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THE HOMOLOGY OF THE JAMES-HOPF MAPS

BY

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If X is a path connected space, there are filtered spaces C_nX and CX which approximate $\Omega^n\Sigma^nX$ and $QX = \lim_{\rightarrow} \Omega^n\Sigma^nX$ respectively [11]. Quotients of successive filtrations are the extended power spaces denoted by $D_{n,q}X$ and D_qX. Snaith [13], generalizing a result of Kahn [6], showed that

$$\Sigma^\infty\Omega^n\Sigma^nX \simeq \bigvee_{q\geq 1} \Sigma^\infty D_{n,q}X$$

where $\Sigma^\infty Y$ denotes the suspension spectrum of a space Y.

Projection onto the q-th wedge summand and adjunction yield the James-Hopf maps

$$j_q : QX \to QD_qX \quad \text{and} \quad j_q : \Omega^n\Sigma^nX \to QD_{n,q}X.$$

It is the purpose of this paper to study the induced maps j_{q*} on homology and from this deduce geometric results. All homology will be with Z_2 coefficients.

Our geometric input is the following. Let $D_0X = S^0$. In [2] it is shown that, via passage to quotients of filtrations, the additive E_∞-structure on CX induces an E_∞-ring structure on $\Pi_{q\geq 0} QD_qX$. Let $j_0 : QX \to QD_0X$ send everything to 1, the identity map in QS^0. The maps j_q piece together to give a map

$$j : QX \to \prod_{q\geq 0} QD_qX.$$

F. Cohen, R. Cohen, May, and Taylor [2] show that the map j is exponential in the sense that it takes the additive E_∞ structure on QX to the multiplicative one on $\Pi_{q\geq 0} QD_qX$. Also $j_1 : QX \to QX$ is homotopic to the identity and the composite

$$X \xrightarrow{\eta} QX \xrightarrow{j_q} QD_qX$$

is nullhomotopic for $q > 1$.

Recall that the rich structure of iterated loop spaces allows one to define operations on their homology. Under these operations, $H_*(QX)$ is generated by $H_*(X)$.

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The image of $H_*(X)$ under j_* is clear, and thus the calculation of j_* in general reduces to the calculation of "multiplicative" homology operations in $H_*(\Pi_{q\geq 0} QD_qX)$. Projection onto the q-th component then completes the calculation of j_{q*}.

This leads us into analysis of the relationships between additive and multiplicative homology operations in E_∞-ring spaces, in particular Tsuchiya's "mixed" Adem relations [14]. The applications illustrate the usefulness of a variety of properties of these formulae.

In §1 we recall the properties of Dyer-Lashof operations and the various formulae which hold in E_∞-ring spaces and review the computations of $H_*(QX; Z_2)$.

In §2 we present some of the more tractable computations of the map j_{q*} and indicate some of the problems that arise in general. As an application, we give a proof of the "delooped" Kahn-Priddy theorem.

In §3 we examine the homology operations appearing in the image of j_{q*}. We prove the following result, the case $n = 1$ of which is due to Kirley [8].

THEOREM 3.1. *Let $2^p \leq n < 2^{p+1}$. In Snaith's splitting, if the map of spectra*

$$\Sigma^\infty\Omega^{n+1}\Sigma^{n+1}X \to \Sigma^\infty D_{n+1,2^t}X$$

desuspends to

$$\Sigma^d\Omega^{n+1}\Sigma^{n+1}X \to \Sigma^d D_{n+1,2^t}X$$

then $d > 2^{t+p+1} - 2^{p+1} + n$.

Finally, Tsuchiya's "mixed" Adem formula was published in [14] without proof. Ensuing discussions between May and Tsuchiya raised questions concerning the validity of this formula (see [3, p. 105]). Thus, in the appendix, we provide a complete derivation of the mixed Adem relations.

I would like to thank Peter May for suggesting the approach taken in this paper.

1. Homology operations in E_∞-ring spaces

Recall that if Y is an $(n + 1)$-fold loop space, there are operations

$$Q_r : H_a(Y) \to H_{2a+r}(Y)$$

defined for $0 \leq r \leq n$. We also let $x * y$ denote the homology product and Δx denote the diagonal of x in $H_*(Y) \otimes H_*(Y)$.

Typically, these Dyer-Lashof operations are reindexed with $Q^r x = Q_{r-a}x$, but for the purposes of this paper it will be more illuminating to use the lower indices.

If Y is an infinite loop space then the operations Q_r are additive. They satisfy the usual Cartan formula with respect to $*$, the diagonal Cartan formula, and Adem relations. The Nishida relations relate their action to the action of the Steenrod algebra. For details of the construction and properties of the Dyer-Lashof operations see [3].

To describe $H_*(QX)$, we recall that there are natural maps

$$\alpha : C_nX \to \Omega^n\Sigma^nX \quad \text{and} \quad \alpha : CX \to QX,$$

compatible with the natural inclusions $\eta : X \to C_nX$ and $\eta : X \to \Omega^n\Sigma^nX$. The usefulness of the spaces C_nX and CX derives from the fact that the maps α are weak homotopy equivalences if X is connected, and group completions in general [11], [12].

CX has an additive E_∞-structure, allowing Dyer-Lashof operations to be defined on $H_*(CX)$, and $H_*(CX)$ can be described as follows. If $I = (i_1, \ldots, i_s)$ is a sequence of nonnegative integers, let Q_I denote the operation $Q_{i_1}Q_{i_2} \cdots Q_{i_s}$. Define $l(I)$, the length of I, by $l(I) = s$. Admissible sequences I will mean nondecreasing sequences with $i_1 > 0$ or the empty sequence. Let

$$T(X) = \{Q_Ix \mid x \in \tilde{H}_*(X), \text{I admissable}\}$$

and let $A(X)$ be the polynomial algebra generated by $T(X)$.

Proposition 1.1 [3]. $H_*(CX) = A(X)$.

