.40) that

$$\zeta_1(t) - \zeta_2(t) = \frac{t^p}{2\alpha} - \frac{pt^{3p-1}}{48\alpha^2} + O(t^{4p}). \tag{8.10}$$

Next we compute

(8.11)

$$\int_{\zeta_2(t)}^{\zeta_1(t)} x[pt^{p-1}w_2(x, t) - z_2(x, t)]\, dx = \int_{\zeta_2(t)}^{\zeta_1(t)} \left[-\frac{pt^{p-1}x}{16\alpha} + \frac{p\alpha x^3}{t} \right] dx + O(t^{5p}).$$

$$= -\frac{pt^{2p-1}}{32\alpha}\,[(\zeta_1(t))^2 - (\zeta_2(t))^2] + \frac{p\alpha}{4t}\,[(\zeta_1(t))^4 - (\zeta_2(t))^4] + O(t^{5p}).$$

We also find, analogously to (8.9), that

$$\int_a^{\zeta_1(t)} x[p(\zeta_1^{-1}(x))^{p-1}w_2(x, \zeta_1^{-1}(x) - z_2(x, \zeta_1^{-1}(x))]\, dx = O(t^{5p}).$$

Substituting this and (8.11) into (5.42) and also recalling (5.43), we get

$$\zeta_1^2(t) - \zeta_2^2(t) = O(t^{5p}); \tag{8.12}$$

here we have used the assumptions that $r \geq 3p$, $q \geq 4p - 1$ and $\sigma \geq 3p - 1$.

Dividing both sides of (8.12) by $\zeta_1 - \zeta_2$ and using (8.10), we get

$$\zeta_1(t) + \zeta_2(t) = O(t^{4p}).$$

Comparing this with (8.10), the assertions (8.1) and (8.2) follow.

References

1. J. V. Breakwell and H. Chernoff, *Sequential tests for the mean of a normal distribution II (large t)*, Ann. Math. Statist., vol. 35 (1964), pp. 162–173.
2. H. Brezis and A. Friedman, *Estimates on the support of solutions of parabolic variational inequalities*, Illinois J. Math., vol. 20 (1976), pp. 82–97.
3. H. Brezis and D. Kinderlehrer, *The smoothness of solutions to non-linear variational inequalities*, Indiana Univ. Math. J., vol. 23 (1974), pp. 831–844.
4. L. A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math., vol. 139 (1977), pp. 155–184.
5. H. Chernoff, *Sequential analysis and optimal design*, Regional Conference Series in Applied Mathematics, #8, SIAM, Philadelphia, 1972.
6. L. C. Evans and B. F. Knerr, *Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities*, Illinois J. Math., vol. 23 (1979), pp. 153–166.
7. A. Friedman, *Free boundary problem for parabolic equations I. Melting of solids*, J. Math. Mech., vol. 8 (1959), pp. 499–517.
8. ———, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
9. ———, *Stochastic differential equations and applications*, vol. 2, Academic Press, New York, 1976.

10. ———, *Variational inequalities in sequential analysis*, SIAM J. Math. Anal., vol. 12 (1981), pp. 385–397.
11. B. F. Jones, *A class of singular integrals*, Amer. J. Math., vol. 86 (1964), pp. 441–462.
12. B. Knerr, *A singular free boundary problem*, Illinois J. Math., vol. 23 (1979), pp. 438–458.

Northwestern University
Evanston, Illinois