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LOCATION OF ZEROS PART II: ORDERED FIELDS

BY

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1. Introduction

In this sequel to [6], we continue the investigation into the behavior of multiplier sequences and n-sequences (of the first kind) begun in Part I. The definitions and notation used in Part I for studying the real numbers will be carried over to arbitrary fields in this part. In particular, a *multiplier sequence* for a field F is an infinite sequence $\Gamma = \{\gamma_0, \gamma_1, \gamma_2, \ldots\}$ of elements of F with the property that if $f(x) = \Sigma a_k x^k$ is a polynomial which splits in F, then $\Gamma[f] = \Sigma \gamma_k a_k x^k$ also splits in F. An *n-sequence* is a finite sequence

$$\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_n\}$$

with the above property for polynomials of degree at most n. A multiplier sequence Γ will be called an *exponential* sequence if $\gamma_k = cr^k$, $k = 0, 1, 2, \ldots$ for some elements c and r in the field. These sequences, together with those such that $\gamma_k = 0$ for $k \neq n, n + 1$ (some fixed n), will be called *trivial* multiplier sequences; they are precisely the multiplier sequences which work for all fields [4]. If $\Gamma = \{\gamma_0, \gamma_1, \gamma_2, \ldots\}$ and s is a positive integer, the sequence $\{\gamma_s, \gamma_{s+1}, \gamma_{s+2}, \ldots\}$ will be called a *shift* of Γ; it is again a multiplier sequence [4, Proposition 2.2]. For further definitions and notation see Part I [6]. We shall refer to results in Part I by using the form Theorem I.2.3 to mean Theorem 2.3 of Part I.

Recall that a field is said to be *formally real* if it can be ordered [2], [11]. In the next section, we explore the extent to which multiplier sequences and n-sequences can be characterized over arbitrary formally real fields in ways similar to that for the real numbers as first proved by Pólya and Schur [4, Theorem 3.1]. This leads us to study certain special classes of formally real fields.

In the third section, we take one of the main results of Part I for the real numbers and try to extend it to arbitrary real closed fields. In Part I we showed that a multiplier sequence Γ for the real numbers can be applied to an arbitrary polynomial with real coefficients, giving a new polynomial

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with no more nonreal roots than the original polynomial. Thus, if the degree is unchanged, multiplier sequences never decrease the number of real roots. We also showed in Part I that this fails for n-sequences in general; therefore Tarski's principle is not applicable, since the hypothesis that Γ be a multiplier sequence requires that it be applied to polynomials of all degrees and is thus not an elementary statement. Section 3 establishes this result for all multiplier sequences in certain real closed fields and for certain multiplier sequences in arbitrary real closed fields.

In Section 4 we look at the converse problem of determining what can be said about the field when its multiplier sequences satisfy this inequality on the change in the number of roots. We are particularly interested in the special case of this inequality for the sequence $\Gamma = \{0, 1, 2, 3, \ldots\}$ since this is roughly equivalent to saying that the field satisfies Rolle's theorem for polynomials. The question of whether any field which satisfies Rolle's theorem is necessarily real closed was raised by Michael Slater in 1972 [14] and a counterexample has just been given by M. J. Pelling [12]. Earlier, I. Kaplansky posed the problem of characterizing the fields for which the above sequence Γ is a multiplier sequence [8, p. 30, Exercise 4]. This problem remains open.

In the final section, we consider the structure of the set of all multiplier sequences for an arbitrary field F (not necessarily formally real). This is a semigroup of linear operators on the space of polynomials over F which leaves invariant the set of polynomials that split in F. We look at a few properties of these semigroups, about which very little is known in general.

2. Multiplier sequences and n-sequences in formally real fields

In [4, Section 3] the problem of determining the multiplier sequences of a formally real field is studied. Only two major results are given there. The first states that if the field has any nontrivial multiplier sequence, then it must be a *pythagorean* field (every sum of squares from the field is a square in the field). The second result characterizes the multiplier sequences of a real closed field and we state it here for future reference. This result has also been obtained by Zervos [16] using entirely different methods.

THEOREM 2.1 [14, Theorem 3.7]. *Let F be a real closed field and let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of elements in F. The polynomial $\Gamma[(x+1)^n]$ splits in F and all of its roots have the same sign if and only if Γ is an n-sequence for F. In particular, Γ is a multiplier sequence for F if and only if for every positive integer n, the polynomial $\Gamma[(x+1)^n]$ splits in F and all its roots have the same sign.*

We shall generalize this theorem to a wider class of fields. We also give a characterization of pythagorean fields and give a partial solution to an open problem stated in [4].

The condition that all of the roots of $\Gamma[(x+1)^n]$ have the same sign tends to unnecessarily complicate the proofs of the results to be presented below. Further complications ensue when the field under consideration has more than one ordering, in which case we must require that the roots all have the same sign with respect to each fixed ordering. To overcome these difficulties, we generalize Remark I.2.3 and note that essentially nothing is lost if we assume that all entries of the sequence Γ are either zero or *totally positive* (positive in all orderings); in fact, we shall see that we may assume that all the entries of Γ are squares. Then, by virtue of this assumption, the coefficients of the polynomial $\Gamma[(x+1)^n]$ are all nonnegative and thus all its roots must be less than or equal to zero in each ordering.

PROPOSITION 2.2. *Let F be any formally real field. Then Γ is a multiplier sequence for F if and only if there exists a multiplier sequence Λ_1, all of whose entries are squares, and an exponential sequence Λ_2 such that $\Gamma = \Lambda_1 \Lambda_2$, where the product is defined componentwise.*

Proof. Theorem 3.4(b) of [4] states that the nonzero entries in Γ involve at most two square classes of F. If Γ is a trivial multiplier sequence, the proposition is clear. Otherwise, let γ_k be the first nonzero entry in Γ. Then [4, Theorem 3.4] implies that $\gamma_{k+1} \neq 0$ and that for any $\gamma_j \neq 0$, either $\gamma_j \gamma_k$ or $\gamma_j \gamma_{k+1}$ is a square in F depending on whether $j \equiv k$ or $k+1$ modulo 2. Now define Λ_2 to be the exponential sequence $\{cr^n\}_{n=0}^{\infty}$, where $r = \gamma_k^{-1} \gamma_{k+1}^{-1}$; and $c = \gamma_k^{-1}$ if k is even and γ_{k+1}^{-1} if k is odd. Then set $\Lambda_1 = \Gamma\Lambda_2^{-1}$, where Λ_2^{-1} is the exponential sequence $\{c^{-1}r^{-n}\}_{n=0}^{\infty}$. A straightforward computation now shows that $\Gamma = \Lambda_1 \Lambda_2$ as desired. The converse is trivial, so the proof is complete.

For any formally real field F, we shall let F^* denote the intersection of all real closures of F inside a fixed algebraic closure $\overline{F}$ of F. Let $\tilde{F}$ be a fixed real closure of F and consider the following three conditions that the field F may satisfy, where Γ denotes a sequence of zeros and totally positive elements of F:

(A_n) If $\Gamma[(x+1)^n]$ splits in F, then Γ is an n-sequence for F.
(B_n) If $\Gamma[(x+1)^n]$ splits in F^*, then Γ is an n-sequence for F.
(C_n) If $\Gamma[(x+1)^n]$ splits in $\tilde{F}$, then Γ is an n-sequence for F.