By construction, it is clear that the image of the map $H_*(C_{n+1}X) \to H_*(CX)$ generated by elements x with $w(x) \leqslant q$. If $x \in H_*(F_qCX)$, we will denote by $\bar{x}$ the corresponding element in $H_*(D_qX)$.

If $x \in H_*(CX)$ is a monomial, we define $w(x)$, the weight of x, inductively: $w(x) = 1$ if $x \in H_*(X) \subset H_*(CX)$, $w(Q_ix) = 2w(x)$, and $w(x * y) = w(x) + w(y)$. This gives an algebraic filtration of $H_*(CX)$ corresponding to the geometric filtration of CX. More precisely, $H_*(F_qCX)$ is the submodule of $H_*(CX)$ generated by elements x with $w(x) \leqslant q$. If $x \in H_*(F_qCX)$, we will denote by $\bar{x}$ the corresponding element in $H_*(D_qX)$.

An E_∞-ring space has both additive and multiplicative structure maps, compatible in the appropriate manner [3]. Typical examples are QS^0, with structure coming from loop addition and the smash product, and $BO \times \mathbf{Z}$, with structure coming from the direct sum and tensor product of vector bundles.

If Y is an E_∞-ring space, then $H_*(Y)$ admits both additive and multiplicative sets of Dyer-Lashof operations, to be denoted by Q_r and $\tilde{Q}_r$, respectively. We let $x * y$ denote the additive homology product and $x \# y$ or xy denote the multiplicative one. Both are commutative.

Let [0] and [1] in $H_0(Y)$ denote the additive and multiplicative units. Note that $Q_0([0]) = \tilde{Q}_0([0]) = [0]$, $\tilde{Q}_0([1]) = [1]$, $Q_r([0]) = \tilde{Q}_r([0]) = 0$ if $r > 0$, and $\tilde{Q}_r([1]) = 0$ if $r > 0$.

In the example we wish to study, $H_*(\Pi_{q\geq 0} QD_qX)$, there are elements of the form $y_0 * y_1 * y_2 * \cdots$ with $y_q \in H_*(QD_qX)$ and all but finitely many of the y_q equal to [0]. If $\bar{x} \in H_*(D_qX)$ is considered as an element of $H_*(\Pi_{q\geq 0} QD_qX)$, then $\tilde{Q}_i\bar{x} = \overline{Q_ix}$. The multiplicative unit in $H_*(\Pi_{q\geq 0} QD_qX)$ is $[1] \in H_0(QS_0)$.

With this notation, to calculate j_* and thus j_{q*} we use the following consequence of the geometry of j.

LEMMA 1.2. *If $x \in H_*(X) \subset H_*(QX)$ and I is a sequences of indices, then $j_*(x) = [1] * x$, and thus*

$$j_*(Q_Ix) = \tilde{Q}_Ij_*(x) = \tilde{Q}_I([1] * x).$$

Also

$$j_*(y * z) = j_*(y) \# j_*(z) \quad \text{for all } y \text{ and } z.$$

Madsen, May and Tsuchiya [10], [3], [14] determined various "mixed" Cartan and Adem relations between the various operations in E_∞-ring spaces. These will be used to evaluate expressions of the form $\tilde{Q}_I([1] * x)$. We express these relations in lower indices.

PROPOSITION 1.3 [3, p. 80]. *Let $\Delta x = \Sigma x' \otimes x''$, $\Delta y = \Sigma y' \otimes y''$.*

(1) $x \# (y * z) = \Sigma(x' \# y) * (x'' \# z)$
(2) $([1] * x) \# ([1] * y) = \Sigma[1] * x' * y' * (x'' \# y'')$

Let P^r denote the dual to the Steenrod operation Sq^r.

PROPOSITION 1.4 [3, p. 81]. $(Q^sx) \# y = \Sigma_i Q^{s+i}(x \# P^iy)$

PROPOSITION 1.5 (the "mixed" Cartan formula) [3, p. 89]. *If $\Delta x = \Sigma x' \otimes x''$ and $\Delta y = \Sigma y' \otimes y''$ then*

$$\tilde{Q}_r(x * y) = \sum_{a+b+c=r} \sum_{\Delta x, \Delta y} \tilde{Q}_ax' * Q_b(x''y') * \tilde{Q}_cy''.$$

Binomial coefficients $\binom{b}{a}$ are defined, for all integers b and a, by

$$(1 + x)^b = \sum_{a=0}^{\infty} \binom{b}{a} x^a \quad \text{and} \quad \binom{b}{a} = 0 \text{ for } a < 0.$$

We interpret $\binom{b}{a}$ as an integer mod 2.

PROPOSITION 1.6 (The "mixed" Adem relations) [14]. *If $\Delta x = \Sigma x' \otimes x''$ then*

$$\tilde{Q}_rQ_sx = \sum_{i,j,k} \sum_{\Delta x} \binom{j+k-r}{s-i-k} Q_i\tilde{Q}_jx' * Q_{r+2s-i-2j-2k}\tilde{Q}_kx''.$$

In the Appendix we provide an elaboration of the details of Tsuchiya's unpublished proof of this formula.

Remark 1.7. Both of these last two formulae will be seen to simplify greatly when homology elements are primitive. In this paper x will be said to be primitive if $\Delta x = [0] \otimes x + x \otimes [0]$.

We list the most useful elementary properties of binomial coefficients. These will be used both explicitly and implicitly throughout the remaining sections.

PROPOSITION 1.8.

(1) $$\binom{b}{a} = \frac{b(b-1)\cdots(b-a+1)}{a!} \text{ if } a > 0, \quad \binom{b}{0} = 1.$$

(2) $$\binom{b}{a} = \binom{a-b-1}{a}.$$

(3) *If* $b \geqslant 0$ *then*

$$\binom{b}{a} = \binom{b}{b-a}.$$

(4) *If* $b = \Sigma_{i\geqslant 0}\, b_i 2^i$ *and* $a = \Sigma_{i\geqslant 0}\, a_i 2^i$ *are the binary expansions of* b *and* a, *then*

$$\binom{b}{a} \equiv \prod_i \binom{b_i}{a_i} \quad \text{mod } 2.$$

In other words, $\binom{b}{a} = 1$ *if and only if* $a_i \leqslant b_i$ *for all* i.

