

## LOCATION OF ZEROS PART I: REAL POLYNOMIALS AND ENTIRE FUNCTIONS

BY

THOMAS CRAVEN<sup>1</sup> AND GEORGE CSORDAS

### 1. Introduction

In the study of the distribution of zeros of polynomials and entire functions the techniques used, roughly speaking, fall into three categories: analytic, geometric and algebraic. In this paper, which represents the first portion of a two-part investigation, we will attempt to exploit the advantages of all three techniques. In Section 2 we will introduce a novel geometric tool (see also [3]) to prove results which, for the most part, are intractable by algebraic or analytic methods. In addition, the geometric theorems are generally stronger than their algebraic counterparts which are derived as corollaries.

In the extensive literature dealing with the location of zeros of real polynomials (and real entire functions) a significant role is played by linear transformations  $T$  which possess the following property:

$$(1.1) \quad Z_c(T[f]) \leq Z_c(f),$$

where  $f$  is a polynomial and  $Z_c(f)$  denotes the number of nonreal zeros of  $f$ , counting multiplicities. If  $T = D = d/dx$ , then (1.1) follows from Rolle's theorem; and if  $T = \varphi(D)$ , where  $\varphi$  is a polynomial with only real zeros, then (1.1) is a consequence of the classical Hermite-Poulain Theorem [15, p. 4]. There are many other linear transformations  $T$  which satisfy (1.1). Indeed, let  $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$  be a real sequence and for an arbitrary real polynomial  $f(x) = \sum_{k=0}^n a_k x^k$  define  $\Gamma[f]$  by

$$(1.2) \quad \Gamma[f(x)] = \sum_{k=0}^n a_k \gamma_k x^k.$$

Now let  $\Gamma = \{Q(k)\}_{k=0}^{\infty}$ , where  $Q(x)$  is a polynomial with only real negative zeros. Then Laguerre's theorem [15, p. 6] asserts that

$$(1.3) \quad Z_c(\Gamma[f]) = Z_c\left(\sum_{k=0}^n a_k Q(k)x^k\right) \leq Z_c(f).$$

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The real sequences  $\Gamma = \{\gamma_k\}$  for which  $\Gamma[f]$  has only real zeros whenever  $f$  is a real polynomial with only real zeros (so that in this case (1.1) reads  $Z_c(\Gamma[f]) = Z_c(f) = 0$ ) have been completely characterized by Pólya and Schur [19]. In this celebrated paper Pólya and Schur introduced the following definition.

**DEFINITION 1.4.** A sequence  $\Gamma = \{\gamma_k\}_{k=0}^\infty$  of real numbers is called a *multiplier sequence of the first kind* if  $\Gamma$  takes every polynomial  $f(x)$  with only real zeros into a polynomial  $\Gamma[f(x)]$  (defined by (1.2)) of the same class. A sequence  $\Gamma = \{\gamma_k\}_{k=0}^\infty$  of real numbers is called a *multiplier sequence of the second kind* if  $\Gamma$  takes every polynomial  $f(x)$ , all of whose zeros are real and of the same sign, into a polynomial all of whose zeros are real.

With the aid of the Schur Composition Theorem [15, p. 26] Pólya and Schur [19]. In this celebrated paper Pólya and Schur introduced the following definition.

**THEOREM 1.5.** Let  $\Gamma = \{\gamma_k\}_{k=0}^\infty$  be a sequence of real numbers. Then  $\Gamma$  is a multiplier sequence of the first kind if and only if the zeros of the polynomials

$$(1.6) \quad \Gamma[(1 + x)^n] = \sum_{k=0}^n \binom{n}{k} \gamma_k x^k, \quad n = 1, 2, 3, \dots,$$

are all real and of the same sign. The sequence  $\Gamma$  is a multiplier sequence of the second kind if and only if the zeros of the polynomials (1.6) are all real.

In [19] Pólya and Schur also established the following transcendental characterization of these sequences.

**THEOREM 1.7.** Let  $\Gamma = \{\gamma_k\}_{k=0}^\infty$ ,  $\gamma_0 \neq 0$ , be a sequence of real numbers. Then in order that  $\Gamma$  be a multiplier sequence of the first kind it is necessary and sufficient that the series

$$\Phi(x) = \sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k$$

converge in the whole plane, and that the entire function  $\Phi(x)$  or  $\Phi(-x)$  can be represented in the form

$$(1.8) \quad \Phi(x) = ce^{\sigma x} \prod_{n=1}^\infty (1 + x/x_n)$$

where  $\sigma \geq 0$ ,  $x_n > 0$ ,  $c \in \mathbf{R}$  and  $\sum_{n=1}^\infty x_n^{-1} < \infty$ . In order that the sequence  $\Gamma$  be a multiplier sequence of the second kind it is necessary and sufficient

that the series

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

converge in the whole plane, and that the entire function  $\Phi(x)$  can be represented in the form

$$(1.9) \quad \Phi(x) = ce^{-\alpha x^2 + \beta x} \prod_{n=1}^{\infty} (1 - x/x_n) e^{x/x_n},$$

where  $\alpha \geq 0$ ,  $\beta$ ,  $c$  and  $x_n$  are real and  $\sum_{n=1}^{\infty} x_n^{-2} < \infty$ .

The following additional terminology will facilitate the description of the problems to be considered in the sequel. A real entire function  $\Phi(x)$  which can be represented in the form (1.9) is said to be of *type II* in the *Laguerre-Pólya class*. Functions of the form (1.8) are termed *type I* in the *Laguerre-Pólya class*. The significance of the Laguerre-Pólya class in the theory of entire functions (see, for example, [12, Chapter 8]) is natural since the functions of type II, and only those, are the uniform limits of polynomials with only real zeros.

The disparate nature of these two kinds of multiplier sequences becomes clear in light of the Fundamental Inequality. This inequality [4] states that a real sequence  $\Gamma = \{\gamma_k\}$  is a multiplier sequence of the first kind if and only if

$$(1.10) \quad Z_c(\Gamma[f]) \leq Z_c(f)$$

for all real polynomials  $f(x)$ . In [4] we have shown by means of an example that (1.10) need not hold if  $\Gamma$  is assumed to be a multiplier sequence of the second kind.

In Section 2 we give a new proof of inequality (1.10) (see Theorem 2.4). In addition, the geometric methods we use in Section 2 enable us to extend an important theorem of Pólya [16] on the reality of roots of polynomials (Theorem 2.5 and Corollary 2.6).

In abstract field theory,  $n$ -sequences were first introduced by Craven in [1]. These sequences are defined as follows.

**DEFINITION 1.11.** Let  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  be a sequence of real numbers. Then  $\Gamma$  is called an  *$n$ -sequence of the first kind* if for every polynomial  $f(x)$  of degree less than or equal to  $n$  and with only real zeros the polynomial  $\Gamma[f(x)]$  also has only real zeros.

If in the above definition we add the stipulation that all the zeros of  $f(x)$  are of the same sign, then  $\Gamma$  will be called an  *$n$ -sequence of the second kind*. Since for the most part we will be dealing with  $n$ -sequences of the first kind, we will use, for the sake of brevity, the term  *$n$ -sequence* to mean

$n$ -sequence of the first kind. The theory of  $n$ -sequences presented in Section 3 will provide the algebraic apparatus necessary for analyzing the location of zeros of certain classes of polynomials and entire functions. Working with  $n$ -sequences instead of multiplier sequences permits one to attempt the investigation of the location of zeros of more general classes of polynomials and entire functions. Besides providing stronger theorems for the real numbers, giving generalizations of several classical theorems, the introduction of  $n$ -sequences allows the use of Tarski's Principle to extend many of our results to all real closed fields and sometimes more general fields. This will be the aim of the sequel to this paper [6], which we shall refer to as Part II. In Part II, we shall be investigating the extent to which the results of this paper can be extended from the real numbers to arbitrary fields, especially ordered fields. As we have seen, this includes the question of which ordered fields satisfy Rolle's theorem and generalizations of it, a topic central to Part II.

One of the most intriguing facts concerning  $n$ -sequences emerges in Section 4, where, among other things, we show (Theorem 4.8 and Example 4.9) that there is an  $n$ -sequence  $\Gamma$  and a polynomial  $f(x)$  of degree  $n$  such that

$$(1.12) \quad Z_c(\Gamma[f]) > Z_c(f).$$

In light of (1.10) this result implies, in particular, that this sequence  $\Gamma$  is not extendible (see definition below) to a multiplier sequence.

**DEFINITION 1.13.** A sequence of real numbers  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  is said to be *extendible to an*  $(n + m)$ -sequence if there are real numbers  $\gamma_{n+1}, \dots, \gamma_{n+m}$  such that the sequence

$$\{\gamma_0, \dots, \gamma_n, \dots, \gamma_{n+m}\}$$

is an  $(n + m)$ -sequence.  $\Gamma$  is said to be *extendible to a multiplier sequence of the first kind* if there is a multiplier sequence of the first kind  $\{\gamma'_k\}$  such that  $\gamma_k = \gamma'_k$  for  $k = 0, 1, \dots, n$ .

In Section 4 we (1) establish several results concerning extendibility of  $n$ -sequences, (2) provide some concrete examples of  $n$ -sequences, (3) raise some open questions and (4) use  $n$ -sequences to provide an equivalent formulation of an open problem involving zero-diminishing transformations.

## 2. Geometric Results Concerning Polynomials

In this section we will (1) extend several results of [3], [4] and generalize a theorem of Pólya [16], (2) give a new proof of the Fundamental Inequality for multiplier sequences of the first kind (see [4]) and (3) provide two sufficient conditions for a given sequence  $\{\gamma_0, \dots, \gamma_n\}$  to be an  $n$ -sequence. We will conclude this section with some results involving multiplier sequences of the first kind and the simplicity of zeros of certain polynomials.

We will follow, as much as possible the nomenclature used in Walker's book *Algebraic curves* [21]. In particular, we use the term *component* to refer to an irreducible factor of the curve

$$F(x, y) \equiv \sum_{k=0}^n b_k x^k f^{(k)}(y),$$

where  $f(x)$  is a real polynomial and  $b_k \in \mathbf{R}$ . Since we deal only with the real points of a not necessarily irreducible algebraic variety, we require a term to refer to the individual parts of the curve  $F(x, y) = 0$ , even though these parts need not be components, or even connected components since two parts may intersect. For this we shall use the word *branch*. Classically, the branches are only defined in a neighborhood of a point on the curve [7, p. 39] where they have the usual analytic meaning. We shall use the word branch in the following global sense: the *branch* of the curve  $F$  containing a given point on the curve is obtained by travelling along the curve in both directions until reaching a singularity or returning to the starting point; at a singularity travel out along the other arc of the same local branch on which you arrived. Thus the branches are the "pieces" into which a circuit in the projective plane [7, p. 50] is broken by removing the line at infinity. We shall see below that the branches in which we are most interested will always go to infinity in both directions. Finally, in compliance with the usual custom in algebraic geometry we will count all roots, branches and components with their multiplicities.

The following theorem is proved in [3, Theorem 3.1].