Theorem 2.1 shows that any real closed field satisfies (A_n), (B_n) and (C_n) for all n. Since $F \subseteq F^* \subseteq \tilde{F}$, we see that ($C_n$) implies ($B_n$), which in turn implies (A_n). The main results of this section concern the conditions (B_n) and (C_n). We know very little about (A_n). Indeed, for $n > 2$, it is not even known whether (A_n) and (B_n) are equivalent, though this seems unlikely. For $n = 2$ we have the following:

THEOREM 2.3. *Let F be a formally real field. The following conditions are equivalent:*

(a) *F satisfies* (A_2).
(b) *F satisfies* (B_2).
(c) *F is pythagorean.*
(d) *There exists a* 2*-sequence* $\{\gamma_0, \gamma_1, \gamma_2\}$ *with* $\gamma_0 \gamma_1 \gamma_2 \neq 0$ *and* $\gamma_1^2 - \gamma_0 \gamma_2 \neq 0$.
(e) *If a polynomial f of degree* 3 *over F splits, then its derivative f' also splits* (*see* [8, p. 30, Exercise 4]).

Proof. We have already noted that (b) implies (a). Assume F satisfies (A_2). Then $\Gamma = \{1, 2, 3\}$ is a 2-sequence for F since

$$\Gamma[(x + 1)^2] = (3x + 1)(x + 1),$$

and thus (a) implies (d). The implication (d) $\Rightarrow$ (c) is contained in [4, Theorem 2.8]. Now assume that F is pythagorean and

$$\Gamma[(x + 1)^2] = \gamma_2 x^2 + 2\gamma_1 x + \gamma_0$$

splits in F^*. Since F is pythagorean, Proposition 7 of [7] implies that F is the intersection of F^* with the quadratic closure of F, and thus $\Gamma[(x + 1)^2]$ splits in F. In particular, we see by computing the discriminant that $\gamma_1^2 - \gamma_0 \gamma_2$ is a square in F. We must show that

$$\Gamma[(x + a)(x + b)] = \gamma_2 x^2 + \gamma_1(a + b)x + \gamma_0 ab$$

splits in F for any choice of $a, b \in F$; that is,

$$\gamma_1^2(a + b)^2 - 4\gamma_0 \gamma_2 ab$$

is a square in F. If any two of $\gamma_0, \gamma_1, \gamma_2$ are zero or if $ab = 0$, this is clear, so assume otherwise. Since F is formally real, the element γ_1 can not be zero [4, Theorem 3.4(a)]. Now,

$$\gamma_1^2(a + b)^2 - 4\gamma_0 \gamma_2 ab$$
$$= (a\gamma_1 + b\gamma_1^{-1}(\gamma_1^2 - 2\gamma_0 \gamma_2))^2 + 4b^2 \gamma_0 \gamma_2 \gamma_1^{-2}(\gamma_1^2 - \gamma_0 \gamma_2)$$

is a sum of squares since $\gamma_1^2 - \gamma_0 \gamma_2$ is a square and $\gamma_0 \gamma_2$ is a square [4, Theorem 3.4(b)], hence the discriminant is a square in the pythagorean field F. Thus (c) implies (b). To obtain the equivalence of (e) with the others, note that (e) is equivalent to the statement that $\Gamma = \{0, 1, 2, 3\}$ is a 3-sequence, since $\Gamma[f(x)] = x f'(x)$ for any polynomial f of degree 3. Thus, (e) implies that $\{1, 2, 3\}$ satisfies (d). Finally, we assume that F is pythagorean. Then (e) holds since

$$\frac{d}{dx}[(x - \alpha)(x - \beta)(x - \gamma)]$$

is a quadratic polynomial with discriminant

$$4(\alpha + \beta + \gamma)^2 - 12(\alpha\beta + \beta\gamma + \alpha\gamma) = 2[(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\alpha - \gamma)^2]$$

which is a sum of squares.

PROPOSITION 2.4. *Let F be a formally real field satisfying* (A_n). *Then every arithmetic sequence Γ of totally positive elements is an n-sequence for F.*

Proof. Let $\gamma_k = a + kb$, where a, $b \in F$ are totally positive. Then

$$\Gamma[(x + 1)^n] = (x + 1)^{n-1}((a + bn)x + a)$$

splits in F, so by (A_n), $\{\gamma_0, \ldots, \gamma_n\}$ is an n-sequence for F.

The following proposition generalizes [3, Theorem 2.2].

PROPOSITION 2.5. *Let F, F^* and $\tilde{F}$ be as before.*

(a) *The field F satisfies* (B_n) *if and only if every polynomial over F of degree n which splits in F^* splits in F.*

(b) *The field F satisfies* (C_n) *if and only if every polynomial over F of degree n which splits in $\tilde{F}$ splits in F.*

Proof. We shall state the proof for (a). Part (b) follows *mutatis mutandis.* Assume first that F satisfies (B_n), and let

$$f(x) = \sum_{i=0}^{n} a_i^{xi}, \quad a_n = 1,$$

be a polynomial over F which splits in F^*. Note that for any ordering of F and any root α of $f(x)$, we have

$$\alpha = -\sum_{i=0}^{n-1} a_i\,\alpha^{i-n} \leqslant \sum_{i=0}^{n-1} |a_i||\alpha^{i-n}| < 1 + \sum_{i=0}^{n-1} |a_i| < 1 + \sum_{i=0}^{n-1} (1 + a_i^2).$$

Without loss of generality, we may replace $f(x)$ by

$$f\left(x + 1 + \sum_{i=0}^{n-1} (1 + a_i^2)\right),$$

so that all of the coefficients of $f(x)$ may be assumed to be totally positive. Now write

$$f(x) = \sum_{i=0}^{n} \gamma_k \binom{n}{k} x^k$$

and let $\Gamma = \{\gamma_0, \ldots, \gamma_n\}$. Then $\Gamma[(x + 1)^n] = f(x)$ which splits in F^*. Then (B_n) implies that Γ is an n-sequence for F, hence $f(x)$ splits in F.

Conversely, assume that every polynomial over F of degree n which splits in F^* splits in F. Let Γ be a sequence of zeros and totally positive elements such that $\Gamma[(x + 1)^n]$ splits in F^*, and let f be a polynomial of degree $m \leqslant n$ splitting in F. By Theorem 2.1, the sequence Γ is an n-sequence for each real closure of F. Therefore $\Gamma[f]$ splits in all real closures of F; hence $(x + 1)^{n-m}\Gamma[f]$ splits in F^* and thus, by hypothesis, also in F. By definition, Γ is an n-sequence and the proof is complete.

Recall that a field is called *euclidean* if every element is either a square or its negative is a square. Equivalently, the field is pythagorean and has a unique ordering, the set of positive elements being the squares.

COROLLARY 2.6. *A field F satisfies* (C_2) *if and only if F is euclidean.*

THEOREM 2.7. *Let F be a formally real field. The following are equivalent.*

(a) *F satisfies* (B_n) *for all* $n = 1, 2, 3, \ldots$.
(b) *If a polynomial over F splits in F^*, then it splits in F.*
(c) $F = F^*$.