2. The James maps and the Kahn-Priddy theorem

F. Cohen posed the following question: For a connected space X, is there a filtration of $H_*(QD_qX)$ such that $j_{q*} : H_*(QX) \to H_*(QD_qX)$, restricted to the subalgebra with generators of weight greater than or equal to q, is a monomorphism of algebras, up to filtration?

In this section, I use good behavior of the mixed Cartan and Adem relations to answer Cohen's question in the affirmative when q is a power of 2, with the assumption that $H_*(X)$ is primitive. The calculations of j_* will then imply a "delooped" Kahn-Priddy theorem.

We first examine the mixed Adem relations. The following elementary lemma plays an important role in all of our calculations.

LEMMA 2.1 ("Pairing" lemma). *In the formula*

$$\tilde{Q}_r Q_s x = \sum_{i,j,k} \sum_{\Delta x} \binom{j+k-r}{s-i-k} Q_i \tilde{Q}_j x' * Q_{r+2s-i-2j-2k} \tilde{Q}_k x'',$$

all terms with $j + k > r$ vanish in pairs, as do all terms with $j + k = r$ except those with $j = k = r/2$, $i = s - r/2$, and $x' = x''$.

Proof. The commutativity of $*$ and the commutativity of Δ imply that, in the sum, a term $Q_i\tilde{Q}_jx' * Q_{r+2s-i-2j-2k}\tilde{Q}_kx''$ appears twice, unless $x' = x''$, $j = k$, and $i = r + 2s - i - 2j - 2k$. This last condition implies that $i = s - j - k + r/2$ so that the corresponding coefficient,

$$\binom{j+k-r}{s-i-k} = \binom{2(j-r/2)}{j-r/2},$$

is 0 unless $j = r/2$.

In all other cases we have terms pairing:

$$\left[\binom{j+k-r}{s-i-k} + \binom{k+j-r}{s-(r+2s-i-2j-2k)-j}\right] Q_i\tilde{Q}_jx' * Q_{r+2s-i-2j-2k}\tilde{Q}_kx''.$$

The coefficient is of the form

$$\left[\binom{b}{a} + \binom{b}{b-a}\right]$$

and is thus 0 unless $j + k - r < 0$.

COROLLARY 2.2. *If x is primitive,*

$$\tilde{Q}_rQ_sx = \sum_{j<r}\left[\binom{j-r}{s-j} + \binom{j-r}{2j-r-s}\right] Q_{r+2s-2j}\tilde{Q}_jx.$$

This corollary should be interpreted as saying that the use of the mixed Adem relations strictly lowers the index of the multiplicative operation.

The following is a consequence of the mixed Cartan formula.

LEMMA 2.3. *If x is primitive, $\tilde{Q}_r([1] * x) = [1] * Q_rx + [1] * \tilde{Q}_rx$.*

If $J = (j_1, \ldots, j_s)$ and $K = (k_1, \ldots, k_t)$ are two sequences, let JK denote the sequence $(j_1, \ldots, j_s, k_1, \ldots, k_t)$.

COROLLARY 2.4. *If x is primitive, then*

$$\tilde{Q}_I([1] * x) = \sum_{J_1K_1\cdots J_mK_m = I} [1] * \tilde{Q}_{J_1}Q_{K_1}\cdots\tilde{Q}_{J_m}Q_{K_m}x.$$

The mixed Adem relations can now be used to change terms

$$\tilde{Q}_{J_1}Q_{K_1}\cdots\tilde{Q}_{J_m}Q_{K_m}x$$

into a linear combination of terms of the form $Q_K\tilde{Q}_Jx$. Then the ordinary

Adem relations can be used to change $Q_K\tilde{Q}_J x$ into a linear combination of terms of the form $Q_K\tilde{Q}_{J'}x$ where J' is nondecreasing.

LEMMA 2.5. *If $I = (i_1, \ldots, i_s)$ is a sequence of nonnegative integers then $\tilde{Q}_I x$ is a linear combination of terms of the form $\tilde{Q}_{I'}x$ such that $l(I') = l(I)$, I' is nondecreasing, and if $I' = (i'_1, \ldots, i'_s)$ then $i'_1 \leq i_1$.*

Proof. If I is not nondecreasing then iterated use of the ordinary Adem relations will make it so. Consider the Adem relation

$$\tilde{Q}_r\tilde{Q}_s x = \sum_j \binom{j-r}{2j-r-s} \tilde{Q}_{r+2s-2j}\,\tilde{Q}_j x.$$

The coefficient $\binom{j-r}{2j-r-s}$ is 0 unless $2j - r - s \geq 0$, so that $r + 2s - 2j \leq s$. Thus if $s < r$ then $r - 2s - 2j < r$. The lemma follows.

Together with this lemma, repeated use of Corollary 2.2 now implies the following proposition.

PROPOSITION 2.6. *Let $I = (i_1, \ldots, i_s)$ be a nondecreasing sequence and let x be primitive. Then*

$$\tilde{Q}_I([1] * x) \equiv \sum_{KJ=I} [1] * Q_K\tilde{Q}_J x$$

modulo elements of the form $[1] * Q_{K'}\tilde{Q}_{J'}x$ such that $l(K') + l(J') = s$, J' is nondecreasing, and if $J' = (j_1, \ldots, j_t)$, then $j_1 < i_{s-t+1}$.

Computations of the James-Hopf maps can now be made. Recall that

$$j_*(Q_I x) = \tilde{Q}_I([1] * x) \quad \text{if } x \in H_*(X) \subset H_*(QX).$$

To compute j_{2^t*}, we project onto $H_*(QD_{2^t}X)$. The terms in the sum of Corollary 2.4 which will contribute to j_{2^t*} will be exactly those with $l(J_1) + \cdots + l(J_m) = t$. Proposition 2.6 thus implies the following result.