**THEOREM 2.1.** *Let  $h(x) = \sum_{k=0}^n b_k x^k$  be a polynomial of degree  $n$  with only real negative zeros and let  $f(y)$  be an arbitrary real polynomial with  $r$  real zeros and degree at most  $n$ . Then the real algebraic curve*

$$(2.2) \quad F(x, y) \equiv \sum_{k=0}^n b_k x^k f^{(k)}(y) = 0$$

*has at least  $r$  intersection points with each line  $sx - ty + u = 0$ , where  $s \geq 0$ ,  $t \geq 0$ ,  $s + t > 0$  and  $u$  is real.*

As an application of this theorem we shall include here a new proof of the Fundamental Inequality for multiplier sequences of the first kind [4, Theorem 3]. Our previous proof was based on a consequence of Theorem 2.1 and on the construction of a complicated family of multiplier sequences [4, Theorem 2]. In contrast, our new proof is shorter and simpler since it directly exploits the geometric content of the conclusion of Theorem 2.1. In order to simplify our arguments we begin with the following remark.

**REMARK 2.3.** If  $\Gamma = \{\gamma_k\}$  is a multiplier sequence of the first kind, then the terms  $\gamma_k$  either all have the same sign or they have alternating signs.

Moreover, the relations  $\gamma_i \gamma_j \neq 0$  and  $\gamma_k = 0$  for any  $k, i < k < j$ , cannot hold at the same time (see, for example, Craven and Csordas [2, Theorem 3.4 (b), p. 807]). For reasons of convenience we shall often assume in the sequel that  $\gamma_k \geq 0$  for all  $k$ . Indeed, if  $\Phi(x) = \Gamma[e^x]$  is a function of type I in the Laguerre-Pólya class, then so is the function  $\Phi(-x)$ .

**THEOREM 2.4 (THE FUNDAMENTAL INEQUALITY).** *Let  $\Gamma = \{\gamma_k\}$  be a multiplier sequence of the first kind and let  $f(x) = \sum_{k=0}^m a_k x^k$  be an arbitrary real polynomial of degree  $m$ . Then*

$$(*) \quad Z_c(\Gamma[f]) \leq Z_c(f).$$

*Proof.* In light of the preceding remarks it suffices to prove the theorem when  $\gamma_k \geq 0$ . We shall first establish inequality (\*) under the additional hypothesis that  $\gamma_0 \neq 0$ . If  $\gamma_0 \neq 0$ , then the following two cases arise: (a)  $\gamma_k \neq 0$  for  $k = 1, \dots, m$  and (b)  $\gamma_k = 0$  for some  $k, 0 < k < m$ .

*Case (a).* Suppose  $\gamma_0 \neq 0$  and  $\gamma_k \neq 0$  for  $k = 1, \dots, m$ . Then, by the algebraic characterization of multiplier sequences of the first kind (see Theorem 1.5), for each positive integer  $n$  the polynomial

$$g_n\left(\frac{x}{n}\right) = \sum_{k=0}^n \binom{n}{k} \gamma_k \frac{x^k}{n^k}$$

has only real, negative zeros. We next fix a positive integer  $n, n > m$ . Then by Theorem 2.1 the real algebraic curve

$$F(x, y) = \sum_{k=0}^m \frac{\gamma_k}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) x^k f^{(k)}(y) = 0$$

has at least  $r$  intersection points with the line  $y = 0$ , where  $r$  denotes the number of real zeros of  $f$ . Since  $f^{(k)}(0) = a_k k!$ , it follows that

$$Z_c\left(\sum_{k=0}^m \gamma_k \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) a_k x^k\right) \leq Z_c(f).$$

Now we take the limit as  $n \rightarrow \infty$ . Under this limiting process the above inequality prevails by Hurwitz's theorem and thus we obtain inequality (\*).

*Case (b).* Suppose  $\gamma_0 \neq 0$  and  $\gamma_k = 0$  for some  $k, 0 < k < m$ . Let  $p$  denote the largest integer  $k, 0 < k < m$ , such that  $\gamma_k \neq 0$ . Thus,  $\gamma_k \neq 0$  for  $k = 0, \dots, p$ , but  $\gamma_{p+1} = \gamma_{p+2} = \cdots = 0$  by Remark 2.3. This observation together with the transcendental characterization of multiplier sequences of the first kind shows that the polynomial

$$\Phi(x) = \sum_{k=0}^p \frac{\gamma_k}{k!} x^k$$

has only real negative zeros. For each fixed  $\varepsilon > 0$  we next consider the

*m*-th degree polynomial

$$(1 + x\varepsilon)^{m-p}\Phi(x) = \sum_{k=0}^m \frac{\gamma_k(\varepsilon)}{k!} x^k$$

and note that

$$\lim_{\varepsilon \rightarrow 0} \gamma_k(\varepsilon) = \gamma_k \text{ for } k = 0, \dots, p \text{ and } \lim_{\varepsilon \rightarrow 0} \gamma_k(\varepsilon) = 0 \text{ for } k > p.$$

As in the previous case we invoke Theorem 2.1 to conclude that for each fixed  $\varepsilon > 0$ ,

$$Z_c \left( \sum_{k=0}^m \gamma_k(\varepsilon) a_k x^k \right) \leq Z_c(f).$$

On the other hand, by Hurwitz's theorem,

$$Z_c \left( \sum_{k=0}^p \gamma_k a_k x^k \right) \leq Z_c \left( \sum_{k=0}^m \gamma_k(\varepsilon) a_k x^k \right)$$

for all sufficiently small  $\varepsilon$ . These inequalities show that, in this case also,  $Z_c(\Gamma[f]) \leq Z_c(f)$ .

Finally, we must prove that inequality (\*) remains valid when  $\Gamma = \{\gamma_k\}$  has some leading zero terms. Accordingly, we now assume that  $\gamma_0 = \gamma_1 = \dots = \gamma_{p-1} = 0$ ,  $\gamma_p \neq 0$ , where  $p - 1 < m$ . (The stipulation that  $p - 1 < m$  is introduced here merely to avoid trivialities.) Let

$$\Phi(x) = x^p \left( \frac{\gamma_p}{p!} + \frac{\gamma_{p+1}}{(p+1)!} x + \dots \right), \quad \gamma_p \neq 0,$$

and note that for each fixed  $\delta > 0$  the function

$$\begin{aligned} \Phi_\delta(x) &= (x + \delta)^p \left( \frac{\gamma_p}{p!} + \frac{\gamma_{p+1}}{(p+1)!} x + \dots \right) \\ &= \sum \frac{\gamma_k(\delta)}{k!} x^k \end{aligned}$$

is of type I in the Laguerre-Pólya class. Consequently, the sequence

$$\Gamma_\delta = \{\gamma_k(\delta)\}, \gamma_0(\delta) = \frac{\delta^p \gamma_p}{p!} \neq 0$$

is a multiplier sequence of the first kind. Thus, by the above argument (Case (a) or Case (b)),

$$Z_c(\Gamma_\delta[f]) \leq Z_c(f).$$

If we let  $\delta \rightarrow 0$ , then, as before, the inequality on the number of nonreal

zeros is preserved. Hence, in all cases  $Z_c(\Gamma[f]) \leq Z_c(f)$  and thus the proof of the theorem is complete.

**THEOREM 2.5.** *Let  $h(x) = \sum_{k=0}^n b_k x^k$  be a real polynomial such that the polynomial*

$$\sum_{k=0}^n \frac{b_k}{(n-k)!} x^k$$

*has only real negative zeros. Let  $f(y) = \sum_{k=0}^m a_k y^k$  be a real polynomial of degree  $m$ ,  $m \leq n$ , with only real zeros. Then the real algebraic curve*

$$F(x, y) = \sum_{k=0}^n b_k x^k f^{(k)}(y) = 0$$

*has  $m$  intersection points with each line  $sx - ty + u = 0$ , where  $s \geq 0$ ,  $t \geq 0$ ,  $s + t > 0$ , and  $u$  is real.*

*Proof.* First note that we may assume without loss of generality that the roots of  $f$  are simple. Consider an arbitrary but fixed line  $y = c$ . Since  $F$  has degree  $m$  there can be no more than  $m$  points in which this line intersects  $F$ . Now suppose that the number of intersection points with this line is less than  $m$ . Let  $\gamma_k = b_k k!$  and note that the sequence  $\Gamma = \{\gamma_0, \dots, \gamma_m\}$  is an  $m$ -sequence (see Section 3 for the various properties of  $m$ -sequences). But then the polynomial

$$\begin{aligned} F(x, c) &= \sum_{k=0}^m b_k x^k f^{(k)}(c) \\ &= \sum_{k=0}^m \gamma_k \frac{f^{(k)}(c)}{k!} x^k \\ &= \Gamma[f(x + c)] \end{aligned}$$

has less than  $m$  real zeros. This contradicts the definition of an  $m$ -sequence and thus we conclude that each horizontal line has exactly  $m$  intersection points with the real curve  $F$ .

If there exists a point  $(x_0, y_0)$  where two branches intersect or where  $dy/dx = 0$  along the curve, then  $F(x, y_0) = \Gamma[f(x + y_0)]$  is a polynomial with a multiple root at  $x_0$ . The polynomial  $f(x + y_0)$  has simple roots, hence a sufficiently small change in the coefficients of  $f$  will cause  $\Gamma[f(x + y_0)]$  to have some nonreal roots while the slightly changed  $f(x + y_0)$  still has only real roots, contradicting the fact that  $\Gamma$  is an  $m$ -sequence.

From this we conclude that the curve consists of  $m$  disjoint branches. In fact, each of the  $m$  branches extends indefinitely with negative slope;



that is, as  $x$  varies from  $-\infty$  to  $+\infty$ ,  $y$  varies monotonically from  $+\infty$  to  $-\infty$ . To see this, we show that the real algebraic curve  $F$  has  $m$  real asymptotes with negative slopes. Set  $y = tx$  in the equation  $F(x, y) = 0$  and compute

$$a_m^{-1} \lim_{|x| \rightarrow \infty} x^{-m} F(x, tx) = \sum_{k=0}^m b_k \frac{m!}{(m-k)!} t^{m-k} \equiv g(t).$$

Since, by assumption, the zeros of

$$\varphi(t) = \sum_{k=0}^n \frac{b_k}{(n-k)!} t^k$$

are all real and negative it follows from Rolle's theorem that the zeros of the polynomial

$$m! \frac{d^{n-m}}{dt^{n-m}} \varphi^*(t) = g(t),$$

where  $\varphi^*(t) = t^n \varphi(t^{-1})$ , are also all real and negative. This implies that the "ends" of each branch of  $F(x, y) = 0$  are asymptotic to lines which have negative slopes. The possible slopes of these lines are given by the zeros of  $g(t)$ . From the characteristics of the branches, it is now clear that lines of nonnegative or infinite slope will each intersect the curve in exactly  $m$  points (counting multiplicity). Hence the proof of the theorem is complete.

The previous theorem generalizes a similar theorem of Pólya [16] in which he uses the more restrictive hypothesis that  $h(x)$  has only real zeros. It also gives a partial generalization of the Hermite-Poulain theorem [15, Satz 3.1] by intersecting with the line  $x = 1$ .

As immediate consequences of Theorem 2.5 we obtain two conditions, each of which is a necessary condition for a given sequence  $\{\gamma_0, \dots, \gamma_n\}$  to be an  $n$ -sequence.

**COROLLARY 2.6.** *Let  $\{\gamma_0, \dots, \gamma_n\}$ ,  $\gamma_k > 0$ , be an  $n$ -sequence and let  $f$  be a polynomial of degree at most  $n$  with only real zeros. Then the polynomials*

$$(a) \sum_{k=0}^n \frac{\gamma_k}{k!} f^{(k)}(x) \quad \text{and} \quad (b) \sum_{k=0}^n \frac{\gamma_k}{k!} x^k f^{(k)}(x)$$

*also have only real zeros.*

*Proof.* Apply Theorem 2.5 to the lines (a)  $x = 1$  and (b)  $y = x$ .