Proof. The equivalence of (a) and (b) follows from Propsosition 2.5(a). The implication (c) $\Rightarrow$ (b) is trivial. Finally, assume that F satisfies (b). The field F^* is a normal extension of F [7, Lemma 1]. Thus any polynomial over F with one root in F^* splits in F^*, and hence splits in F by hypothesis. Therefore $F = F^*$.

COROLLARY 2.8. *Let F be a formally real field such that $F = F^*$. If Γ is an infinite sequence of zeros and totally positive elements and $\Gamma[(x + 1)^n]$ splits in F for all n, then Γ is a multiplier sequence for F.*

THEOREM 2.9. *Let F be an ordered field with real closure $\tilde{F}$. The following are equivalent:*

(a) *F satisfies* (C_n) *for all* $n = 1, 2, 3, \ldots$.
(b) *If a polynomial over F splits in $\tilde{F}$, then it splits in F.*
(c) *F has no normal extension to which its ordering extends.*

Proof. The equivalence of (a) and (b) follows from Proposition 2.5(b). The equivalence of (b) and (c) follows immediately from the definition of a normal extension.

Fields which are intersections of real closed fields arise naturally in many contexts [1], [3], [7], [15], particularly in quadratic form theory. There is a minimal such field, namely $\mathbf{Q}^*$, consisting of all algebraic numbers whose minimal polynomial over $\mathbf{Q}$ has only real roots [3]. A large class of examples are the hereditarily pythagorean fields studied in [1, Chapter 3]. In particular, this includes the field of iterated Laurent series over a real closed field F, denoted $F((t_1))((t_2)) \cdots ((t_n))$. Thus we see that any multiplier sequence for the real numbers $\mathbf{R}$ is also a multiplier sequence for $\mathbf{R}((t))$ by Corollary 2.8. Our results also extend easily to rings of formal power series (or indeed, to any unique factorization domain whose field of fractions is an intersection of real closed fields).

COROLLARY 2.10. *Let $R = F[[t]]$ be the ring of formal power series over a field $F = F^*$, and let Γ be any sequence of squares in R. If $\Gamma[(x + 1)^n]$ is a product of linear factors in $R[x]$, then for any $f(x) \in R[x]$ of degree at most n which is a product of linear factors, the polynomial $\Gamma[f(x)]$ is again a product of linear factors in $R[x]$.*

Proof. By hypothesis, the polynomial $\Gamma[(x + 1)^n]$ splits in $F((t))$, which by [1, Chapter 3, §2] is an intersection of real closed fields. By Theorem 2.7, the polynomial $\Gamma[f(x)]$ splits over $F((t))$. Since R is a unique factorization domain, Gauss' lemma implies that $\Gamma[f(x)]$ factors into linear factors in $R[x]$.

COROLLARY 2.11. *Let Γ be an n-sequence for $\mathbf{R}$ with all entries in $\mathbf{Q}$. Then Γ is an n-sequence for any intersection of real closed fields.*

Proof. The polynomial $\Gamma[(x + 1)^n]$ splits in $\mathbf{R}$ and the roots are algebraic over $\mathbf{Q}$, so it splits in every real closure of $\mathbf{Q}$, hence in $\mathbf{Q}^*$. Thus it splits in any intersection of real closed fields, so Γ is an n-sequence.

Next we consider Open Question 5 from [4]: let $\{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence for F. Under what conditions on F is the sequence $\{c\gamma_k + k\gamma_{k-1}\}_{k=0}^{\infty}$, $c \in F$, $\gamma_{-1} = 0$, again a multiplier sequence for F. We restrict our consideration to formally real fields and totally positive sequences. For the real numbers, any $c \geqslant 0$ works; for if we set

$$\Phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k!,$$

an entire function of type I in the Laguerre-Pólya class, then

$$(x + c)\,\Phi(x) = \sum_{k=0}^{\infty}(c\gamma_k + k\gamma_{k-1})x^k/k!$$

is again such an entire function, so that the sequence

$$\{c\gamma_k + k\gamma_{k-1}\}_{k=0}^{\infty}.$$

is also a multiplier sequence [4, Theorem 3.2]. Using §3 of Part I to get a theorem in terms of n-sequences, we can apply Tarski's principle to obtain this result for any real closed field.

THEOREM 2.12. *Let F be a real closed field and let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence of nonnegative elements. Then for any $c \geq 0$, the sequence $\{c\gamma_k + k\gamma_{k-1}\}_{k=0}^{\infty}$, $\gamma_{-1} = 0$, is also a multiplier sequence for F.*

Proof. Fix $n \geq 1$ and consider the n-sequence $\{\gamma_0, \ldots, \gamma_n\}$. By Corollary I.3.10 and Tarski's principle, the first $n + 1$ of the Taylor coefficients of

$$(x + c)\sum_{k=0}^{n} \gamma_k x^k/k! = \sum_{k=0}^{n}(c\gamma_k + k\gamma_{k-1})x^k/k! + \gamma_n x^{n+1}/n!$$

form an n-sequence. Since n was arbitrary, the sequence

$$\{c\gamma_k + k\gamma_{k-1}\}_{k=0}^{\infty}$$

is a multiplier sequence.

COROLLARY 2.13. *Let F be any formally real field satisfying $F = F^*$ and let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence of totally positive elements. Then for any totally positive element $c \in F$, the sequence $\Lambda = \{c\gamma_k + k\gamma_{k-1}\}_{k=0}^{\infty}$ is also a multiplier sequence for F.*

Proof. The polynomial $\Gamma[(x + 1)^n]$ splits in F, hence Γ is a multiplier sequence for every real closure of F. Thus $\Lambda[(x + 1)^n]$ splits in every real closure of F, hence in F^*, so Λ is a multiplier sequence by Corollary 2.8.

While we do not know whether the converse of the above corollary holds, the existence of such multiplier sequences is a rather strong condition and some consequences are given in the following proposition.

PROPOSITION 2.14. *Let F be an ordered field for which, given any multiplier sequence $\{\gamma_k\}$ of totally positive elements, the sequence $\{c\gamma_k + k\gamma_{k-1}\}$, $c \geq 0$ is again a multiplier sequence. Then every arithmetic sequence of totally positive elements is a multiplier sequence for F and F is a pythagorean field.*

Proof. Let $\gamma_k = 1$ for all k. Then given a, b totally positive in F, set $c = ab^{-1}$ to obtain $\{a + kb\}_{k=0}^{\infty}$ as a multiplier sequence for F. The field F is pythagorean by Theorem 2.3.

3. The fundamental inequality

In Theorem I.2.4 we have seen that for any multiplier sequence Γ of the real numbers $\mathbf{R}$ and any polynomial f with real coefficients, the polynomial

$\Gamma[f]$ has no more nonreal roots than f; in the notation of Part I, this is the inequality

$$(3.1) \qquad Z_c(\Gamma[f]) \leqslant Z_c(f).$$

We now extend the meaning of $Z_c(f)$. Given any polynomial f with coefficients in a field F, by $Z_c(f)$ we mean the number of roots of f which do not lie in F. It is necessary to count in this manner because the operator Γ can change the degree of f.