PROPOSITION 2.7. *Suppose that $x \in H_*(X)$ is primitive and $I = (i_1, \ldots, i_s)$ is a nondecreasing sequence.*

(1) *$j_{q*}(Q_I x) = 0$ unless $q = 2^t$ with $t \leq s$.*

(2) *If $t \leq s$, let $I = KJ$ with $l(J) = t$. Then $j_{2^t*}(Q_I x) \equiv Q_K\overline{Q_J x}$ modulo elements of the form $Q_{K'}\overline{Q_{J'}x}$ such that $l(K') = l(K)$, $l(J') = l(J)$, J' is nondecreasing, and if $J' = (j_1, \ldots, j_k)$ then $j_1 < i_{s-t+1}$.*

The behavior of j_{2^t*} with respect to the product $*$ is more complicated. It is convenient to define a weight function on finite sums of elements in $H_*(\Pi_{q\geq 0} QD_qX)$ of the form $[\varepsilon] * y_1 * y_2 * y_3 \cdots$ where y_q is a monomial in $H_*(QD_qX)$, $\varepsilon = 0$ or 1, and all but finitely many of the y_q are equal to

[0]. Inductively we define $w'(y) = k$ if $y = [\varepsilon] * \bar{x}$ with

$$\bar{x} \in H_*(D_kX) \subset H_*\left(\prod_{q \geq 0} QD_qX\right),$$

$$w'(y * x) = w'(y) + w'(x), \quad w'(Q_ix) = 2w'(x)$$

$$\text{and} \quad w'(x + y) = \min\{w'(x), w'(y)\}.$$

LEMMA 2.8. (1) *If* $\Delta x = \Sigma x' \otimes x''$ *then* $w'(x') + w'(x'') \geq w'(x)$.
(2) $w'(P^r x) \geq w'(x)$ *if* $P^r x \neq 0$
(3) $w'(x * y) = w'(x) + w'(y)$ *and* $w'(Q_ix) = 2w'(x)$
(4) $w'(x \# y) \geq w'(x) + w'(y)$ *and* $w'(\tilde{Q}_ix) \geq 2w'(x)$.

COROLLARY 2.9. *If I is nonempty and x is primitive then*

$$([1] * Q_Ix) \# ([1] * y) \equiv [1] * (Q_Ix * y)$$

modulo terms of higher weight.

Proof. We may assume that $I = (i)$, a sequence of length 1. By Proposition 1.3,

$$([1] * Q_ix) \# ([1] * y) = [1] * (Q_ix * y) + \sum_{\Delta y}[1] * y' * (Q_ix \# y'').$$

Proposition 1.4 then implies that $Q_ix \# y''$ is a linear combination of terms of the form $Q_s(x \# P^r y'')$ so that $w'([1] * y' * (Q_ix \# y'')) > w'([1] * Q_Ix * y)$.

If X is a connected space, let

$$T_{2^t}(X) = \{Q_Ix | x \in \tilde{H}_*(X), l(I) \geq t, \text{ and } I \text{ is nondecreasing}\}.$$

Let $A_{2^t}(X)$ be the polynomial algebra generated by $T_{2^t}(X)$. Then $A_{2^t}(X)$ is a subalgebra of $H_*(QX)$.

The two weight functions w and w' induce decreasing filtrations on $H_*(QX)$ and $H_*(QD_{2^t}X)$ respectively, and the map j_* is filtration preserving, by construction and Lemma 2.8.

THEOREM 2.10. *If $H_*(X)$ is primitive then $H_*(QD_{2^t}X)$ becomes an $A_{2^t}(X)$-module, up to filtration, via the map j_{2^t*}. The map*

$$j_{2^t*} : H_*(QX) \to H_*(QD_{2^t}X)$$

is a map of $A_{2^t}(X)$-modules, and the composite

$$A_{2^t}(X) \to H_*(QX) \to H_*(QD_{2^t}X)$$

is a monomorphism of algebras, up to filtration.

Proof. As an immediate consequence of Proposition 2.7 and Corollary 2.9, if $l(I) \geqslant t$ then $j_{2^t*}(Q_I x * y) \equiv j_{2^t*}(Q_I x) * j_{2^t*}(y)$ modulo elements of higher weight. Also Proposition 2.7 implies that the generating set $T_{2^t}(X)$ is mapped monomorphically to an algebraically independent set in $H_*(Q_{2^t}X)$. The theorem follows.

Example 2.11. If $X = S^1$ and $x \in H_1(S^1)$ is the generator, then

$$\tilde{Q}_3([1] * x) \# \tilde{Q}_1([1] * x)$$
$$= ([1] * Q_3 x + [1] * \tilde{Q}_3 x) \# ([1] * Q_1 x + [1] * \tilde{Q}_1 x).$$

This can be expanded by Proposition 1.3(2). The only terms which can contribute to j_{2*} are $[1] * \tilde{Q}_3 x * \tilde{Q}_1 x$ and $[1] * (Q_3 x \# Q_1 x) = [1] * Q_0 Q_0 (x \# x) = [1] * (x \# x)^4$. We conclude that

$$j_{2*}(Q_3 x) = \overline{Q_3 x}, j_{2*}(Q_1 x) = \overline{Q_1 x},$$

but

$$j_{2*}(Q_3 x * Q_1 x) = \overline{Q_3 x} * \overline{Q_1 x} + \overline{x * x}^4.$$

Note that $\overline{x * x}^4$ is in filtration 8 while $\overline{Q_3 x} * \overline{Q_1 x}$ is in filtration 4.

Now suppose that X is a connected infinite loop space with structure map

$$QX \xrightarrow{\theta} X.$$

θ is an infinite loop map, and thus Y, the fiber of θ, is an infinite loop space.