In [3, Theorem 2.3; also Statement 2.4] we have proved that if  $h(x) = \sum_{k=0}^n b_k x^k$ ,  $n \geq 1$ ,  $b_0 \neq 0$ ,  $b_n = 1$ , has only real roots and  $f(y)$  is an arbitrary real polynomial, then the branches of

$$F(x, y) \equiv \sum_{k=0}^n b_k x^k f^{(k)}(y) = 0$$

which intersect the  $y$ -axis cannot meet each other at any point off the  $y$ -axis. We shall now conclude this section with two propositions which show how this result on the simplicity of roots of polynomials can be extended to multiplier sequences of the first kind. While the first of these propositions is motivated in part by the aforementioned result, its proof, as we shall see below, does not require the geometric machinery introduced in this section.

**PROPOSITION 2.7.** *Let  $\Gamma = \{\gamma_k\}$ ,  $\gamma_k > 0$ , be a multiplier sequence of the first kind. Then*

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \neq (\text{poly})e^{\alpha x}$$

*if and only if for any polynomial  $f(x)$  with only real roots the polynomial  $\Gamma[f(x)]$  has only simple real roots except possibly at  $x = 0$ .*

*Proof.* Since  $\Gamma$  is a multiplier sequence of the first kind,  $\Phi(x)$  is a function of type I in the Laguerre-Pólya class (see Theorem 1.5). We shall first suppose that  $\Phi(x) \neq (\text{poly})e^{\alpha x}$ . Then  $\Phi(x)$  has an infinite number of roots and in this case it is known [8] that for each  $n$  the polynomial  $g_n(x) = \Gamma[(1 + x)^n]$  has only simple negative real roots. Now let  $f(x)$  be a polynomial of degree  $n$  with only real roots and assume that  $\Gamma[f(x)]$  has a multiple root other than  $x = 0$ . Then there is a sufficiently small change in the first  $n + 1$  terms of  $\Gamma$  such that (1) a nonzero double root of  $\Gamma[f(x)]$  becomes a pair of nonreal roots (that is, if  $\Gamma'$  denotes the resulting new sequence, then  $\Gamma'[f(x)]$  has a pair of nonreal roots); and (2)  $\Gamma'[(1 + x)^n]$  has only real roots. The second claim follows from the above cited fact that  $\Gamma[(1 + x)^n]$  has only simple negative real roots. But then by Theorem 3.1 below (see also [2, Theorem 3.7]),  $\Gamma'$  is still an  $n$ -sequence. Therefore,  $\Gamma'[f(x)]$  has only real roots. This is the desired contradiction.

We will prove the converse implication also by an argument by contradiction. Thus suppose that

$$\begin{aligned} \Phi(x) &= \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k = \left( \sum_{m=0}^p b_m x^m \right) e^{\alpha x}, \quad b_p \neq 0, \\ &= \sum_{m=0}^p b_m \left( \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \alpha^k \frac{x^{k+m}}{(k+m)!} \right). \end{aligned}$$

Since  $\gamma_k > 0$  for all  $k$ ,  $\alpha \neq 0$  and hence we may assume without loss of

generality that  $\alpha = 1$ . Then, for  $n \geq p + 2$ , the polynomial

$$\begin{aligned} \Gamma[(1 + x)^n] &= \sum_{k=0}^n \binom{n}{k} \gamma_k x^k \\ &= \sum_{m=0}^p b_m x^m \sum_{k=0}^n \binom{n}{k} k(k-1) \cdots (k-m+1) x^{k-m} \\ &= \sum_{m=0}^p b_m x^m D^m (x+1)^n, \quad D = d/dx, \\ &= \sum_{m=0}^p b_m x^m n(n-1) \cdots (n-m+1) (x+1)^{n-m} \end{aligned}$$

has  $x = -1$  as a multiple root. This contradicts our assumption that for any polynomial  $f(x)$  with only real roots the polynomial  $\Gamma[f(x)]$  has only *simple* real roots (except possibly at  $x = 0$ ). Thus,  $\Phi(x) \neq (\text{poly})e^{\alpha x}$  and the proof of the proposition is complete.

PROPOSITION 2.8. *Let*

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \quad \gamma_0 \neq 0, \gamma_k \geq 0,$$

*be a function of type I in the Laguerre-Pólya class and let  $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ . Let  $f(x)$  be a real polynomial and suppose that its real roots are simple. Then two branches of*

$$F(x, y) \equiv \sum \frac{\gamma_k}{k!} x^{kf^{(k)}(y)} = 0$$

*which intersect the  $y$ -axis cannot meet each other at any point off the  $y$ -axis. In particular, the polynomial  $\Gamma[f(x)]$  has at least as many distinct real roots of odd multiplicity as  $f(x)$  has real roots.*

*Proof.* If  $\gamma_n \neq 0$  but  $\gamma_{n+1} = 0$  for some  $n$ , then it follows from Remark 2.3 that  $\Phi(x)$  is a polynomial

$$\sum_{k=0}^n \frac{\gamma_k}{k!} x^k$$

all of whose roots are real and negative. In this case the proposition is a consequence of a previously established result (see [3, Theorem 2.3]). Thus we may assume that  $\gamma_k > 0$  for all  $k$ . Now suppose that two branches which cross the  $y$ -axis do meet at a point  $(x_0, y_0)$  off the  $y$ -axis. Thus the polynomial

$$h(x) = \sum \frac{\gamma_k}{k!} f^{(k)}(y_0) x^k$$

has a multiple root at  $x = x_0$ . But then there exists an arbitrarily small change in the coefficients of  $h(x)$  such that (1) the multiple root at  $x_0$  can be assumed to be a double root, and (2) the double root will become a pair of nonreal roots. Since  $\gamma_k > 0$ , this can be accomplished by varying the factors  $f^{(k)}(y_0)$ . Moreover, a sufficiently small change will keep the nonreal roots of  $f$  nonreal and keep the real roots both real and simple. Thus we may assume that  $F(x, y)$  has a branch which crosses the  $y$ -axis at two distinct points. Now if we approximate  $\Phi(x)$  by

$$g_n\left(\frac{x}{n}\right) = \sum_{k=0}^n \binom{n}{k} \gamma_k \frac{x^k}{n^k},$$

then, for sufficiently large  $n$ , the qualitative behavior of this branch will be the same; that is, there will be a branch which crosses the  $y$ -axis at two distinct points. Since  $g_n(x)$  has only real roots, this contradicts [3, Theorem 2.3]. Therefore  $h(x)$  cannot have a multiple root  $x = x_0$ . Finally, we consider the intersection of the branches of  $F(x, y) = 0$  with the line  $y = 0$ , obtaining the polynomial  $\Gamma[f(x)] = F(x, 0)$ . If we compute the slopes of the asymptotes to the curve as in the proof of Theorem 2.5, we see that they are all either negative or infinite. Thus, each branch of the curve which crosses the  $y$ -axis (at a real root of  $f$ ) also crosses the  $x$ -axis in at least one point, giving rise to at least one root of  $\Gamma[f]$  of odd multiplicity.

### 3. The Theory of $n$ -Sequences

In the course of our investigation of  $n$ -sequences, we will provide here several characterizations of these sequences (Theorem 3.1, Theorem 3.4, Theorem 3.16 and Corollary 3.21). While multiplier sequences and  $n$ -sequences have many properties in common (Lemma 3.2), there are some major differences (see also Section 4) which account for the intricate character of many  $n$ -sequences. Indeed, it appears that for  $n$ -sequences there is no direct analogue of the transcendental characterization of the multiplier sequences of the first kind (see Theorem 1.5). Nevertheless, our strategy will be to focus our attention on the class of polynomials of the form

$$\sum_{k=0}^n \frac{\gamma_k}{k!} x^k,$$

where  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence. In Corollary 3.8 we will note the curious fact that the distribution of the zeros of a polynomial  $\varphi(x)$  in this class can be arbitrary as long as the zeros of  $\varphi$  lie in a double sector, symmetric with respect to the real line, with vertex at the origin and whose angle opening is sufficiently small. But by exploiting the similarities that exist between the linear operators  $D = d/dx$  and  $\Gamma$ , where  $\Gamma$  is an  $n$ -sequence, we are able to solve several problems surrounding this class of polynomials.

In addition we also establish some results (see, for example, Corollary 3.10 and Theorem 3.15) concerning certain real entire functions

$$\sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k,$$

where  $\{\alpha_0, \dots, \alpha_n\}$  is an  $n$ -sequence. Finally, we discuss the connection between  $n$ -sequences of the second kind and a longstanding conjecture of Pólya in the theory of differentiation.

For a fixed positive integer  $n$ , let  $\mathcal{P}_n$  denote the set of all polynomials  $f(x)$ ,  $\deg f(x) \leq n$ , which have only real zeros. Then the following natural question arises. What condition must a sequence  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  of real numbers satisfy in order that  $\Gamma[f(x)] \in \mathcal{P}_n$  for every  $f(x) \in \mathcal{P}_n$ ? The remarkable fact is that we need only to examine the action of  $\Gamma$  on a single polynomial. This is the content of our first result and its proof is an immediate consequence of [2, Theorem 3.7].

**THEOREM 3.1 (ALGEBRAIC CHARACTERIZATION OF  $n$ -SEQUENCES).** *Let*

$$\Gamma = \{\gamma_0, \dots, \gamma_n\}$$

*be a sequence of real numbers. Then  $\Gamma$  is an  $n$ -sequence if and only if the zeros of the polynomial  $\Gamma[(1 + x)^n]$  are all real and of the same sign.*

Theorem 3.1 as well as the geometric results of Section 2 suggest that  $n$ -sequences have several properties in common with multiplier sequences of the first kind. In the next lemma we provide a partial list of such properties.

**LEMMA 3.2.** *Let  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  be an  $n$ -sequence.*

- (a)  $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$  for  $k = 1, \dots, n - 1$ .
- (b) *The nonzero terms  $\gamma_k$  either all have the same sign or they have alternating signs. Moreover, the relations  $\gamma_k\gamma_m \neq 0$  and  $\gamma_j = 0$ , for any  $j, k < j < m$ , cannot hold at the same time.*
- (c) *The sequence  $\Gamma^* = \{\gamma_n, \dots, \gamma_0\}$  is an  $n$ -sequence and the sequence  $\Gamma_1 = \{\gamma_1, \dots, \gamma_n\}$  is an  $(n - 1)$ -sequence.*
- (d) *The sequence*

$$\left\{ \frac{\gamma_0}{n!}, \frac{\gamma_1}{(n-1)!}, \dots, \frac{\gamma_n}{0!}, 0, 0, \dots \right\}$$

*is a multiplier sequence of the first kind.*

- (e) *The sequence*

$$\{\lambda_0\gamma_0, \dots, \lambda_n\gamma_n\}$$

*is an  $n$ -sequence, where  $\Lambda = \{\lambda_0, \dots, \lambda_n\}$  is any  $n$ -sequence.*

Since Lemma 3.2 follows from Theorem 3.1 and the arguments presented in [2], we will omit the proof here. In light of part (b) of the lemma, in the sequel we will frequently assume that the terms  $\gamma_k$  of an  $n$ -sequence are nonnegative.