In this section we shall investigate the validity of inequality (3.1) for multiplier sequences in real closed fields. First note that it holds for any real closed subfield of **R**. For if R is a real closed subfield of **R** with a multiplier sequence Γ, then Γ is also a multiplier sequence for **R** by Theorem 2.1. Thus inequality (3.1) holds with respect to **R**. But the roots of $f \in R[x]$ and $\Gamma[f]$ are all algebraic over R, hence the meaning of Z_c is the same with respect to both **R** and the real closed field R.

On the other hand, if one looks carefully at the proof of Theorem I.2.4, one sees that besides needing a real closed field, the main requirement for that proof is the use of a limit as integers n approach infinity, so that the proof works only for a real closed field which is *archimedean* (every element of the field is less than some integer). But this again gives us precisely the real closed subfields of **R**. For nonarchimedean real closed fields the question is open, though we shall give several partial results. Note that the statement is not elementary, so Tarski's principle cannot be used. Furthermore, we saw in Theorem I.4.8 that the inequality fails in general for n-sequences. One remaining hope is to use the constants $c(n)$ defined in Section I.4 in order to keep the problem within the real numbers where analytic methods can be used.

PROPOSITION 3.2. *Let* $c(n) = \inf\{m \geqslant n \mid$ *for all* m*-sequences* Γ *of* **R** *and all polynomials* $f \in \mathbf{R}[x]$ *of degree* n, $Z_c(\Gamma[f]) \leqslant Z_c(f)\}$. *If* $c(n) < \infty$ *for all* n, *then the inequality* (3.1) *holds for all multiplier sequences over any real closed field.*

Proof. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence for a real closed field F and let $f \in F[x]$ be a polynomial of degree n. Let $m = c(n)$ and consider the m-sequence $\Lambda = \{\gamma_0, \ldots, \gamma_m\}$. The statement that (3.1) holds in **R** for all m-sequences and all polynomials of degree n is elementary, hence it holds for F by Tarski's principle. Therefore $Z_c(\Gamma[f]) = Z_c(\Lambda[f]) \leqslant Z_c(f)$.

The remainder of this section will be spent looking at cases in which we can show that inequality (3.1) is valid. Our next few results use Tarski's principle. We then apply valuation theory to obtain a class of multiplier sequences which work. Finally we examine some sequences of infinitesimals which have no counterpart over the real numbers. A very simple application of Tarski's principle is to note that if Γ is a multiplier sequence for a real closed field R and all of the elements of Γ lie in $\mathbf{R}$, then the inequality holds for Γ since the statement that it holds up to degree n (for any fixed n) in $\mathbf{R}$ is elementary.

THEOREM 3.3. *Let R be any real closed field, let α be a positive element of R, and let g be a polynomial over R with only real negative roots. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be defined by the formal power series*

$$\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = g(x)\, e^{\alpha x} = g(x) \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} x^k.$$

Then $Z_c(\Gamma[f]) \leq Z_c(f)$ for all $f \in R[x]$.

Proof. Let $g(x) = \sum_{k=0}^{n} a_k\, x^k/k!$. Then

$$\gamma_k = \sum_{j=0}^{k} \binom{k}{j} a_j \alpha^{k-j}. \tag{3.4}$$

Pólya has shown that sequences defined in this way (i.e., $g(x)$ has only real negative roots and each γ_k is defined in terms of its coefficients by (3.4)) over the real numbers are multiplier sequences [4, Theorem 3.2], and hence satisfy the inequality by Theorem I.2.4. For a polynomial f of fixed degree m, the elements $\gamma_0, \ldots, \gamma_m$ need only satisfy finitely many elementary conditions, so that Tarski's principle can be applied to obtain the inequality for such sequences Γ over an arbitrary real closed field.

THEOREM 3.5. *Let $\Gamma = \{\gamma_0, \ldots, \gamma_n\}$ be an n-sequence for a real closed field R. Let $f(x) = \sum_{k=0}^{m} a_k x^k$ be a polynomial over R with degree $m \leq n$. Then*

$$Z_c\left(\sum_{k=0}^{m} \frac{\gamma_k}{(m-k)!} a_k x^k\right) \leq Z_c(f).$$

Proof. Since Γ is an n-sequence, the polynomial

$$\sum_{k=0}^{m} \frac{\gamma_k}{k!(m-k)!} x^k$$

splits in R by Theorem 2.1. The theorem then follows via Tarski's principle from Corollary 3.5 of [5].

PROPOSITION 3.6. *Let R be a real closed field and let $g(x)$ be a polynomial over R which splits in R and has all of its roots outside the open interval $(0, n)$. Then*

$$\Gamma = \{g(0), g(1), \ldots, g(n)\}$$

is an n-sequence such that $Z_c(\Gamma[f]) \leq Z_c(f)$ for all polynomials f of degree *at most n. In particular, if the roots of $g(x)$ are all negative in R, then*

$$\Gamma = \{g(0), g(1), g(2), \ldots\}$$

is a multiplier sequence satisfying inequality (3.1).

Proof. This is an old theorem of Laguerre for the real numbers and the statement is clearly elementary, so Tarski's principle applies. (In fact, the first of the two proofs given by Obreschkoff in [10, Satz 3.2] for Laguerre's theorem is valid for any real closed field, so Tarski's principle is not really necessary here.)

We next look at results which can be obtained using valuation theory and refer the reader to [11, §7] for elementary results on real valuations and real places. Let F be any ordered field. Then F has a unique place into the real numbers compatible with the ordering: if v denotes the corresponding (additive) valuation and A_v the valuation ring, then the positive elements of F which are units in A_v reduce to an ordering of the residue field, a subfield of $\mathbf{R}$, modulo the valuation ideal m_v. We shall generally denote the image of an element $a \in A_v$ in the residue field by $\bar{a}$.

LEMMA 3.7. *Let F be an ordered field with real valuation v as above. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence of totally positive elements of F such that $\gamma_k \in A_v$ for each k. Then $\bar{\Gamma} = \{\bar{\gamma}_k\}_{k=0}^{\infty}$ is a multiplier sequence for* $\mathbf{R}$.

Proof. We need to show that for each n, the polynomial $\bar{\Gamma}[(x+1)^n]$ has only real roots. By hypothesis, the polynomial $\Gamma[(x+1)^n]$ splits in F, so its roots map to elements of $\mathbf{R}$ or infinity under the real place. Thus $\bar{\Gamma}[(x+1)^n]$ can have no complex roots and therefore $\bar{\Gamma}$ is a multiplier sequence for $\mathbf{R}$.