We construct an infinite loop map $g : QD_2X \to Y$. The composition

$$X \xrightarrow{\eta} QX \xrightarrow{\theta} X$$

is the identity. Consider the diagram

$$\begin{array}{ccccccc} & & & & & \overset{g'}{\nearrow} & Y \\ & & & & & & \downarrow i \\ X & \xrightarrow{\eta} & F_2CX & \longrightarrow & QX & \xrightarrow{1-\eta\theta} & QX \\ & & & & & & \downarrow \theta \\ & & & & & & X \end{array}.$$

The composite

$$QX \xrightarrow{1-\eta\theta} QX \xrightarrow{\theta} X$$

is nullhomotopic, as is the composite

$$X \xrightarrow{\eta} QX \xrightarrow{1-\eta\theta} QX.$$

Thus we can find a lifting $g' : QX \to Y$ such that

$$X \xrightarrow{\eta} QX \xrightarrow{g'} X$$

is trivial (assuming that η is a cofibration, see [9]). Then

$$X \xrightarrow{\eta} F_2CX \to D_2X$$

is a cofibration sequence so that $g' \circ \eta$ trivial implies that g' factors through a map $g'' : D_2X \to Y$. Let $g : QD_2X \to Y$ be the infinite loop map extending g''.

THEOREM 2.12. *If $X = S^1$, the composite*

$$Y \xrightarrow{i} QS^1 \xrightarrow{j_2} Q\Sigma \mathbf{R}P^\infty \xrightarrow{g} Y$$

is an equivalence, localized at the prime 2.

The following lemma should be contrasted to Corollary 2.2.

LEMMA 2.13.

$$\sum_j \left[\binom{j-r}{s-j} + \binom{j-r}{2j-r-s}\right] Q_{r+2s-2j}Q_j x = 0.$$

Proof. The equation

$$Q_rQ_s = \sum_j \binom{j-r}{2j-r-s} Q_{r+2s-2j}Q_j x$$

is an Adem relation [3]. The other sum also equals Q_rQ_sx by the following computation:

$$\begin{aligned}
\sum_j \binom{j-r}{s-j} Q_{r+2s-2j}Q_j x &= \sum_j \sum_i \binom{j-r}{s-j}\binom{i-r-2s+2j}{2i-r-2s+j} Q_{r+2s-2i}Q_i x \\
&= \sum_i \sum_j \binom{j-r}{s-j}\binom{i-j-1}{2i-r-2s+j} Q_{r+2s-2i}Q_i x \\
&= \sum_i \sum_k \binom{s-r-k}{k}\binom{i-s-1+k}{2i-r-s-k} Q_{r+2s-2i}Q_i x \\
&= \sum_i \binom{i-r}{2i-r-s} Q_{r+2s-2i}Q_i x \\
&= Q_rQ_sx.
\end{aligned}$$

The second-to-last equality made use of Adem's formula [1]:

$$\sum_k \binom{a-k}{k}\binom{b+k}{c-k} \equiv \binom{a+b+1}{c} \mod 2.$$

Proof of Theorem 2.12. As spaces $QS^1 \approx S^1 \times Y$, and as $A_2(S^1)$-modules,

$$H_*(QS^1) \simeq H_*(S^1) \otimes A_2(S^1).$$

Let $x \in H_1(S^1)$ denote the homology generator. Then $i_* g_*(\overline{Q_i x}) = Q_i x$, by construction. This, together with the fact that g is an infinite loop map, completely specifies g_* and implies that $i_* : H_*(Y) \to H_*(QS^1)$ is an injection of algebras embedding $H_*(Y)$ isomorphically as $A_2(S^1)$. Certainly, g_* is filtration preserving. If $I = (I', i)$ then, reasoning as in Proposition 2.7, we have $j_{2*}(Q_I x) \equiv Q_{I'}\overline{Q_i x}$ modulo elements in the kernel of g_*, by virtue of Lemma 2.13. We conclude that $g_* j_{2*} i_*(Q_I x) = Q_I x$.

By Theorem 2.10, $j_{2*}i_*$ is an algebra map, up to filtration. Therefore, up to filtration, the composite $g_* j_{2*} i_*$ is a map of algebras, which is the identity on a generating set. Thus the composite is an isomorphism on Z_2-homology. Since Y is simply connected, this composite is a 2-local homotopy equivalence.

Note that Y is the universal cover of QS^1, so that $\Omega Y = Q_0 S^0$. Thus we have the following corollary.

COROLLARY 2.14 (Kahn-Priddy Theorem) [7].

$$Q_0 S^0 \xrightarrow{\Omega j_2} Q\mathbf{R}P^\infty \xrightarrow{\Omega g} Q_0 S^0$$

is a homotopy equivalence, localized at 2.

Remark 2.15. In this last corollary, the equivalence visibly deloops once, but does not deloop twice. For example,

$$g_* j_{2*}(Q_3 x) = Q_3 x, \; g_* j_{2*}(Q_1 x) = Q_1 x,$$

but

$$g_* j_{2*}(Q_3 x * Q_1 x) = Q_3 x * Q_1 x + x^8.$$

See Example 2.11.

Remark 2.16. For a general connected infinite loop space of finite type, it should still be true that the composite

$$Y \xrightarrow{i} QX \xrightarrow{j_2} QD_2 X \xrightarrow{g} Y$$

is an equivalence, localized at 2. Indeed this is essentially the theorem of Kahn and Finkelstein announced in [5]. In this generality, however, the homology calculations of j_{2*} are much more complicated.

3. Applications to stable splittings

In the last section it was shown that use of the mixed Adem relations lowers the indices of the multiplicative homology operations. In this section we study the maximal increase in the indices of the additive operations.

THEOREM 3.1. *Let $2^p \leq n < 2^{p+1}$ and let X be any connected space with $\tilde{H}_*(X; Z_2) \neq 0$. If the map of spectra*

$$\Sigma^\infty \Omega^{n+1}\Sigma^{n+1}X \to \Sigma^\infty D_{n+1,2^t}X$$

desuspends to $\Sigma^d \Omega^{n+1}\Sigma^{n+1}X \to \Sigma^d D_{n+1,2^t}X$ then $d > 2^{t+p+1} - 2^{p+1} + n$.