Our next immediate goal is to show that the conditions (a) or (b) of Corollary 2.6 are both necessary and sufficient for a sequence  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  to be an  $n$ -sequence. To this end our investigation will be focused on the class of polynomials

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k$$

whose Taylor coefficients  $\{\gamma_0, \dots, \gamma_n\}$  form an  $n$ -sequence. The most obvious examples in this class of polynomials are (1) the polynomials which have only real negative zeros and (2) the  $n$ -th Taylor polynomials associated with functions of type I in the Laguerre-Pólya class. We hasten to add, however, that the examples just cited by no means exhaust the class of polynomials whose Taylor coefficients form an  $n$ -sequence (see Corollary 3.8 and Section 4). As we shall see below, a complete description of the distribution of zeros of the polynomials in this class is, in general, very difficult. Indeed, it is for this reason that for  $n$ -sequences there seems to be no direct analogue of the well-known transcendental characterization of multiplier sequences of the first kind. On the other hand, we will show below that some of the problems surrounding the above class of polynomials will become tractable if we take into account the similarities that exist between the linear operators  $D = d/dx$  and  $\Gamma$ , where  $\Gamma$  is a multiplier sequence of the first kind or more generally  $\Gamma$  is an  $n$ -sequence. (These similarities were studied in [4] for multiplier sequences of the first kind.)

Preliminaries aside, we shall first recall the following classical result due to Walsh [22] (see also Marden [13, p. 81]).

**THEOREM 3.3.** *Let*

$$f(x) = \sum_{j=0}^n a_j x^j = a_n \prod_{j=1}^n (x - \alpha_j),$$

$$g(x) = \sum_{j=0}^n b_j x^j = b_n \prod_{j=1}^n (x - \beta_j),$$

and

$$h(x) = \sum_{j=0}^n (n-j)! b_{n-j} f^{(j)}(x) = \sum_{j=0}^n (n-j)! a_{n-j} g^{(j)}(x).$$

*If all the zeros of  $f(x)$  lie in a circular region  $C$ , then all the zeros of  $h(x)$*

lie in the point set  $S$  consisting of the  $n$  circular regions obtained by translating  $C$  in the amount and direction of the vectors  $\beta_j$ .

*Remarks.* (1) By a circular region we mean the closure of the interior or the exterior of a circle, or a closed half-plane.

(2) Theorem 3.3 is a direct consequence of the Grace Apolarity Theorem (see, for example, Marden [13, p. 61]).

(3) If both  $f(x)$  and  $g(x)$  have only real zeros, then it follows from Theorem 3.3 that  $h(x)$  also has only real zeros. To see this we note that every zero  $x_0$  of  $h(x)$  has the form  $x_0 = c + \beta_j$  for some  $j$  and some  $c \in C$ , where  $C$  is a circular region which contains all the zeros of  $f(x)$ . Now apply the theorem to the circular regions  $\{z | \text{Im } z \leq 0\}$  and  $\{z | \text{Im } z \geq 0\}$ .

**THEOREM 3.4.** *Let*

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k, \quad \gamma_k \geq 0.$$

*Then  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence if and only if*

$$(3.5) \quad Z_c(\varphi(D)f(x)) = 0$$

*for all polynomials  $f(x)$  in  $\mathcal{P}_n$ .*

*Proof.* Suppose  $\Gamma$  is an  $n$ -sequence. We shall first prove (3.5) under the additional hypothesis that all the coefficients  $\gamma_k$  are positive. By Theorem 3.1 the polynomial  $g_n(x) = \Gamma[(1 + x)^n]$  has only real zeros and *a fortiori* the zeros of the polynomial

$$g_n^*(x) = x^n g_n(x) \left(\frac{1}{x}\right) = \sum_{k=0}^n \binom{n}{k} \gamma_{n-k} x^k$$

are also all real. Now if we apply Theorem 3.3 to  $g_n^*(x)$  and  $f(x)$  we obtain that the polynomial

$$\sum_{k=0}^n (n - k)! \binom{n}{k} \gamma_k f^{(k)}(x) = n! \varphi(D)f(x)$$

also has only real zeros.

We next consider the case when some but not all of the terms  $\gamma_k$  are zero. By Lemma 3.2 part (b) we may suppose that  $\gamma_0 = \dots = \gamma_{p-1} = 0$ ,  $\gamma_p \neq 0$ ,  $\gamma_{p+j} \neq 0$  for  $j = 0, 1, \dots, n - q - p$ , and  $\gamma_{n-q+1} = \dots = \gamma_n = 0$ . Then once again by Theorem 3.1 the polynomial

$$x^p \left[ \binom{n}{p} \gamma_p + \dots + \binom{n}{n - q} \gamma_{n-q} x^{n-p-q} \right]$$

has only real zeros. For  $\varepsilon > 0$  let

$$A_\varepsilon(x) = (x + \varepsilon)^p \left[ \binom{n}{p} \gamma_p + \dots + \binom{n}{n-q} \gamma_{n-q} x^{n-p-q} \right] (1 + x\varepsilon)^q$$

$$= \sum_{k=0}^n \binom{n}{k} \gamma_k(\varepsilon) x^k,$$

where  $\gamma_k(\varepsilon) \rightarrow 0$  for  $k = 0, \dots, p - 1$  and  $k = n - q + 1, \dots, n$  and  $\gamma_k(\varepsilon) \rightarrow \gamma_k$  for  $k = p, \dots, n - q$ , as  $\varepsilon \rightarrow 0$ . Since  $\gamma_k(\varepsilon) > 0$ , we can apply the previous argument to the polynomials  $A_\varepsilon^*(x) = x^n A_\varepsilon(1/x)$  and  $f(x)$ . Hence, for each  $\varepsilon > 0$  the polynomial

$$\sum_{k=0}^n (n - k)! \binom{n}{k} \gamma_k(\varepsilon) f^{(k)}(x)$$

has only real zeros. Finally, if we let  $\varepsilon \rightarrow 0$ , then by Hurwitz's theorem we conclude that the polynomial  $\varphi(D)f(x)$  also has only real zeros.

Conversely, suppose that  $Z_c(\varphi(D)f(x)) = 0$  for all polynomials  $f(x)$  in  $\mathcal{P}_n$ . Thus in particular the polynomial

$$\varphi(D)x^n = \sum_{k=0}^n \binom{n}{k} \gamma_k x^{n-k}$$

has only real zeros and consequently by Theorem 3.1 the sequence  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence.

The next corollary illustrates how  $n$ -sequences can be used to generate new multiplier sequences of the first kind.

**COROLLARY 3.6.** *Let  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$ ,  $\gamma_k > 0$ , be an  $n$ -sequence and let  $f(x) = \sum_{k=0}^n a_k x^k$ ,  $a_k > 0$ , be a polynomial in  $\mathcal{P}_n$ . Let*

$$\Lambda = \{\lambda_0, \dots, \lambda_n, 0, 0, \dots\} \quad \text{where } \lambda_k = \Gamma[f^{(k)}](1);$$

*that is,  $\lambda_k$  is the value of the polynomial  $\Gamma[f^{(k)}]$  at  $x = 1$ . Then  $\Lambda$  is a multiplier sequence of the first kind.*

*Proof.* It suffices to show that the polynomial

$$\sum_{k=0}^n \lambda_k \frac{x^k}{k!} = \sum_{k=0}^n \Gamma[f^{(k)}](1) \frac{x^k}{k!}$$

$$= \sum_{k=0}^n \left( \sum_{j=0}^{n-k} \gamma_j a_{j+k} \frac{(j+k)!}{j!} \right) \frac{x^k}{k!}$$

has only real, negative zeros. If we set

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k,$$



then by Theorem 3.4 the polynomial  $\varphi(D)f(x)$  has only real zeros. On the other hand a calculation shows that

$$\begin{aligned}\varphi(D)f(x) &= \sum_{k=0}^n a_k \sum_{j=0}^k \binom{k}{j} \gamma_j x^{k-j} \\ &= \sum_{j=0}^n \sum_{k=j}^n a_k \binom{k}{j} \gamma_{k-j} x^j \\ &= \sum_{j=0}^n \left( \sum_{k=0}^{n-j} a_{k+j} \frac{(k+j)!}{k!} \gamma_k \right) \frac{x^j}{j!} \\ &= \sum_{j=0}^n \frac{\lambda_j}{j!} x^j.\end{aligned}$$

This completes the proof of the corollary.

We pause for a moment to compare the foregoing results with the following classical theorem due to Hermite, Poulain and Fujiwara (see for example, Obreschkoff [15, p. 275]).

**THEOREM 3.7.** *Let*

$$\Phi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k.$$

*Then  $\Phi(x)$  has only real zeros if and only if for every polynomial  $f$ ,  $\deg f = m \geq n$ , with only real zeros, the polynomial*

$$\Phi(D)f(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} f^{(k)}(x)$$

*also has only real zeros.*

Thus we see that Theorem 3.4 supplements the above theorem since our theorem applies to the case when the degree of the polynomial  $f(x)$  is less than or equal to  $n$ .

We shall next show that if a real polynomial  $f(x)$  of degree  $n$  has all its zeros in a double sector  $S$ , whose vertex is at the origin and whose angle opening is sufficiently small, then the Taylor coefficients of  $f$  form an  $n$ -sequence. The precise formulation of this observation is as follows.

**COROLLARY 3.8.** *Let*

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k, \quad \gamma_k \geq 0, \quad n \geq 2,$$

*and suppose that the zeros of  $\varphi$  lie in the double sector*

$$S = \{z = x + iy \mid |y| \leq |x|/\sqrt{n-1}\}.$$

Then  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence.

*Proof.* Let  $f(x)$  be an arbitrary polynomial in  $\mathcal{P}_n$ . Since all the zeros of  $\phi(x)$  lie in the double sector  $S$ , it follows from a theorem of Obreschkoff [15, p. 273] that the polynomial  $\varphi(D)f(x)$  has only real zeros. Thus by Theorem 3.4 the sequence  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence.

The import of Corollary 3.8 is two-fold. First it provides us with a large number of examples of  $n$ -sequences. Second, it shows that the distribution of the zeros of

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k,$$

where  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence, is arbitrary as long as the zeros of  $\varphi(x)$  lie in the double sector  $S$ . In particular, this corollary asserts that if  $f(x)$  is an arbitrary real polynomial of degree  $n$ , then the sequence

$$\{f(\lambda), f'(\lambda), \dots, f^{(n)}(\lambda)\}$$

is an  $n$ -sequence for all  $\lambda$ , provided  $|\lambda|$  is sufficiently large.

*Remark 3.9.* It is easy to see that the product  $\Phi(x) = \Phi_1(x) \cdot \Phi_2(x)$  of two functions, where  $\Phi_1(x)$  and  $\Phi_2(x)$  are functions of type  $I$  in the Laguerre-Pólya class, is also a function in this class. Thus, in particular, the Taylor coefficients of  $\Phi(x)$  form a multiplier sequence of the first kind. We will demonstrate below that if the Taylor coefficients of the polynomials  $\varphi_j(x)$ ,  $j = 1, 2$ , form an  $n$ -sequence, and if

$$\varphi_1(x) \cdot \varphi_2(x) = \sum \frac{\alpha_k}{k!} x^k,$$

then the sequence  $\{\alpha_0, \dots, \alpha_n\}$  is an  $n$ -sequence. This result will be of great importance in Part II for extending theorems for  $\mathbf{R}$  to arbitrary real closed fields.