THEOREM 3.8. *Let R be a real closed field. Let v be the unique valuation with residue field $F \subseteq \mathbf{R}$. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence for R with $v(\gamma_k) = 0$ for each k. Let f be a monic polynomial in $A_v[x]$ such that $\bar{f} \in F[x]$ has no multiple roots in F. Then $Z_c(\Gamma[f]) \leq Z_c(f)$.*

Proof. By the previous lemma, $\bar{\Gamma}$ is a multiplier sequence for $\mathbf{R}$. Furthermore, since the elements of Γ are all units in the valuation ring, the elements of $\bar{\Gamma}$ are nonzero. Now F is a real closed subfield of $\mathbf{R}$ [11, Theorem 8.6], hence $\bar{f}$ has no multiple roots in $\mathbf{R}$, and we can apply Proposition I.2.8 to conclude that $\bar{\Gamma}[\bar{f}]$ has at least as many distinct real roots with odd multiplicity as $\bar{f}$ has real roots. Since R is real closed, the valued field (R, v) is henselian [11, Theorem 8.6]. But then Hensel's Lemma guarantees that $\Gamma[f]$ (respectively, f) has roots in R corresponding to those of $\bar{\Gamma}[\bar{f}]$

(respectively, $\bar{f}$) in $\mathbf{R}$ which have odd multiplicity, and therefore $Z_c(\Gamma[f]) \leq Z_c(f)$ since $Z_c(\bar{\Gamma}[\bar{f}]) \leq Z_c(\bar{f})$ for $\mathbf{R}$.

We conclude this section by looking at sequences $\{\gamma_k\}_{k=0}^{\infty}$ such that each $\gamma_k \geq 0$ but the sequence decreases sufficiently rapidly to guarantee that the inequality holds. We shall make use of infinitesimal elements (with respect to $\mathbf{Q}$); that is, elements of the field which are less than every positive rational number and greater than every negative rational number [2].

THEOREM 3.9. *Let R be a nonarchimedean real closed field and let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of positive elements of R such that γ_0 is infinitesimal and $\gamma_k\gamma_{k-1}^{-k}$ is infinitesimal for each $k = 1, 2, 3, \ldots$ Then Γ is a multiplier sequence for R which satisfies $Z_c(\Gamma[f]) \leq Z_c(f)$ for all polynomials $f \in R[x]$. In fact, $Z_c(\Gamma[f]) = 0$ for any polynomial f, all of whose coefficients are positive, not infinitesimal and not infinite over* $\mathbf{Q}$ (i.e., *are positive units in the valuation ring for the unique place into* $\mathbf{R}$).

Proof. Let $h(x) = \sum_{k=0}^{n} a_k\gamma_k x^k$ be an arbitrary polynomial with each a_k positive, not infinitesimal and not infinite over $\mathbf{Q}$. The hypothesis that $\gamma_k\gamma_{k-1}^{-k}$ is infinitesimal implies that the $(k+1)$-*st* term dominates in the expression for $h(-\gamma_k^{-1})$, that is,

$$\begin{aligned} h(0) &= a_0\gamma_0 > 0, \\ h(-\gamma_1^{-1}) &= a_0\gamma_0 - a_1 + a_2\gamma_2\gamma_1^{-2} - \cdots < 0, \\ h(-\gamma_2^{-1}) &= a_0\gamma_0 - a_1\gamma_1\gamma_2^{-1} + a_2\gamma_2^{-1} - a_3\gamma_3\gamma_2^{-3} + \cdots > 0, \\ &\vdots \\ h(-\gamma_n^{-1}) &= a_0\gamma_0 - a_1\gamma_1\gamma_n^{-1} + \cdots + (-1)^n a_n\gamma_n^{-n+1}. \end{aligned}$$

Thus the polynomial h has n sign changes and hence n roots in R by the mean value theorem [2, §2, Exercise 13]. Therefore the last statement of the theorem holds. In particular, if $a_k = (k!)^{-1}$, we obtain

$$Z_c\left(\sum_{k=0}^{n} \gamma_k x^k/k!\right) = 0$$

for each positive integer n. The inequality now follows immediately from [5, Corollary 3.5] and an application of Tarski's principle. In particular, it works for $f(x) = (x+1)^n$, so Γ is a multiplier sequence by Theorem 2.1.

4. Rolle's theorem and generalizations

In this section we consider ordered fields other than real closed fields and ask when the inequality (3.1) of the previous section holds. We first note that this is closely related to the question of which fields satisfy Rolle's theorem for polynomials. (Rolle's theorem is known to hold for all real closed fields [2, §2, Exercise 12].) If one ignores the conclusion that the root of the derivative lie between the roots of the polynomial, this is equivalent to a special case of our inequality.

LEMMA 4.1. *Let F be an ordered field for which Rolle's theorem holds for polynomials. Let Γ be the arithmetic sequence $\Gamma = \{0, 1, 2, 3, \ldots\}$. Then $Z_c(\Gamma[f]) \leq Z_c(f)$ for all polynomials $f \in F[x]$.*

Proof. This follows immediately from the fact that $\Gamma[f(x)] = xf'(x)$.

In particular, $\Gamma = \{0, 1, 2, \ldots\}$ is a multiplier sequence for any field satisfying Rolle's theorem, so that such a field is necessarily pythagorean by Theorem 2.3. In fact, much more can be said.

THEOREM 4.2. *Let F be an ordered field which satisfies Rolle's theorem. Then $Z_c(\Gamma[f]) \leq Z_c(f)$ for every arithmetic sequence Γ of positive rational numbers.*

Proof. Let $\Gamma = \{a + bk\}_{k=0}^{\infty}$, $a, b \geq 0$. The result follows from Lemma 4.1 if $a = 0$ and is clear if $b = 0$, so assume $a > 0$ and $b > 0$. We may replace Γ by $b^{-1}\Gamma = \{ab^{-1} + k\}_{k=0}^{\infty}$; set $\alpha = ab^{-1}$. Then $\Gamma[f] = \alpha f(x) + xf'(x)$. Assume $\alpha = mn^{-1}$ with m, n positive integers. Now consider $n\Gamma[f] = mf(x) + nxf'(x)$ and

$$\frac{d}{dx}(x^m f(x)^n) = x^{m-1} f(x)^{n-1}(mf(x) + nxf'(x)).$$

By Rolle's theorem, applied to $x^m f^n$, together with the fact that the multiple roots of f are also roots of f' of multiplicity one less, we see that $mf + nxf'$ has at least as many roots in F as f does. Therefore $Z_c(\Gamma[f]) \leq Z_c(f)$ and the theorem is proved.

COROLLARY 4.3. *If an ordered field F satisfies Rolle's theorem, then for all integers $n \geq 2$, the field F contains an n-th root of each positive rational number.*

Proof. Let $n \geq 2$ be fixed and let m be any positive integer. It will suffice to show that F contains an n-th root of m. Let $r = m^{-1}$ and consider the arithmetic sequence

$$\Gamma = \{r^n, r^n + (r - r^n)n^{-1}, r^n + 2(r - r^n)n^{-1}, \ldots, r, \ldots\}$$

of positive rational numbers. Let $f(x) = x^n - 1$. Since f has a root in F, the polynomial $\Gamma[f(x)] = rx^n - r^n = r(x^n - r^{n-1})$ has a root $\alpha \in F$ by Theorem 4.2. Therefore αr^{-1}, an n-th root of m, also lies in F, and the proof is complete.

With somewhat stronger hypotheses, we can prove the field is real closed. A formally real field F is called *hereditarily pythagorean* if F and every formally real algebraic extension of F is pythagorean. These fields have been studied in depth by Becker [1].