Proof. By adjunction, to prove the theorem it suffices to show that if there were a factorization

$$\begin{array}{ccc} \Omega^{n+1}\Sigma^{n+1}X & \dashrightarrow & \Omega^d\Sigma^d D_{n+1,2^t}X \\ \downarrow & & \downarrow \\ QX & \xrightarrow{j_{2^t}} & QD_{2^t}X \end{array}$$

then $d > 2^{t+p+1} - 2^{p+1} + n$.

Recall that the image of $H_*\Omega^{n+1}\Sigma^{n+1}X \to H_*QX$ is the polynomial algebra generated by $Q_I x$ with I an admissible sequence of indices all less than $n + 1$. Consider the image under $j_{2^t *}$ of elements in H_*QX involving only Dyer-Lashof operations Q_i with $i \leq n$. A lower bound for d can be found by studying the maximal indices of the additive operations which appear. The theorem follows from the following proposition.

PROPOSITION 3.2. *Let $2^p \leq n < 2^{p+1}$ and let $r_n = 2^{p+1} - n$. Let*

$$I_{n,t} = (\overbrace{r_n, r_n, \ldots, r_n}^{t \text{ times}}, n).$$

If $x \in H_*(X)$ is primitive then $j_{2^t *}(Q_{I_{n,t}}x) = Q_{2^{t+p+1}-2^{p+1}+n}x^{2^t}$ modulo elements of the form $Q_i\overline{Q_J x}$ with $l(J) = t$ and $i < 2^{t+p+1} - 2^{p+1} + n$. Moreover, this result cannot be improved by replacing $I_{n,t}$ by any other sequence I of indices all less than or equal to n.

Proof. Corollary 2.2, Corollary 2.4, and the discussion following Proposition 2.6 imply that it suffices to study the effect of commuting t or fewer multiplicative Dyer-Lashof operations past an additive one, with the condition that all indices be less than or equal to n. Induction on t, Corollary 2.2, and the following lemma prove the proposition.

LEMMA 3.3. *Suppose that $2^p \leq n < 2^{p+1}$ and the following conditions hold:*

(1) $r \geq 0, s \geq 0, j \geq 0$,
(2) $r \leq n$,
(3) $s \leq 2^{t+p} - 2^{p+1} + n$,
(4) $$\binom{j-r}{s-j} + \binom{j-r}{2j-r-s} \neq 0.$$

Then $r + 2s - 2j \leqslant 2^{t+p+1} - 2^{p+1} + n$. *The equality is realized if* $j = 0$, $r = 2^{p+1} - n$, *and* $s = 2^{t+p} - 2^{p+1} + n$.

Proof. First note that the second term in the coefficient

$$\binom{j-r}{s-j} + \binom{j-r}{2j-r-s}$$

is zero unless $2j - r - s \geqslant 0$. But then

$$r + 2s - 2j \leqslant s \leqslant 2^{t+p} - 2^{p+1} + n \leqslant 2^{t+p+1} - 2^{p+1} + n.$$

Thus we can replace condition (4) by condition (4′):

$$\binom{j-r}{s-j} \neq 0. \tag{4′}$$

If $j = 0$, $r = 2^{p+1} - n$, and $s = 2^{t+p} - 2^{p-1} + n$, then

$$\binom{j-r}{s-j} = \binom{n-2^{p+1}}{2^{t+p} - 2^{p+1} + n} = \binom{2^{t+p} - 1}{2^{t+p} - 2^{p+1} + n} \neq 0,$$

so that (4′) holds. Also $r + 2s - 2j = 2^{t+p+1} - 2^{p+1} + n$, as claimed.

To prove that this is best possible we first show that we can assume that $j = 0$. We do this by induction on n. When $n = 2^p$ we clearly have the best possible result, since in this case $2^{p+1} - n = n$. Now suppose that both $r \leqslant n$ and $s \leqslant 2^{t+p} - 2^{p+1} + n$. If $j > 0$ then

$$\binom{j-r}{s-j} = \binom{(j-1)-(r-1)}{(s-1)-(j-1)},$$

and, by inductive hypothesis, this will be nonzero only if

$$(r-1) + 2(s-1) - 2(j-1) \leqslant 2^{t+p+1} - 2^{p+1} + (n-1).$$

But then $r + 2s - 2j \leqslant 2^{t+p+1} - 2^{p+1} + n$.

Thus if the lemma is not true then there will be a minimal counterexample with $j = 0$, so that

$$\binom{j-r}{s-j} = \binom{-r}{s}.$$

So suppose that r and s satisfy conditions (1), (2), and (3) and that

$$r + 2s > 2^{t+p+1} - 2^{p+1} + n.$$

Then

$$\begin{aligned} r + s - 1 &\geqslant 2^{t+p+1} - 2^{p+1} + n - s \\ &\geqslant 2^{t+p+1} - 2^{p+1} + n - (2^{t+p} - 2^{p+1} + n) \\ &= 2^{t+p}. \end{aligned}$$

Also $r + s + 1 - 2^{t+p} \leqslant s$ so that $0 \leqslant r + s - 1 - 2^{t+p} < s$. Finally $s < 2^{t+p}$ so that

$$\binom{-r}{s} = \binom{r+s-1}{s} = \binom{r+s-1-2^{t+p}}{s} = 0.$$

This completes the proof of the lemma, and thus the proofs of Proposition 3.2 and Theorem 3.1.

Remark 3.4. The hypothesis that x is primitive in Proposition 3.2 was made only to aid in the clarity of exposition. The use of nonprimitive elements will not improve the result given here.

Remark 3.5. Theorem 3.1 and Proposition 3.2 were proved by Kirley [8] in the special case when $n = 1$, using homology of groups techniques. This special case suffices to prove the following corollary.

COROLLARY 3.6. *If $n \geqslant 2$, then Snaith's stable splitting*

$$\Sigma^\infty \Omega^n \Sigma^n X = \bigvee_{q \geqslant 1} \Sigma^\infty D_{n,q} X$$

does not admit a desuspension

$$\Sigma^d \Omega^n \Sigma^n X = \bigvee_{q \geqslant 1} \Sigma^d D_{n,q} X$$

for any finite d.