**COROLLARY 3.10.** *Let*

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k, \quad \gamma_k \geq 0,$$

where  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence. Let

$$\hat{\varphi}(x) = \sum \frac{\beta_k}{k!} x^k, \quad \beta_k \geq 0,$$

where  $\{\beta_0, \dots, \beta_m\}$  is an  $m$ -sequence,  $m \geq n$ . If

$$F(x) = \varphi(x)\hat{\varphi}(x) = \sum \frac{\alpha_k}{k!} x^k,$$

then the sequence  $\{\alpha_0, \dots, \alpha_n\}$  is an  $n$ -sequence.

*Proof.* Let

$$\psi(x) = \sum_{k=0}^n \frac{\alpha_k}{k!} x^k$$

and let  $f(x)$  be an arbitrary polynomial in  $\mathcal{P}_n$ . Then by Corollary 3.5 it suffices to show that  $\psi(D)f(x)$  is also in  $\mathcal{P}_n$ . Consider

$$F(D)f(x) = \varphi(D)\widehat{\varphi}(D)f(x) = \psi(D)f(x).$$

Since  $h(x) = \widehat{\varphi}(D)f(x)$  is a polynomial in  $\mathcal{P}_n$ , it follows that  $\varphi(D)h(x) = \psi(D)f(x)$  also has only real zeros. Hence by Theorem 3.4 the sequence  $\{\alpha_0, \dots, \alpha_n\}$  is an  $n$ -sequence.

**COROLLARY 3.11.** *Let*

$$\varphi(x) = \sum \frac{\gamma_k}{k!} x^k, \quad \gamma_k \geq 0,$$

where  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence. Let

$$g_k(t) = \sum_{j=0}^k \binom{k}{j} \gamma_j t^j \quad \text{and} \quad g_k^*(t) = \sum_{j=0}^k \binom{k}{j} \gamma_j t^{k-j}.$$

Then for each fixed  $t > 0$ , the sequences

$$\{g_0(t), g_1(t), \dots, g_n(t)\} \quad \text{and} \quad \{g_0^*(t), g_1^*(t), \dots, g_n^*(t)\}$$

are  $n$ -sequences.

*Proof.* For each fixed  $t > 0$ , consider the products

$$e^{xt}\varphi(x) = \sum \frac{g_k^*(t)}{k!} x^k \quad \text{and} \quad e^x\varphi(xt) = \sum \frac{g_k(t)}{k!} x^k$$

and apply Corollary 3.10.

In a certain sense Corollary 3.10 is best possible. In other words, if

$$F(x) = \varphi(x)\widehat{\varphi}(x) = \sum \frac{\alpha_k}{k!} x^k,$$

where  $\varphi(x)$  and  $\widehat{\varphi}(x)$  are the functions defined in Corollary 3.10, then, in general,  $\{\alpha_1, \dots, \alpha_{n+1}\}$  is not an  $n$ -sequence. This is illustrated below and is motivated, in part, by the following lemma.

**LEMMA 3.12.** *If  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$ ,  $\gamma_k > 0$ , is an  $n$ -sequence and  $\lambda \geq 0$ , then  $\lambda\Gamma + \Gamma_1 = \{\lambda\gamma_0 + \gamma_1, \dots, \lambda\gamma_{n-1} + \gamma_n\}$  is an  $(n - 1)$ -sequence.*

*Proof.* Let

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k$$

and consider the product

$$e^{\lambda x} \varphi(x) = \sum_{k=0}^{\infty} g_k^*(\lambda) \frac{x^k}{k!}, \quad \lambda \geq 0.$$

By Corollary 3.11,  $\{g_0^*(\lambda), \dots, g_n^*(\lambda)\}$  is an  $n$ -sequence, and, by Lemma 3.2 (c),  $\{g_1^*(\lambda), \dots, g_n^*(\lambda)\}$  is an  $(n - 1)$ -sequence. Let

$$\psi(x) = \sum_{k=0}^{n-1} g_{k+1}^*(\lambda) \frac{x^k}{k!} \quad \text{and} \quad F(x) = e^{-\lambda x} \psi(x) = \sum \alpha_k \frac{x^k}{k!}.$$

Then  $\alpha_k = \alpha_k(\lambda) > 0, k = 0, \dots, n - 1$ , since a computation shows that

$$\alpha_k = \sum_{j=0}^k \binom{k}{j} g_{j+1}^*(\lambda) (-\lambda)^{k-j} = \lambda \gamma_k + \gamma_{k+1}$$

for  $k = 0, \dots, n - 1$ . Thus, it suffices to verify that  $\{\alpha_0, \dots, \alpha_{n-1}\}$  is an  $(n - 1)$ -sequence. If  $f(x)$  is an arbitrary polynomial in  $\mathcal{P}_{n-1}$ , then by Theorem 3.4 the polynomial

$$F(D)f(x) = \sum_{k=0}^{n-1} \frac{\alpha_k}{k!} f^{(k)}(x) = e^{-\lambda D} \psi(D)f(x) = \psi(D)f(x - \lambda)$$

has only real zeros. Thus, another application of Theorem 3.4 shows that  $\{\alpha_0, \dots, \alpha_{n-1}\}$  is an  $(n - 1)$ -sequence.

*Example 3.13.* Let  $\Gamma = \{1, 7/2, 61/6, 21, 36\}$ . Then a computation shows that  $\Gamma$  is a 4-sequence (see also Section 4). By Lemma 3.12 (use  $\lambda = 1$ ) the sequence  $\{9/2, 82/6, 187/6, 57\}$  is a 3-sequence. However, we shall show now that  $\Lambda = \{9/2, 82/6, 187/6, 57, 36\}$  is not a 4-sequence. By Theorem 3.1 it suffices to check that the polynomial

$$h(x) = \Lambda[(1 + x)^4] = \frac{9}{2} + \frac{164}{3}x + 187x^2 + 228x^3 + 36x^4$$

does not have only real zeros. In fact, tedious but elementary computations show that the derivative  $h'(x)$  has two nonreal zeros. Thus it follows that  $h(x)$  cannot have only real zeros.

Now set

$$\begin{aligned} \varphi(x) &= \Gamma \left[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \right] \\ &= 1 + \frac{7}{2}x + \frac{61}{6} \frac{x^2}{2!} + 21 \frac{x^3}{3!} + 36 \frac{x^4}{4!} \end{aligned}$$

where  $\Gamma$  is the 4-sequence introduced above, and consider the product

$$F(x) = e^x \varphi(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k.$$

By Corollary 3.10  $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$  is a 4-sequence. We shall now show that  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$  is not a 4-sequence. In order to simplify the ensuing computations we call attention to the following slight extension of Theorem 3.4. If

$$G(x) = \sum_{k=0}^{\infty} \frac{\beta_k}{k!} x^k, \quad \beta_k > 0,$$

is an arbitrary real entire function, then  $\{\beta_0, \dots, \beta_n\}$  is an  $n$ -sequence if and only if the polynomial

$$G(D)f(x) = \sum \frac{\beta_k}{k!} f^{(k)}(x)$$

has only real zeros for every  $f(x)$  in  $\mathcal{P}_n$ . Now if we let  $f(x) = (x - 1)^4$  and compute  $F'(D)(x - 1)^4$ , we obtain

$$\begin{aligned} F'(D)(x - 1)^4 &= e^{D[\varphi(D) + \varphi'(D)]}(x - 1)^4 \\ &= [\varphi(D) + \varphi'(D)]x^4 \\ &= \frac{9}{2}x^4 + \frac{164}{3}x^3 + 187x^2 + 228x + 36 \\ &= h^*(x). \end{aligned}$$

But we have seen above that the polynomial  $h(x)$  does not have only real zeros. Consequently,  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$  is not a 4-sequence.

*Remark 3.14.* The above example also establishes the following surprising fact. The class of functions of the form

$$(*) \quad F(x) = \varphi(x)\Phi(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k$$

where

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k,$$

$\{\gamma_0, \dots, \gamma_n\}$ ,  $\gamma_k > 0$ , is an  $n$ -sequence and  $\Phi(x)$  is a function with nonnegative coefficients of type I in the Laguerre-Pólya class, is *not* closed under differentiation. Thus, in general,  $F'(x)$  does not admit a factorization of the above form. But  $Z_c(F'(x)) \leq Z_c(\varphi)$ , so that if we factor  $F'(x)$  in the form

$$F'(x) = \varphi_1(x)\Phi_1(x) = \sum_{k=0}^{\infty} \frac{\alpha_{k+1}}{k!} x^k,$$

where  $\Phi_1$  is a function of type I in the Laguerre-Pólya class, then  $\deg \varphi_1 \leq n$  and  $\{\alpha_1, \dots, \alpha_n\}$  is an  $(n - 1)$ -sequence.

It is also instructive to examine these ideas in light of the classical theory of differentiation. By the celebrated 4/3-Theorem of Pólya [18], if  $F(x)$  is an entire function of the form (\*), then there is a positive integer  $p$  such that  $F^{(p)}(x)$  has only real zeros all of which have the same sign. Thus,  $F^{(p)}(x)$  is a function of type I in the Laguerre-Pólya class and, *a fortiori*,  $\{\alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n}\}$  is an  $n$ -sequence for all integers  $m \geq p$ .

In our next result we impose a condition on  $\varphi(x)$  which will guarantee that the sequence  $\{\alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n}\}$  is an  $n$ -sequence for every nonnegative integer  $m$ , while the sequence  $\{\alpha_0, \alpha_1, \dots, \alpha_n, \dots\}$  is *not*, in general, a multiplier sequence.

**THEOREM 3.15.** *Suppose all the zeros of the polynomial*

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!}, \quad \gamma_k > 0, n > 1,$$

*lie in the sector*

$$S = \{z = x + iy \mid |y| \leq |x|/\sqrt{n-1}, x \leq 0\}.$$

*Let*

$$F(x) = \varphi(x)\Phi(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k,$$

*where*

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\beta_k}{k!} x^k, \quad \beta_k > 0,$$

*is an entire function of type I in the Laguerre-Pólya class. Then for every nonnegative integer  $m$  the sequence*

$$\{\alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n}\}$$

*is an  $n$ -sequence. Moreover, there exists a positive integer  $p$  such that the sequence  $\{\alpha_p, \alpha_{p+1}, \dots\}$  is a multiplier sequence of the first kind.*

*Proof.* By Corollary 3.8,  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence and hence by Corollary 3.10,  $\{\alpha_0, \dots, \alpha_n\}$  is also an  $n$ -sequence. Since the zeros of  $F(x)$  lie in the sector  $S$  the zeros of  $F^{(m)}(x)$  also lie in  $S$ . This assertion may be deduced from Levin [12, Theorem 2, p. 331] and the Gauss-Lucas Theorem (see, for example, Marden [13]). It is also clear that  $F^{(m)}(x)$  possesses at most  $n$  nonreal zeros. Since  $\beta_k > 0$  for all  $k$ ,  $F^{(m)}(x)$  has at least  $n$  zeros. (Just consider the two cases (a)  $\Phi$  has an infinite number of zeros and (b)  $\Phi(x) = p(x)e^{\alpha x}$ , where  $\alpha > 0$  and  $p(x)$  is a polynomial with only real negative zeros.) Thus for each fixed positive integer  $m$ , we have the factorization

$$F^{(m)}(x) = q_m(x)\Phi_m(x),$$

where  $q_m(x)$  is a polynomial of degree at most  $n$  and the zeros of  $q_m(x)$  lie in  $S$  and  $\Phi_m(x)$  is an entire function of type I in the Laguerre-Pólya class. Now a short argument together with Corollary 3.8 and Corollary 3.10 imply that  $\{\alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n}\}$  is an  $n$ -sequence. The second assertion of the theorem follows directly from Pólya's 4/3-Theorem [18].