THEOREM 4.4. *Let F be a hereditarily pythagorean field and assume $Z_c(\Gamma[f]) \leqslant Z_c(f)$ holds for all polynomials and every arithmetic sequence Γ of totally positive elements of F. Then F is real closed.*

Proof. Let $\alpha \in F$ be arbitrary with $\alpha^2 < 1$ in all orderings. Let

$$\Gamma = \{\alpha^2, (3\alpha^2 + 1)/4, (\alpha^2 + 1)/2, (\alpha^2 + 3)/4, 1, \ldots\}$$

and set $f(x) = x^4 - 1$. Since $\alpha^2 < 1$, the arithmetic sequence Γ has elements which are totally positive. Since f has two roots in F, the inequality implies

$$\Gamma[f] = x^4 - \alpha^2 = (x^2 + \alpha)(x^2 - \alpha)$$

also has two roots in F; that is, either α is a square or its negative is a square in F. Given any nonzero element $\beta \in F$, let $\alpha = \beta(1 + \beta^2)^{-2}$ so that $\alpha^2 < 1$ in all orderings. Then β also is either a square or its negative is a square. Therefore F is euclidean and hence has a unique ordering.

Now assume $0 < \alpha < 1$ and let Γ equal

$$\{\alpha^n, \alpha^n + (\alpha - \alpha^n)/n, \ldots, \alpha, \ldots\},$$

an arithmetic sequence of totally positive elements of F. Let $f(x) = x^n - 1$. Since f has a root in F, so does $\Gamma[f] = \alpha x^n - \alpha^n$ and therefore α has an n-th root in F. Since any positive element can be replaced by its multiplicative inverse, this holds for all $\alpha > 0$. By [1, Chapter 3, Theorem 13], the field F is hereditarily pythagorean if and only if the real closure of F is a radical extension. But F has no proper formally real radical extension, hence F is real closed. This completes the proof.

For future reference, we record the following proposition which was established in the proof of the previous theorem.

PROPOSITION 4.5. *Let F be a formally real field such that*

$$Z_c(\Gamma[f]) \leqslant Z_c(f)$$

for all polynomials $f \in F[x]$ and all arithmetic sequences of totally positive elements. Then F is euclidean and is closed under taking nth roots for all odd integers n.

Remark 4.6. (i) If we require that the inequality hold only for the multiplier sequences that the field happens to have, then we obtain a large class of fields. If F is ordered and not pythagorean, then the only multiplier sequences F has are the trivial ones and for these $Z_c(\Gamma[f])$ equals either $Z_c(f)$ or zero so that the inequality always holds.

(ii) We know from Section 2 that intersections of real closed fields have many multiplier sequences. In particular, all arithmetic sequences of totally positive elements are multiplier sequences. Thus we obtain many examples where the inequality fails. In $\mathbf{Q}^*$, even Rolle's theorem fails: use $f(x) = 2x^4 - 2x$. By Theorem 4.4, the inequality fails for any hereditarily pythagorean field which is not real closed. By Proposition 4.5, it fails for any field which is an intersection of real closed fields and has a solvable formally real extension.

We now take a different approach and see what information we can obtain using valuation theory.

THEOREM 4.7. *Let F be an ordered field satisfying Rolle's theorem and let v be a real valuation on F with valuation ring A_v, valuation ideal m_v, value group G_v and residue field F_v endowed with the induced ordering.*

(a) *The field F_v satisfies Rolle's theorem.*
(b) *For any odd natural number n, the value group G_v is n-divisible.*

Proof. Let $\bar{f} \in F_v[x]$ have roots $\bar{a} < \bar{b}$ in F_v, so we can write

$$\bar{f}(x) = (x - \bar{a})(x - \bar{b})\bar{g}(x).$$

Lift this to $f(x) = (x - a)(x - b)g(x) \in F[x]$. Since F satisfies Rolle's theorem, there exists an element $c \in F$ with $a < c < b$ and $f'(c) = 0$. But then $\bar{f}' \in F_v[x]$ has $\bar{c}$ as a root and $\bar{a} < \bar{c} < \bar{b}$, so F_v also satisfies Rolle's theorem.

For part (b), assume G_v is not n-divisible. Then there exists an element $\gamma \in G_v$ with $\gamma \notin nG_v$. We may assume $\gamma > 0$ and $\gamma \notin mG_v$ for any m dividing n. Let $t \in F$ with $v(t) = \gamma$ and consider

$$\begin{aligned} f(x) &= \left(\sum_{i=0}^{(n-1)/2} x^{2i}\right)(x + t/2 - 1)(x + t/2 + 1) \\ &= x^{n+1} + t \sum_{i=0}^{(n-1)/2} x^{2i+1} + \frac{t^2}{4} \sum_{i=0}^{(n-1)/2} x^{2i} - 1 \end{aligned}$$

which has exactly two roots in F since the roots of $\Sigma_{i=0}^{(n-1)/2} x^{2i}$ are nontrivial roots of 1, hence not in the formally real field F. We have

$$f'(x) = (n + 1)x^n + t \sum_{i=0}^{(n-1)/2} (2i + 1)x^{2i} + \frac{t^2}{2} \sum_{i=1}^{(n-1)/2} ix^{2i-1}.$$

Let α be a root of f' and let w, A_w, m_w, and G_w be the extensions to $F(\alpha)$ of v, A_v, m_v and G_v. Since $t \in m_v$, we have $\alpha \in m_w$, hence $w(\alpha) > 0$. Since $f'(\alpha) = 0$, at least two of the terms in

$$(n + 1)\alpha^n + t \sum(2i + 1)\alpha^{2i} + \frac{t^2}{2} \sum i\alpha^{2i-1}$$

have the same value in G_w. But $w(t) + 2i\, w(\alpha) > w(t)$ if $i > 0$ and $2w(t) + (2i - 1)\, w(\alpha) > w(t)$. Therefore $w(t)$, the value of the constant term, must equal $nw(\alpha)$, the value of the leading term; that is, $w(t) = nw(\alpha) \in nG_w$. Thus the ramification index is n, and the degree of $F(\alpha)$ over F is n. It follows that $f'(x)$ is irreducible over F; in particular, it has no root in F and Rolle's theorem fails.

Example 4.8. The field $\mathbf{R}((t))(t^{1/2}, t^{1/4}, t^{1/8}, \ldots)$ does not satisfy Rolle's theorem. Its natural valuation with residue field $\mathbf{R}$ has a value group which is divisible only by 2. On the other hand, the field $\mathbf{R}((t))(t^{1/3}, t^{1/5}, t^{1/7}, \ldots)$ does satisfy Rolle's theorem, since the proof for the example given by M. J. Pelling in [12] can be easily adapted to this field. This field has two orderings; the value group for the unique place into $\mathbf{R}$ is n-divisible for every odd n, but not 2-divisible.