Embeddings enter into the F. Cohen, May, and Taylor construction of the James-Hopf maps [4]. They show that if $F(\mathbf{R}^n, q)/\Sigma_q$ embeds in $\mathbf{R}^d$ then there is a James-Hopf map

$$j_q : \Omega^n \Sigma^n X \to \Omega^d \Sigma^d D_{n,q} X.$$

Here $F(\mathbf{R}^n, q)$ denotes the configuration space of q-tuples of distinct points in $\mathbf{R}^n$.

COROLLARY 3.7. *Let $2^p \leqslant n < 2^{p+1}$. Then $F(\mathbf{R}^{n+1}, 2^t)/\Sigma_{2^t}$ does not embed in $\mathbf{R}^{2^{t+p+1}-2^{p+1}+n}$.*

Remark 3.8. In [4] it is shown that the embedding dimension of $F(\mathbf{R}^{n+1}, 2)/\Sigma_2$ is $n + 2$ more than the immersion dimension of $\mathbf{R}P^n$. Thus Corollary 3.7 implies that $\mathbf{R}P^n$ doesn't immerse in $\mathbf{R}^{2^{p+1}-2}$. This is precisely the result detected by the Steifel Whitney classes on $\mathbf{R}P^n$.

Appendix. The mixed Adem relations

In this appendix we derive the mixed Adem relations. As a starting point we assume only formulae in [3] and [14] which are published with proofs.

Our basic line of reasoning follows Tsuchiya's unpublished proof. For easy comparison with [3] and [14] we use upper indices in this section.

LEMMA A1. *For arbitrary integers a, b and c, the following identities hold:*

(1) $$\sum_k \binom{a}{k}\binom{b}{c-k} = \binom{a+b}{c},$$

(2) $$\sum_k (-1)^k \binom{c}{k}\binom{b-k}{a} = \binom{b-c}{a-c} \quad \text{if } c \geqslant 0.$$

Proof. $(1+t)^a(1+t)^b = (1+t)^{a+b}$, and the comparison of the coefficients of t^c yields identity (1), when $c \geqslant 0$. If $c < 0$, both sides of (1) are clearly 0.

To prove (2), we have

$$t^c(1+t)^{b-c} = (1+t)^b\left(\frac{t}{1+t}\right)^c = (1+t)^b(1+(-1-t)^{-1})^c$$
$$= \sum_k (-1)^k \binom{c}{k}(1+t)^{b-k}.$$

If $a \geqslant 0$ we can equate coefficients of t^a, which yields

$$\binom{b-c}{a-c} = \sum_k (-1)^k \binom{c}{k}\binom{b-k}{a}.$$

If $a < 0$ and $c \geqslant 0$, then both sides of (2) are clearly 0.

LEMMA A2 [3, p. 80]. *If* $\Delta z = \Sigma z' \otimes z''$ *then*
$$(x * y) \# z = \Sigma(-1)^{\deg y \deg z'} xz' * yz''.$$

PROPOSITION A3 (Nishida relations) [3, p. 6].

$$P^r Q^s x = \sum_i \binom{s-r}{r-2i} Q^{s-r+i} P^i x.$$

Let χ denote conjugation in the Steenrod algebra.

PROPOSITION A4 [3], [14].

$$Q^s[1] \# x = \Sigma_r Q^{s+r} P^r x \quad \text{and} \quad Q^s x = \Sigma_t Q^{s+t}[1] \# \chi P^t x.$$

PROPOSITION A5 [3, p. 105]. *If* $r > s > 0$ *then*

$$\tilde{Q}^r Q^s[1] = \sum_{i=0}^{s} \binom{r-i-1}{s-i} Q^i[1] * Q^{r+s-i}[1].$$

It is convenient to define $\tilde{Q}^r$, Q^r, and P^r to be 0 if $r < 0$. Our first observation is that all restrictions on r, s, and i in the last three propositions are unnecessary.

PROPOSITION A6. *For* **all** *integers r and s, we have the following formulae:*

(1) $$P^rQ^sx = \sum_i \binom{s-r}{2-ri} Q^{s-r+i}P^ix,$$

(2) $$Q^s[1] \# x = \sum_r Q^{s+r}P^rx, \quad Q^sx = \sum_t Q^{s+t}[1] \# \chi P^tx$$

(3) $$\tilde{Q}^rQ^s[1] = \sum_i \binom{r-i-1}{s-i} Q^i[1] * Q^{r+s-i}[1].$$

Proof of (1). If $r<0$ then $P^rQ^sx = 0$. On the right hand side, $Q^{s-r+i}P^ix = 0$ unless $i \geq 0$, and, in that case,

$$\binom{s-r}{r-2i} = 0.$$

If $s < 0$ and $r \geq 0$ then $P^rQ^s = 0$ and, by excess arguments, $Q^{s-r+i}P^ix = 0$.

Proof of (2). If $s < 0$, then $Q^s[1] \# x = 0$ and, by excess arguments $Q^{s+r}P^rx = 0$ for all r. Also, it is true that $\Sigma_{r+t=k} P^r\chi P^t = 0$ if $k \neq 0$ and is P^0 if $k = 0$, even when r, t, and k are allowed to be negative. Thus

$$\sum_t Q^{s+t}[1] \# \chi P^tx = \sum_{t,r} Q^{s+t+r}P^r\chi P^tx = Q^sx.$$

Proof of (3). Consider the right hand side of the formula. $Q^i[1] = 0$ unless $i \geq 0$, and

$$\binom{r-i-1}{s-i} = 0$$

unless $s \geq i$, so we can assume that $0 \leq i \leq s$, as in Proposition A.5.

If $s < 0$, the left hand side of the equation, $\tilde{Q}^rQ^s[1]$, is 0. If $i \geq 0$ then

$$\binom{r-i-1}{s-i} = 0,$$

so that the right hand side is also 0.