We shall next provide a new proof of Corollary 2.6 part (b). Moreover, we shall show that condition (b) of Corollary 2.6 is both necessary and sufficient for a sequence  $\{\gamma_0, \dots, \gamma_n\}$ ,  $\gamma_k > 0$ , to be an  $n$ -sequence.

**THEOREM 3.16.** *Let*

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k, \quad \gamma_k > 0.$$

*Then  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence if and only if the polynomial*

$$F(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k f^{(k)}(x)$$

*has only real zeros for all  $f$  in  $\mathcal{P}_n$ .*

*Proof.* Suppose

$$F(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k f^{(k)}(x)$$

has only real zeros for all  $f$  in  $\mathcal{P}_n$ . If we set  $f(x) = (1 + x)^n$ , then by assumption the polynomial

$$F(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k D^k(1 + x)^n = \sum_{k=0}^n \binom{n}{k} \gamma_k x^k (1 + x)^{n-k}$$

has only real zeros. Hence

$$F^*(x) = x^n F(x^{-1}) = \sum_{k=0}^n \binom{n}{k} \gamma_k (1 + x)^{n-k}$$

also has only real zeros and thus it follows from Theorem 3.1 that  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence.

Conversely, suppose that  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$ ,  $\gamma_k > 0$ , is an  $n$ -sequence. Let

$$f(x) = \sum_{k=0}^m a_k x^k, \quad \deg f = m \leq n,$$

be an arbitrary polynomial in  $\mathcal{P}_n$ . Then by Corollary 3.11,

$$\{g_0^*(t), \dots, g_n^*(t)\}$$

is also an  $n$ -sequence for each fixed  $t > 0$  and consequently the polynomial  $\sum_{k=0}^m a_k g_k^*(t)x^k$  has only real zeros. But now an elementary computation shows that, for each fixed  $t > 0$ ,

$$\begin{aligned}
 \sum_{k=0}^m a_k g_k^*(t)x^k &= \sum_{k=0}^m a_k x^k \sum_{j=0}^k \binom{k}{j} \gamma_j t^{k-j} \\
 (3.17) \qquad \qquad \qquad &= \sum_{j=0}^m \frac{\gamma_j}{j!} \sum_{k=j}^m a_k \frac{k!}{(k-j)!} x^k t^{k-j} \\
 &= \sum_{j=0}^m \frac{\gamma_j}{j!} x^j f^{(j)}(xt).
 \end{aligned}$$

Hence for  $t = 1$  we obtain the desired conclusion.

In the theory of multiplier sequences the canonical examples are furnished by Laguerre's theorem (see, for example, Obreschkoff [15, p. 6]). This theorem asserts that if  $Q(x)$  is a polynomial all whose zeros are real and lie outside the open interval  $(0, n)$  and if  $f(x) = \sum_{k=0}^n a_k x^k$  is any real polynomial of degree  $n$ , then

$$(3.18) \qquad \qquad \qquad Z_c \left( \sum_{k=0}^n a_k Q(k)x^k \right) \leq Z_c(f).$$

We note also that this inequality can also be expressed in terms of the differential operator  $\theta = x d/dx$ , since

$$(3.19) \qquad \qquad \qquad Q(\theta)f(x) = \sum_{k=0}^n a_k Q(k)x^k.$$

Thus, in particular, the sequence  $\{Q(0), \dots, Q(n)\}$  is an  $n$ -sequence. If the zeros of  $Q(x)$  are all real and negative, then the sequence  $\{Q(k)\}_{k=0}^\infty$  is a multiplier sequence of the first kind. This leads to the following open problem. If

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k,$$

where  $\{\gamma_0, \dots, \gamma_n\}$ ,  $\gamma_k > 0$ , is an  $n$ -sequence, is  $\{\varphi(0), \dots, \varphi(n)\}$  also an  $n$ -sequence? It is curious, however, that a related problem can be readily resolved (see Corollary 3.21 below) with the aid of the polynomial  $\bar{\varphi}(x)$ , where

$$\bar{\varphi}(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x(x-1)\cdots(x-k+1).$$

If we let  $s(m, k)$  denote the Stirling numbers of the first kind (see, for example, Riordan [20]), then an elementary computation shows that

$$(3.20) \qquad \qquad \qquad \bar{\varphi}(x) = \sum_{k=0}^n \left( \sum_{m=k}^n \frac{\gamma_m}{m!} s(m, k) \right) x^k.$$



Let

$$\alpha_k = \sum_{m=k}^n \frac{\gamma_m}{m!} s(m, k).$$

Then, in general, the sequence  $\{\alpha_0, \dots, \alpha_n\}$  is *not* an  $n$ -sequence (consider, for example,  $\varphi(x) = (1 + x)^2$ ). Consequently, in light of Theorem 3.4 the differential operator  $\bar{\varphi}(D)$  will not take, in general, the polynomials in  $\mathcal{P}_n$  into  $\mathcal{P}_n$ . This situation is remedied, as the following corollary shows, if we replace  $D$  by  $\theta = xD$  and apply Theorem 3.16.

**COROLLARY 3.21.** *Let  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$ ,  $\gamma_k > 0$ , and set*

$$\bar{\varphi}(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x(x-1)\cdots(x-k+1).$$

*Then the following statements are equivalent.*

- (a)  $\Gamma$  is an  $n$ -sequence.
- (b)  $Z_c(\bar{\varphi}(\theta)f(x)) = 0$  for all  $f$  in  $\mathcal{P}_n$ .
- (c) The sequence  $\{\bar{\varphi}(0), \dots, \bar{\varphi}(n)\}$  is an  $n$ -sequence.

*Proof.* We shall show that (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c)  $\Rightarrow$  (a). If  $\Gamma$  is an  $n$ -sequence, then by Corollary 3.11,  $\{g_0(1), \dots, g_n(1)\}$  is also an  $n$ -sequence. Hence if  $f(x) = \sum_{k=0}^n a_k x^k$  is any polynomial in  $\mathcal{P}_n$ , then by (3.17), (3.19) and the standard methods of the difference calculus (see, for example, Riordan [20]), we have

$$\begin{aligned} \sum_{k=0}^n a_k g_k(1) x^k &= \sum_{k=0}^n \frac{\gamma_k}{k!} x^k f^{(k)}(x) \\ &= \bar{\varphi}(\theta)f(x) \\ &= \sum_{k=0}^n a_k \bar{\varphi}(k) x^k. \end{aligned}$$

Thus the implications (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c)  $\Rightarrow$  (a) follow from Theorem 3.16.

We conclude this section with a few remarks concerning  $n$ -sequences of the second kind. (A sequence

$$\Gamma = \{\gamma_0, \dots, \gamma_n\}$$

is an  $n$ -sequence of the second kind if for any real polynomial  $f(x)$ ,  $\deg f(x) \leq n$ , all whose zeros are real and of the same sign, the polynomial  $\Gamma[f(x)]$  has only real zeros.) Several of our results in this section extend, *mutatis mutandis*, to  $n$ -sequences of the second kind. There are, however, major differences between these two types of sequences. Indeed, in [4] it

was shown that there is a multiplier sequence  $\Gamma$  of the second kind and a real polynomial  $f(x)$  such that

$$Z_c(\Gamma[f(x)]) > Z_c(f(x)).$$

In contrast to this phenomenon, if  $\Gamma$  is an  $n$ -sequence of the first kind which extends to a multiplier sequence of the first kind, then by Theorem 2.4

$$Z_c(\Gamma[f(x)]) \leq Z_c(f(x))$$

for any real polynomial  $f(x)$  of degree less than or equal to  $n$ .

Finally, we shall also mention here the connection between  $n$ -sequences of the second kind and a 50-year old conjecture of Pólya [17] in the theory of the distribution of zeros of real entire functions. Let

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\beta_k}{k!} x^k$$

be a function of type II in the Laguerre-Pólya class and let

$$\varphi(x) = \sum_{k=0}^n \frac{\gamma_k}{k!} x^k,$$

where  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence of the second kind. Let

$$F(x) = \varphi(x)\Phi(x) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k.$$

Then an argument analogous to the proof of Corollary 3.10 shows that the sequence  $\{\alpha_0, \dots, \alpha_n\}$  is an  $n$ -sequence of the second kind. In this situation Pólya's conjecture states that the derivatives  $F^{(m)}(x)$  from a certain one onward have no nonreal zeros; that is, for all sufficiently large  $m$  the sequence  $\{\alpha_m, \alpha_{m+1}, \dots\}$  is a multiplier sequence of the second kind.

#### 4. Extendibility and Examples

We begin this section by presenting some necessary conditions for a given  $n$ -sequence to be extendible to an  $(n + 1)$ -sequence (Theorem 4.1). We have seen in Section 2 (Proposition 2.7) that if

$$\Phi(x) = \sum \frac{\gamma_k}{k!} x^k, \quad \gamma_k > 0,$$

is a function of type I in the Laguerre-Pólya class with an infinite number of zeros and if  $\Gamma = \{\gamma_k\}$ , then for any  $f(x) \in \mathcal{P}_n$ ,  $f(0) \neq 0$ ,  $n = 1, 2, 3, \dots$ , the zeros of the polynomial  $\Gamma[f]$  are all simple. This result suggests a certain connection between the extendibility of an  $n$ -sequence  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  and the multiplicity of the zeros of the polynomial  $\Gamma[(1 + x)^n]$ . In Theorem 4.1 we capitalize on this idea and provide a necessary condition for a given  $n$ -sequence to be extendible to an  $(n + 1)$ -sequence.

THEOREM 4.1. *Let*

$$h(x) = \sum_{k=0}^n \binom{n}{k} \gamma_k x^k$$

*be a polynomial with only real negative zeros and suppose  $\alpha, \beta, \alpha < \beta$ , are two consecutive zeros of  $h(x)$  such that the multiplicities of  $\alpha$  and  $\beta$  are each greater than 1. Then  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence which does not extend to an  $(n + 1)$ -sequence.*

*Proof.* By the algebraic characterization of  $n$ -sequences (Theorem 3.1), it is clear that  $\Gamma$  is an  $n$ -sequence. Now suppose that  $\Gamma$  extends to the  $(n + 1)$ -sequence  $\{\gamma_0, \dots, \gamma_n, \gamma_{n+1}\}$ . Then another application of Theorem 3.1 shows that the polynomial

$$f(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} \gamma_k x^k$$

has only real zeros. If we set  $\psi(x) = f^*(x) = x^{n+1}f(1/x)$ , then it follows from the hypotheses that  $\alpha^{-1}$  and  $\beta^{-1}$  are two consecutive multiple zeros of the polynomial

$$\psi'(x) = (n + 1) \sum_{k=0}^n \binom{n}{k} \gamma_{n-k} x^k = (n + 1)h^*(x)$$

But then, by Rolle's theorem,  $f(x)$  cannot have only real zeros. This is the desired contradiction and hence the proof of the theorem is complete.

We shall next use the idea of Theorem 4.1 to establish a necessary condition for a real entire function to belong to the Laguerre-Pólya class.