5. Multiplier sequences in arbitrary fields

Let F be any field. We may consider the n-sequences for F as linear operators on the polynomial ring $F[x]$. In fact, they are precisely those linear operators $L: F[x] \to F[x]$ which preserve the natural grading on $F[x]$ (that is, $L(x^k) = \lambda_k x^k$ for some eigenvalue λ_k in F) and leave invariant the multiplicative subset of $F[x]$ consisting of all polynomials which split over F. Thus the n-sequences are closely related to the multiplicative structure of the ring $F[x]$ as well as its structure as an F-vector space. By [4, Proposition 2.2], the set $\mathcal{S}_n(F)$ of all n-sequences for F is a commutative semigroup containing 0 and 1. If $n \geqslant m$, there is a natural mapping from $\mathcal{S}_n(F)$ to $\mathcal{S}_m(F)$ defined by truncating the n-sequence to an m-sequence. Note that these mappings are generally not surjective. In real closed fields, for example, our characterization of n-sequences shows that there always exist n-sequences which cannot be extended to $(n + 1)$-sequences. For this inverse system of semigroups we can also consider the inverse limit $\mathcal{S}(F) = \lim_{\leftarrow} \mathcal{S}_n(F)$ which consists of all multiplier sequences for F. In general, a major problem is to characterize the image of $\mathcal{S}(F)$ in $\mathcal{S}_n(F)$; that is, to determine which n-sequences can be extended to multiplier sequences. Our next theorem takes Proposition I.4.5, rephrases it in this general terminology instead of entire functions, and provides a strictly algebraic proof.

THEOREM 5.1. *Let F be a formally real field such that $F = F^*$. Let $r \in F$ be any totally positive element of F such that $r \leqslant 1$ in all orderings of F. Then there exist multiplier sequences Γ_1 and Γ_2 of F such that*

Γ_1 is an arithmetic sequence, Γ_2 is an exponential sequence and $\Gamma_1\Gamma_2 = \{1, 1, r, \ldots\}$. In particular, the natural map $\mathcal{S}(F) \to \mathcal{S}_2(F)$ is surjective. Furthermore, an element of $\mathcal{S}_2(F)$ has a unique element of $\mathcal{S}(F)$ mapping onto it if and only if it has the form $\{a, b, 0\}$ or $\{a, ab, ab^2\}$.

Proof. Set $s = (1 + r + \sqrt{1 - r})r^{-1}$, again a totally positive element of F. Let $\Gamma_1 = \{1 + ks\}_{k=0}^{\infty}$, which is a multiplier sequence for F by Corollary 2.13 and let $\Gamma_2 = \{(1 + s)^{-k}\}_{k=0}^{\infty}$, which is an exponential sequence. Then a straightforward calculation shows that the product $\Gamma_1\Gamma_2$ has the proper form. Given any element $\{a, b, c\} \in \mathcal{S}_2(F)$, Proposition 2.2 shows that we may assume a, b and c are squares in F since $\mathcal{S}(F)$ contains all exponential sequences. Similarly, we can multiply by

$$\{a^{-1}, b^{-1}, ab^{-2}\} = \{a^{-1}, a^{-1}(ab^{-1}), a^{-1}(ab^{-1})^2\}$$

to replace $\{a, b, c\}$ by a sequence of the form $\{1, 1, r\}$ where r is a square, hence totally positive. Since this is a 2-sequence, the polynomial $1 + 2x + rx^2$ must split in F, so its discriminant $4 - 4r$ must be a square; hence $r \leq 1$ in all orderings of F. Thus $\mathcal{S}(F) \to \mathcal{S}_2(F)$ is surjective.

Finally, if $\{a, b, c\} \in \mathcal{S}_2(F)$, then [4, Theorem 3.4(a)] shows that the only element of $\mathcal{S}(F)$ mapping to it is $\{a, b, 0, 0, 0, \ldots\}$. If $\{a, ab, ab^2\} \in \mathcal{S}_2(F)$, we may assume $a = b = 1$ by normalizing as before. Let $\Gamma = \{1, 1, 1, r, \ldots\} \in \mathcal{S}(F)$. Then $r \neq 0$ by [4, Lemma 2.9]. Applying Γ to $(x + 1)^3$ gives

$$rx^3 + 3x^2 + 3x + 1,$$

which splits if and only if $x^3 + 3x^2 + 3x + r$ does. But if $r \neq 1$, this latter polynomial has at most one root in any formally real field, as can be seen by replacing x by $x - 1$. Thus r must equal 1 since Γ is a multiplier sequence. Considering a sequence of n ones followed by an element r, we obtain $r = 1$ in the same manner by applying the sequence to $x^{n-3}(x + 1)^3$. Therefore the only sequence in $\mathcal{S}(F)$ mapping to $\{1, 1, 1\}$ is the sequence of all ones. On the other hand, given any element of $\mathcal{S}_2(F)$ not in one of these two forms, we may assume it has the form $\{1, 1, r\}$ where $r \neq 0, 1$. Given any sequence $\Gamma = \{1, 1, \gamma_2, \gamma_3, \ldots\}$ in $\mathcal{S}(F)$, we can apply the first part of the theorem to obtain sequences Γ_1 and Γ_2 for $\{1, 1, r\gamma_2^{-1}\}$, so that $\Gamma\Gamma_1\Gamma_2$ maps onto $\{1, 1, r\}$. Thus every element of $\mathcal{S}(F)$ maps onto $\{1, 1, r\}$ modulo arithmetic and exponential sequences. This completes the proof of the theorem.

The abundance of sequences Γ as used in the previous proof is shown by Corollary 2.11 and Remark I.4.4. Furthermore, this latter remark gives abundant examples of n-sequences which extend in infinitely many ways to multiplier sequences.

Besides extendability questions, another general problem is to determine the extent to which $\mathcal{S}(F)$ characterizes F. There are many partial results along this line in [4] and the present paper. For example, formally real fields which are not pythagorean have only trivial multiplier sequences [4, Theorem 3.5], so $\mathcal{S}(F)$ is rather uninteresting. We also have the following interesting characterization of algebraically closed fields.

THEOREM 5.2. *Let F be any field with more than two elements. The following conditions are equivalent:*

(1) *F is algebraically closed.*
(2) $\{1, 1, \lambda, \lambda^2, \lambda^3, \ldots\} \in \mathscr{S}(F)$ *for all* $\lambda \neq 0$ *in* F.
(3) $\{\lambda, 1, 1, 1, \ldots\}$ *is a multiplier sequence for all* $\lambda \neq 0$ *in* F.

Proof. Clearly (1) implies (2) and (3). The equivalence of (2) and (3) is evident since the sequences differ only by a factor of an exponential sequence. Now assume (3) holds and let $\Gamma_\lambda = \{\lambda, 1, 1, 1, \ldots\}$. Assume F is not algebraically closed. Choose an element not in F such that its minimal polynomial $f(x) = \Sigma_{k=0}^{n} a_k x^k$ has minimum degree $n \geqslant 2$. Thus all polynomials in $F[x]$ of degree less than n split over F. Now, for each λ in F, consider the polynomial

$$f_\lambda(x) = a_0\lambda + a_1 x + \cdots + a_n x^n.$$

Clearly there is some $\mu \neq 0$ in F such that $f_\mu(x)$ is not irreducible; indeed, by [4, §4], (3) cannot hold for a finite field, and thus there exists $y \in F$ such that

$$\mu = -a_0^{-1}(a_1 y + \cdots + a_n y^n) \neq 0$$

and $f_\mu(y) = 0$. Therefore $f_\mu(x)$ factors into polynomials of degree less than n and hence $f_\mu(x)$ splits. Since Γ_λ is a multiplier sequence for all λ, in particular $\lambda = \mu^{-1}$, the polynomial $f(x) = \Gamma_{\mu^{-1}}[f_\mu(x)]$ splits, giving us the desired contradiction.