Finally, the commutativity of $*$ implies that, in the sum, a term

$$Q^i[1] * Q^{r+s-i}[1]$$

appears twice, unless $i = r + s - i$. Note that

$$\binom{r-i-1}{s-i} = \binom{s-r}{s-i}.$$

In this last case, $i = (r + s)/2$, so that the corresponding coefficient

$$\binom{s - r}{s - i} = \binom{s - r}{(s - r)/2} \text{ is 0 unless } r = s.$$

In all other cases we have terms pairing:

$$\left[\binom{s - r}{s - i} + \binom{s - r}{s - (r + s - i)}\right] Q^i[1] * Q^{r+s-i}[1].$$

The coefficient is of the form

$$\left[\binom{b}{a} + \binom{b}{b - a}\right]$$

and is thus 0 unless $s - r < 0$.

If $r < s$ then $\tilde{Q}^r Q^s[1] = 0$. Also $\tilde{Q}^r Q^r[1] = Q^r[1] * Q^r[1]$ [3, p. 82]. We have just shown that the right hand side of the formula agrees with these results.

We can now prove the mixed Adem relations.

THEOREM A7. *Let $\Delta x = \Sigma x' \otimes x''$. For all integers r and s,*

$$\tilde{Q}^r Q^s x = \sum_{i,j,k} \sum_{\Delta x} \binom{r - i - 2k - 1}{s + j - i - k} Q^i \tilde{Q}^j x' * Q^{r+s-i-j-k} \tilde{Q}^k x''.$$

Proof.

$$\begin{aligned}
\tilde{Q}^r Q^s x &= \sum_t \tilde{Q}^r [Q^{s+t}[1] \# \chi P^t x] \\
&= \sum_{t,a} \tilde{Q}^{r-a} Q^{s+t}[1] \# \tilde{Q}^a \chi P^t x \\
&= \sum_{t,a,b} \binom{r - a - b - 1}{s + t - b} [Q^b[1] * Q^{r+s+t-a-b}[1]] \# \tilde{Q}^a \chi P^t x \\
&= \sum_{\substack{t,a,b\\c,d}} \sum_{\Delta x} \binom{r - a - b - 1}{s + t - b} [Q^b[1] \# \tilde{Q}^c \chi P^d x'] \\
&\quad * [Q^{r+s+t-a-b}[1] \# \tilde{Q}^{a-c} \chi P^{t-d} x''] \\
&= \sum_{\substack{t,a,b,\ c\\d,u,v}} \sum_{\Delta x} \binom{r - a - b - 1}{s + t - b} Q^{b+u} P^u \tilde{Q}^c \chi P^d x' \\
&\quad * Q^{r+s+t-a-b+v} P^v \tilde{Q}^{a-c} \chi P^{t-d} x''
\end{aligned}$$

$$= \sum_{\substack{t,a,b,c,d,\\ u,v,m,n}} \sum_{\Delta x} \binom{r-a-b-1}{s+t-b} \binom{c-u}{u-2m} \binom{a-c-v}{v-2n}$$

$$\times Q^{b+u}\tilde{Q}^{c-u+m}P^m \chi P^d x' * Q^{r+s+t-a-b+v}\tilde{Q}^{a-c-v+n}P^n \chi P^{t-d} x''$$

$$= \sum_{\substack{d,e,i,j,\\ k,m,n}} \sum_{\Delta x} \left[\sum_{u,v} \binom{r+m+n-i-j-k-v-1}{s+d+e+u-i} \binom{j-m}{u-2m} \right.$$

$$\left. \times \binom{k-n}{v-2n} \right] Q^i \tilde{Q}^j P^m \chi P^d x' * Q^{r+s+m+n+d+e-i-j-k} \tilde{Q}^k P^n \chi P^e x''$$

where we have let $i = b + u$, $j = c - u + m$, $k = a - c - v + n$, and $e = t - d$ and have eliminated the variables $t = d + e$, $a = j + k + u + v - m - n$, $b = i - u$, and $c = j + u - m$.

By excess arguments, $\tilde{Q}^j P^m$ is 0 unless $j \geqslant m$, in which case

$$\binom{j-m}{u-2m} = \binom{j-m}{j+m-u}.$$

Then

$$\sum_{u,v} \binom{r+m+n-i-j-k-v-1}{s+d+e+u-i} \binom{j-m}{j+m-u} \binom{k-n}{v-2n}$$

$$= \sum_{u,v} \binom{r+m+n-i-j-k-v-1}{s+d+e+u-i}$$

$$\times \binom{j-m}{(j+m+s+d+e-i)-(s+d+e+u-i)} \binom{k-n}{v-2n}$$

$$= \sum_{v} \binom{r+n-i-k-v-1}{j+m+s+d+e-i} \binom{k-n}{v-2n} \quad \text{(by Lemma A1 (1))}$$

$$= \sum_{v} \binom{k-n}{v-2n} \binom{(r-n-i-k-1)-(v-2n)}{j+m+s+d+e-i}$$

$$= \binom{r-i-2k-1}{j+m+n+d+e+s-i-k} \quad \text{if } k-n \geqslant 0$$

(by Lemma A1 (2)).

If $k - n < 0$ then $\tilde{Q}^k P^n = 0$, by excess arguments. Thus

$$\tilde{Q}^r Q^s x = \sum_{\substack{d,e,i,j,\\ k,m,n}} \sum_{\Delta x} \binom{r-i-2k-1}{j+m+n+d+e+s-i-k} Q^i \tilde{Q}^j P^m \chi P^d x'$$

$$* Q^{r+s+m+n+d+e-i-j-k} \tilde{Q}^k P^n \chi P^e x''.$$

Since m and d appear combined as $m + d$ and since $\Sigma_{m+d=q} P^m \chi P^d x' = 0$ for $q \neq 0$, all terms vanish except those with $m = d = 0$. Similarly we can assume that $n = e = 0$. Thus

$$\tilde{Q}^r Q^s x = \sum_{i,j,k} \sum_{\Delta x} \binom{r - i - 2k - 1}{j + s - i - k} Q^i \tilde{Q}^j x' * Q^{r+s-i-j-k} \tilde{Q}^k x''.$$

Remark. In lower indices this formula is that given in Proposition 1.5.

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