THEOREM 4.2. *Let  $\Psi(x)$  be a real entire function and suppose  $\Psi'(x)$  has only real zeros. If  $\Psi'(x)$  has two consecutive zeros  $\alpha < \beta$  such that the multiplicities of  $\alpha$  and  $\beta$  are greater than 1, then  $\Psi(x)$  cannot be a function in the Laguerre-Pólya class.*

*Proof (Reductio ad absurdum).* Suppose that  $\Psi(x)$  belongs to the Laguerre-Pólya class. Then we claim that every multiple zero  $\alpha$  of  $\Psi'(x)$  (say,  $\Psi'(\alpha) = \Psi''(\alpha) = \dots = \Psi^{(m)}(\alpha) = 0, \Psi^{(m+1)}(\alpha) \neq 0$ ) is a multiple zero of  $\Psi(x)$ . Since the function  $\Psi(x + \alpha)$  is also in the Laguerre-Pólya class it follows that all the polynomials

$$g_n(x) = \sum_{k=0}^n \binom{n}{k} \Psi^{(k)}(\alpha) x^k, \quad n = 1, 2, \dots,$$

have only real zeros. If  $\Psi(\alpha) \neq 0$ , then the polynomial

$$g_{m+1}(x) = \sum_{k=0}^{m+1} \binom{m+1}{k} \Psi^{(k)}(\alpha) x^k, \quad \Psi(\alpha)\Psi^{(m+1)}(\alpha) \neq 0,$$

has only real zeros and has two consecutive coefficients equal to zero. Since this is clearly impossible (see, for example, Levin [12, p. 337]) we conclude that  $\Psi(\alpha) = 0$ . The same argument shows that  $\Psi(\beta) = 0$ . But then by Rolle's theorem  $\Psi'(c) = 0$  for some  $c, \alpha < c < \beta$ . This conclusion contradicts the assumption that  $\alpha$  and  $\beta$  are consecutive zeros of  $\Psi(x)$ .

The Fundamental Inequality (Theorem 2.4) shows that a necessary condition for a given  $n$ -sequence  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  to be extendible to a multiplier sequence is that

$$Z_c(\Gamma[f]) \leq Z_c(f)$$

for every real polynomial  $f(x)$  of degree less than or equal to  $n$ . The complicated character of certain classes of  $n$ -sequences is further demonstrated by the surprising fact that the aforementioned condition is not a sufficient condition for the extendibility of  $n$ -sequences. We shall establish this assertion in the following theorem.

**THEOREM 4.3.** *For each positive integer  $n, n \geq 3$ , there is an  $n$ -sequence  $\Gamma = \{\gamma_0, \dots, \gamma_n\}$  such that (a)  $Z_c(\Gamma[f]) \leq Z_c(f)$ , for all real polynomials of degree  $\leq n$  and (b)  $\Gamma$  does not extend to an  $(n + 1)$ -sequence.*

*Proof.* Fix a positive integer  $n, n \geq 3$ . Let

$$Q(x) = (n - x)(x + \alpha)^{n-1}, \quad \alpha > 0,$$

and set  $\Gamma = \{Q(0), \dots, Q(n)\}$ . Then by Laguerre's theorem (see, for example, Obreschkoff [15, p. 6]) for every real polynomial  $f(x)$  of degree less than or equal to  $n$ , we have  $Z_c(\Gamma[f]) \leq Z_c(f)$ . Thus, in particular,  $\Gamma$  is an  $n$ -sequence.

Next suppose that  $\Gamma$  extends to an  $(n + 1)$ -sequence for every choice of  $\alpha, \alpha > 0$ . Since  $Q(k) > 0$ , for  $k = 0, \dots, n - 1$ , and  $Q(n) = 0$ , it follows from Lemma 3.2 part (b) that if  $\Gamma$  extends to an  $(n + 1)$ -sequence, then  $\gamma_{n+1} = 0$ . Consequently by Theorem 3.1 the polynomial

$$g(x) = \Gamma[(1 + x)^{n+1}] = \sum_{k=0}^{n-1} \binom{n+1}{k} Q(k)x^k,$$

$\gamma_n = \gamma_{n+1} = 0$ , has only real (negative) zeros. On the other hand a calculation shows that the polynomial  $g^{(n-3)}(x)$  has two nonreal zeros for all sufficiently large values of  $\alpha$ . This is the desired contradiction and whence the assertion of the theorem follows.

*Remark 4.4.* Heretofore we have only cited necessary conditions for a given  $n$ -sequence to be extendible to an  $(n + 1)$ -sequence or to a multiplier sequence. We shall now invoke the classical results of Hardy [H] and Hutchinson [10] to provide a sufficient condition for extendibility. Let

$\{\gamma_0, \dots, \gamma_n\}$  be a sequence of positive real numbers with  $\gamma_0 = 1$ . If

$$\gamma_{k-1}^2 \geq 4(1 - 1/k)\gamma_k\gamma_{k-2}, \quad k = 2, \dots, n,$$

then  $\{\gamma_0, \dots, \gamma_n\}$  is an  $n$ -sequence which is extendible to a multiplier sequence of the first kind. Moreover, we can use these Turán-type inequalities to generate successively the subsequent terms  $\gamma_{n+1}, \gamma_{n+2}, \dots$ . The resulting entire function

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

is of order zero and has only real, simple and negative zeros. Furthermore, for each positive integer  $m$  the associated Taylor polynomial

$$\varphi_m(x) = \sum_{k=0}^m \frac{\gamma_k}{k!} x^k$$

has only real, simple and negative zeros (see, for example, Hutchinson [10]). Thus, it is easy to see from these considerations that if a given  $n$ -sequence is extendible to a multiplier sequence, then such an extension, in general, is *not* unique. The restrictions imposed on the  $\gamma_k$ 's by these Turán-type inequalities are, however, too severe and consequently the problem of extendibility of an arbitrary  $n$ -sequence remains open.

The question of extendibility of an  $n$ -sequence  $\{\gamma_0, \dots, \gamma_n\}$  to an  $(n + m)$ -sequence  $\{\gamma_0, \dots, \gamma_n, \gamma_{n+1}, \dots, \gamma_{n+m}\}$  is closely related to the classical problem involving  $\mathbf{R}$ -continuable polynomials (see, Meiman [14]). This problem may be formulated as follows. Let  $f(x) = 1 + a_1x + \dots + a_nx^n$  be a real polynomial of degree  $n$ . Find necessary and sufficient conditions on the  $(n + 1)$  numbers  $1, a_1, \dots, a_n$ , such that there is a polynomial

$$\psi(x) = f(x) + b_1x^{n+1} + \dots + b_mx^{n+m}$$

all whose zeros are real.

Theorem 4.3 shows that for each positive integer  $n, n \geq 3$ , there is an  $n$ -sequence  $\Gamma$  which does not extend to an  $(n + 1)$ -sequence. We shall now briefly consider the case when  $n = 2$ . This case is extremal in the sense that every 2-sequence is extendible to a multiplier sequence of the first kind (see Proposition 4.5 below). This observation is interesting, for it leads to the following novel characterization of  $e^x$ . If

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k,$$

is a function of type I in the Laguerre-Pólya class, and if  $\gamma_0 = \gamma_1 = \gamma_2 = 1$ , then  $\Phi(x) = e^x$ .

**PROPOSITION 4.5.** *Every 2-sequence  $\{\gamma_0, \gamma_1, \gamma_2\}$  is extendible to a multiplier sequence of the first kind. If  $\gamma_0 = \gamma_1 = 1$ , then to each real number  $r$ ,*

$0 \leq r \leq 1$ , there corresponds a function  $\Phi(x)$  of type I in the Laguerre-Pólya class such that  $\Phi(0) = \Phi'(0) = 1$  and  $\Phi''(0) = r$ . Moreover, if  $r = 0$  or  $r = 1$ , then  $\Phi(x)$  is uniquely determined. If  $r = 0$ , then  $\Phi(x) = 1 + x$ ; while if  $r = 1$ , then  $\Phi(x) = e^x$ .

*Proof.* Let  $\Gamma = \{\gamma_0, \gamma_1, \gamma_2\}$ ,  $\gamma_k \geq 0$ , be a 2-sequence. In order to avoid trivialities we shall assume that  $\gamma_0 > 0$  and  $\gamma_1 > 0$ . Furthermore, we may also assume the normalization condition  $\gamma_0 = 1$ . Then it is easy to see that  $\{1, \gamma_1, \gamma_2\}$  is a 2-sequence if and only if  $\{1, 1, \gamma_2\gamma_1^{-2}\}$  is a 2-sequence. Note that by Lemma 3.2 (a),  $0 \leq \gamma_2\gamma_1^{-2} \leq 1$ . Thus, it suffices to show that the 2-sequence  $\{1, 1, r\}$ ,  $0 \leq r \leq 1$ , is extendible to a multiplier sequence of the first kind. First suppose that  $0 < r < 1$ . For each fixed positive real number  $s$ , let  $\alpha_k = 1 + ks$  and  $\beta_k = (1 + s)^{-k}$ ,  $k = 0, 1, 2, \dots$ . Then  $\{\alpha_k\}_{k=0}^\infty$  and  $\{\beta_k\}_{k=0}^\infty$  are both multiplier sequences of the first kind since they are generated by the functions  $(1 + sx)e^x$  and  $\exp(x/(1 + s))$  respectively. Thus, the composite sequence

$$\{\alpha_0\beta_0, \alpha_1\beta_1, \alpha_2\beta_2, \dots\} = \{1, 1, (1 + 2s)(1 + s)^{-2}, \dots\}$$

is also a multiplier sequence of the first kind. Now a simple computation shows that for  $s = [(1 - r) + \sqrt{1 - r}]r^{-1}$  the above sequence becomes  $\{1, 1, r, \dots\}$ . The corresponding entire function

$$\Phi(x) = 1 + x + r \frac{x^2}{2!} + \dots$$

is of type I in the Laguerre-Pólya class and satisfies the required conditions  $\Phi(0) = \Phi'(0) = 1$  and  $\Phi''(0) = r$ .

If  $r = 0$ , then by Remark 2.3  $\Phi(x) = 1 + x$ . Finally, suppose

$$\Phi(x) = \sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k,$$

where  $\gamma_0 = \gamma_1 = \gamma_2 = 1$ , is a function of type I in the Laguerre-Pólya class. Then for each fixed real number  $t$  the function

$$e^x \cdot \Phi(xt) = \sum_{k=0}^\infty g_k(t) \frac{x^k}{k!}$$

is also a function in the Laguerre-Pólya class. Consequently, for every  $t$  the Turán inequality

$$\Delta_k(t) \equiv g_k^2(t) - g_{k-1}(t)g_{k+1}(t) \geq 0$$

holds for  $k = 1, 2, \dots$ . If  $\gamma_0 = \gamma_1 = \gamma_2 = 1$ , then

$$\Delta_2(t) = (1 + t)t^3(1 - \gamma_3)$$

and thus the inequality  $\Delta_2(t) \geq 0$ , for all  $t$ , implies that  $\gamma_3 = 1$ . Now the

function

$$\Phi'(x) = \gamma_1 + \gamma_2 x + \gamma_3 \frac{x^2}{2!} + \dots$$

is again in the Laguerre-Pólya class. Therefore, a repetition of the above steps in conjunction with a simple induction argument shows that  $\gamma_n = 1$  for all  $n$ . That is, if  $\gamma_0 = \gamma_1 = \gamma_2 = 1$ , then  $\Phi(x) = e^x$ .