We were led to studying intersections of real closed fields by requiring that certain sequences be multiplier sequences for the field. In fact, given any field F and sequence Γ, one can look at the smallest field containing F for which Γ is a multiplier sequence. The following simple result gives some cases in which the smallest such field is the algebraic closure $\overline{F}$ of F. Note that the condition that $\overline{F}$ be generated by a countable number of elements over F is not only satisfied by all countable fields, but also by such interesting fields as $\mathbf{R}((t))$ whose algebraic closure is $\mathbf{C}((t))(t^{1/2}, t^{1/3}, t^{1/4}, \ldots)$ [14, Chapter IV, §2, Proposition 8].

PROPOSITION 5.3. *Let F be a field such that $\overline{F}$ is a countably generated extension of F. Then there exists a sequence Γ of elements of F such that the smallest field K containing F, with Γ a multiplier sequence for K, is $\overline{F}$.*

Proof. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be elements of $\overline{F}$ such that

$$\overline{F} = F(\alpha_1, \alpha_2, \alpha_3, \ldots).$$

Let the minimal polynomial of α_k over F be $f_k(x) = \Sigma_{i=0}^{n_k} a_{ik} x^i$ and set Γ

equal to the sequence

$$\left\{a_{01}\binom{n_1}{0}^{-1}, a_{11}\binom{n_1}{1}^{-1}, \ldots, a_{n_1 1}\binom{n_1}{n_1}^{-1},\right.$$
$$\left. a_{02}\binom{n_2}{0}^{-1}, a_{12}\binom{n_2}{1}^{-1}, \ldots, a_{n_2 2}\binom{n_2}{n_2}^{-1}, \ldots\right\}.$$

Any field for which Γ is a multiplier sequence must contain α_1 which is a root of $\Gamma[(x+1)^{n_1}] = f_1(x)$, and α_2 which is a root of

$$\Gamma[x^{n_1+1}(x+1)^{n_2}] = x^{n_1+1}f_2(x),$$

and so forth. Thus the smallest field containing F for which this sequence is a multiplier sequence is $\overline{F}$.

Remark 5.4. In view of the previous two sections, we note that the collection of multiplier sequences which satisfy the Fundamental Inequality (3.1) form a subsemigroup of $\mathcal{S}(F)$. It is not difficult to show that this subsemigroup contains the trivial multiplier sequences and whenever it contains $\{\gamma_0, \gamma_1, \gamma_2, \ldots\}$, it also contains $\{\gamma_p, \gamma_{p+1}, \gamma_{p+2}, \ldots\}$ for any $p \geqslant 0$.

In the remainder of this section we shall consider the existence of inverses for elements in $\mathcal{S}(F)$. The main results concern fields of characteristic $p \neq 0$ which are algebraic over their prime subfield $\mathbf{F}_p$, the field with p elements. These are mainly of interest for infinite algebraic extensions since all multiplier sequences over finite fields have been characterized in [4, Section 4].

Let Γ be an element of $\mathcal{S}(F)$. If Γ has an inverse in $\mathcal{S}(F)$, it will be denoted by Γ^{-1}. More generally, we shall write Γ^g for the sequence whose k-th entry is γ_k^{-1} if $\gamma_k \neq 0$ and 0 if $\gamma_k = 0$. If Γ^g is in $\mathcal{S}(F)$, it will be called the *generalized inverse* of Γ. Note that if all entries of Γ are nonzero, then $\Gamma^{-1} = \Gamma^g$.

THEOREM 5.5. *Let F be any algebraic extension of $\mathbf{F}_p$, and assume Γ is a multiplier sequence for F. Then Γ^g is also a multiplier sequence for F.*

Proof. Let $f(x) \in F[x]$ be any polynomial which splits over F and let n be the degree of $f(x)$. Since F is algebraic over $\mathbf{F}_p$, every nonzero element of F has multiplicative order $p^m - 1$ for some m. For each k, let m_k be the order of γ_k if $\gamma_k \neq 0$ and let $m_k = 1$ if $\gamma_k = 0$. Set r equal to the product $m_1 m_2 \cdots m_n$. Then the product of multiplier sequences Γ^{r-1} is again a multiplier sequence. Since Γ^{r-1} agrees with Γ^g in its first n entries, we know that $\Gamma^g[f(x)] = \Gamma^{r-1}[f(x)]$ splits in F. Since $f(x)$ was arbitrary, the sequence Γ^g is a multiplier sequence.

COROLLARY 5.6. *Let F be any algebraic extension of $\mathbf{F}_p$ and let $\Gamma \in \mathcal{S}(F)$. Then the sequence $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$, $\lambda_k = 1$ if $\gamma_k \neq 0$ and $\lambda_k = 0$ if $\gamma_k = 0$, is an idempotent element of $\mathcal{S}(F)$.*

Proof. The sequence Λ is in $\mathcal{S}(F)$ since it equals the product $\Gamma\Gamma^g$.

Remark 5.7. (1) The previous theorem implies that if F is an algebraic extension of $\mathbf{F}_p$, then $\mathcal{S}(F)$ is a completely regular inverse semigroup [9].

(2) Of course, trivial multiplier sequences always have generalized inverses. It is quite rare for every element of $\mathcal{S}(F)$ to have one. In fact, it can be easily shown that if the characteristic of F is not 2 and Γ, $\Gamma^g \in \mathcal{S}(F)$, where Γ has three consecutive nonzero terms γ_k, γ_{k+1}, γ_{k+2} with $\gamma_{k+1}^2 - \gamma_k\gamma_{k+2} \neq 0$, then F is quadratically closed. (Use [4, Theorem 2.8] and the fact that a pythagorean field which is not formally real is quadratically closed.)

From Theorem 5.5 we also obtain the following interesting result.

PROPOSITION 5.8. *Let F be any algebraic extension of $\mathbf{F}_p$. Then Γ is a multiplier sequence for F if and only if Γ^r and Γ^s are multiplier sequences for two relatively prime positive integers r and s.*

Proof. Since r and s are relatively prime, there exist integers a and b such that $ar + bs = 1$. Theorem 5.5 implies that Γ^{ar} and Γ^{bs} are both multiplier sequences, where a negative power of Γ should be interpreted as a positive power of Γ^g. Therefore the product $\Gamma^{ar}\,\Gamma^{bs} = \Gamma$ is a multiplier sequence.

Remark 5.9. The condition on $\mathcal{S}(F)$ given by the previous proposition is much weaker than the existence of generalized inverses. Nevertheless, it still fails to hold for $\mathbf{R}$ (or any real closed field). Using Theorem 2.1 it can be shown that the sequence $\Gamma = \{1, 2, 2\sqrt{2}, 1, 0, 0, 0, \ldots\}$ is not a multiplier sequence for $\mathbf{R}$, although Γ^2 and Γ^3 are both in $\mathcal{S}(\mathbf{R})$.

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