As an application of Proposition 4.5 we shall prove the following result (see also [4, Theorem 6]).

**THEOREM 4.6.** *Let*

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k,$$

where  $\Gamma = \{\gamma_k\}$ ,  $\gamma_k \geq 0$ , is a multiplier sequence of the first kind. Then

$$(4.7) \quad Z_c(\Gamma[f(x + \lambda)]) = Z_c(\Gamma[f]),$$

for all real polynomials  $f(x)$  and for all real numbers  $\lambda$ , if and only if  $\Phi(x) = \beta x^m e^{\alpha x}$  or  $\Phi(x) = \beta x^m(1 + \alpha x)$ , where  $\alpha, \beta \geq 0$  and  $m$  is a nonnegative integer.

*Proof.* If  $\Phi(x) = \beta x^m e^{\alpha x}$  (or  $\Phi(x) = \beta x^m(1 + \alpha x)$ ) then it is easy to verify that the corresponding multiplier sequence  $\Gamma$  satisfies condition (4.7).

Conversely, suppose that (4.7) holds. We shall consider two cases: (a)  $\gamma_0 \neq 0$  and (b)  $\gamma_0 = \gamma_1 = \dots = \gamma_{m-1} = 0$ , but  $\gamma_m \neq 0$ .

(a) Suppose that  $\gamma_0 \neq 0$ . If  $\gamma_1 = 0$ , then by Remark 2.3,  $0 = \gamma_2 = \gamma_3 = \dots$  and thus  $\Phi(x) \equiv \gamma_0$ . Another appeal to Remark 2.3 yields  $\Phi(x) = \gamma_0 + \gamma_1 x$ , if  $\gamma_2 = 0$ . Consequently, we may assume that  $\gamma_k > 0$  for  $k = 0, 1, 2$ . If we now apply  $\Gamma$  to  $f(x) = 1 + x^2$ , then by virtue of condition (4.7), we obtain  $\gamma_1^2 = \gamma_0 \gamma_2$ . This equality implies that the multiplier sequence corresponding to the function

$$\Phi_1(x) \equiv \gamma_0^{-1} \Phi(\gamma_0 \gamma_1^{-1} x) = 1 + x + \frac{x^2}{2!} + \dots$$

begins with three consecutive ones. Thus, by Proposition 4.5,  $\Phi_1(x) = e^x$ , that is,  $\Phi(x)$  has the desired form.

(b) Suppose that  $\gamma_0 = \gamma_1 = \dots = \gamma_{m-1} = 0$ , but  $\gamma_m \neq 0$ . If  $\gamma_{m+1} = 0$  or  $\gamma_{m+2} = 0$ , then, as we have seen above, the argument is easy and so we shall also assume that  $\gamma_{m+1} \gamma_{m+2} \neq 0$ . Let

$$\begin{aligned} \Phi(x) &= \frac{x^m}{m!} \left[ \gamma_m + \frac{\gamma_{m+1}}{m+1} x + \frac{2\gamma_{m+2}}{(m+2)(m+1)} \frac{x^2}{2!} + \dots \right] \\ &\equiv \frac{x^m}{m!} \Phi_2(x). \end{aligned}$$

In this case we apply  $\Gamma$  to  $f(x) = x^m(1 + x^2)$  and conclude by virtue of (4.7) that

$$\frac{\gamma_{m+1}^2}{(m + 1)^2} = \frac{2\gamma_{m+2}\gamma_m}{(m + 2)(m + 1)}.$$

Thus, the argument used in (a) shows that  $\Phi_2(x) = \gamma_m \exp\{\gamma_m \gamma_{m+1}^{-1} x\}$  and *a fortiori*  $\Phi(x)$  has once again the required form. Thus the proof of the theorem is complete.

We shall next cite a few examples which not only elucidate the differences that exist between multiplier sequences of the first kind and  $n$ -sequences, but also reveal some of the more surprising features of  $n$ -sequences. To begin with we shall consider the following natural question. Does every  $n$ -sequence satisfy the Fundamental Inequality? That is, if  $\Gamma$  is an  $n$ -sequence and if  $f(x)$  is any real polynomial of degree  $n$ , then is it always true that

$$Z_c(\Gamma[f]) \leq Z_c(f)?$$

Although the results presented thus far tend to suggest an affirmative answer to this question, one of the most perplexing facts about  $n$ -sequences is that, in general, they do not satisfy this inequality. We shall state our answer to the above question in the form of an existence theorem.

**THEOREM 4.8.** *There is a 4-sequence  $\Gamma$  and a polynomial  $f(x)$  of degree 4 such that  $Z_c(\Gamma[f]) > Z_c(f)$ .*

Since the existence of the example called for by Theorem 4.8 does not appear to be transparent we will include here the following specific illustration.

*Example 4.9.* Let

$$\varphi(x) = 1 + \frac{7}{2}x + \frac{61}{12}x^2 + \frac{7}{2}x^3 + \frac{3}{2}x^4$$

and observe that

$$4! \left[ \frac{3}{2} + \frac{7}{2}x + \frac{61}{12} \frac{x^2}{2!} + \frac{7}{2} \frac{x^3}{3!} + \frac{x^4}{4!} \right] = (x + 1)^2(x + 6)^2.$$

Thus, by Theorem 3.1 and Lemma 3.2 (c),  $\Gamma = \{1, 7/2, 61/6, 21, 36\}$  is a 4-sequence. Next, let  $f(x) = x^4 + 9x^2 - 6$  and note that  $f(x)$  and *a fortiori* the polynomial

$$f(x + 4) = x^4 + 16x^3 + 105x^2 + 327x + 394$$

have exactly two real zeros. Now a computation shows that

$$\begin{aligned} \Gamma[f(x + 4)] &= 36x^4 + 336x^3 + \frac{2135}{2}x^2 + 1148x + 394 \\ &= (6x^2 + 28x + 19)^2 + \left( \frac{111}{2}x^2 + 84x + 33 \right). \end{aligned}$$



From this expression we can readily deduce that  $Z_c(\Gamma[f(x + 4)]) = 4$ , while  $Z_c(f(x + 4)) = 2$ .

*Example 4.10.* Let  $\varphi(x)$  and  $f(x)$  be the polynomials defined in Example 4.9. Then

$$\varphi(D)f(x) = \left(x^2 + 7x + \frac{21}{2}\right)^2 + \frac{45}{4},$$

so that

$$4 = Z_c(\varphi(D)f(x)) > Z_c(f) = 2.$$

*Example 4.11.* In this example we will show that in a certain sense the Fundamental Inequality is best possible (see also Craven and Csordas [4, Corollary 12] and the Schur Composition Theorem, Obreschkoff [15]). Let  $\varphi(x)$  and  $f(x)$  be the polynomials defined in Example 4.9 and let

$$A(x) = \sum_{k=0}^4 \binom{4}{k} a_k x^k = f(x + 4) \quad \text{and} \quad B(x) = \sum_{k=0}^4 \binom{4}{k} b_k x^k = x^4 \varphi\left(\frac{1}{x}\right).$$

Then elementary considerations imply that

$$4 = Z_c\left(\sum_{k=0}^4 \binom{4}{k} a_k b_k x^k\right) > Z_c\left(\sum_{k=0}^4 \binom{4}{k} a_k x^k\right) = 2.$$

On the other hand, since the sequence

$$\left\{ \frac{b_0}{4!}, \frac{b_1}{3!}, \frac{b_2}{2!}, \frac{b_3}{1!}, 0, 0, \dots \right\}$$

is a multiplier sequence of the first kind, it follows from the Fundamental Inequality that

$$Z_c\left(\sum_{k=0}^4 \binom{4}{k} a_k \frac{b_k}{(4-k)!} x^k\right) \leq Z_c(A(x)) = 2.$$

While Theorem 4.8 is surprising, it also leads to several open questions which we shall presently discuss here. For each positive integer  $n$ ,  $n \geq 4$ , is there an  $n$ -sequence and a polynomial  $f(x)$  of degree  $n$  such that

$$Z_c(\Gamma[f]) > Z_c(f)?$$

Examples are also known for  $n = 6$  and  $8$ . The 4-sequence defined in Example 4.9 is curious not only because it satisfies the inequality of Theorem 4.8 but also because for any polynomial  $f(x)$  of degree  $\leq 3$

$$Z_c(\Gamma[f]) \leq Z_c(f).$$

This raises the following general question. If  $\Gamma$  is an  $m$ -sequence and if  $f(x)$  is a polynomial of degree  $n$ ,  $n < m$ , then is it true that

$$Z_c(\Gamma[f]) \leq Z_c(f)?$$

An affirmative answer to the last question posed suggests the following problem. For each positive integer  $n$ ,  $n \geq 2$ , let

$$c(n) = \inf \{m \geq n \mid \text{for all } m\text{-sequences } \Gamma \text{ and for all real polynomials } f(x) \text{ of degree } n, Z_c(\Gamma[f]) \leq Z_c(f)\}.$$

Is  $c(n) < \infty$ ? It is easy to see that  $c(2) = 2$  and  $c(3) = 3$ , while Example 4.9 shows that  $c(4) > 4$ .

Finally, we shall conclude this paper with an application of the theory of  $n$ -sequences. Let  $\Gamma$  be a linear transformation defined on polynomials by

$$T \left( \sum_{k=0}^n a_k x^k \right) = \sum_{k=0}^n a_k \lambda_k x^k$$

with the eigenvalues  $\lambda_k$  real,  $k = 0, 1, 2, \dots$ . In [11, p. 382], Karlin raised the following question: When is a sequence  $\lambda_0, \lambda_1, \lambda_2, \dots$  a zero-diminishing sequence; that is for which transformations  $T$  is  $Z_R(T[f]) \leq Z_R(f)$  for all real polynomials  $f$ , where  $Z_R(f)$  denotes the number real zeros of  $f$  counting multiplicities. In [5] the authors provided a complete answer to this question and they raised the problem regarding finite sequences  $\{\lambda_0, \dots, \lambda_n\}$  which are zero-diminishing for all polynomials of degree at most  $n$ . An equivalent formulation of this problem is given by the following proposition, which takes on added significance in view of Example 4.9.

PROPOSITION 4.12. *Let*

$$T = \{\lambda_0, \dots, \lambda_n\}, \quad \lambda_k > 0,$$

and let

$$\Gamma = \{\gamma_0, \dots, \gamma_n\}, \quad \gamma_k = \lambda_k^{-1}.$$

Then  $T$  is zero-diminishing for all real polynomials of degree at most  $n$  if and only if  $\Gamma$  is an  $n$ -sequence such that  $Z_c(\Gamma[f]) \leq Z_c(f)$  for all real polynomials of degree at most  $n$ .

*Proof.* Setting  $g = \Gamma[f]$ , we see that the condition  $Z_c(\Gamma[f]) \leq Z_c(f)$  for all polynomials  $f$  of degree at most  $n$ , is equivalent to  $Z_c(g) \leq Z_c(T[g])$  for all polynomials  $g$  of degree at most  $n$ . But then this is equivalent to  $Z_R(T[g]) \leq Z_R(g)$ , since  $\lambda_k > 0$  for all  $k$  implies  $T[g]$  and  $g$  have the same degree.

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UNIVERSITY OF HAWAII AT MANOA  
HONOLULU, HAWAII