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ON THE QUADRATIC VARIATION PROCESS OF A CONTINUOUS MARTINGALE

BY

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In this article we give a simple proof of the existence of the quadratic variation process of a continuous local martingale by providing an explicit expression for it.

Let $(\Omega, \mathscr{B})$ be a fixed measurable space and let $\mathscr{G} = (\mathscr{G}_t)_{t\geqslant 0}$ be an increasing family of sub σ-fields of $\mathscr{B}$. Let M be a continuous $\mathscr{G}$ adapted process such that $M(0) = 0$.

Let $K_n(t, w) = j$, if there exists t_i such that

$$0 = t_o < t_1 < \dots < t_j \leqslant t < t_{j+1},$$
$$|M(t_i) - M(t_{i+1})| = 2^{-n}, \quad 0 \leqslant i < j,$$

and,

$$|M(t_i) - M(s)| < 2^{-n} \quad \text{if } s \in [t_i, t_{i+1}),\ 0 \leqslant i \leqslant j.$$

Let

$$\begin{aligned} X'(t, w) &= \limsup_n \frac{K_n(t, w)}{2^{2n}}, \\ U(w) &= \inf\{t > 0 : X'(t-, w) \neq X'(t^+, w)\} \\ X''(t, w) &= X'(t-, w), \quad \text{and} \\ X(t, w) &= X''(t \wedge U(w), w). \end{aligned}$$

THEOREM. *X is a continuous $\mathscr{G}$ adapted increasing process. Further, for all P such that $(M(t), \mathscr{G}_t, P)$ is a local martingale,*

$$(M^2(t) - X(t), \mathscr{G}_t, P)$$

is also a local martingale.

Proof. Fix a P such that M is a P-local Martingale.
Let $\{T_i^n : i \geqslant 1\}$, $n \geqslant 1$, be defined by

$$T_o^n = 0, \quad T_{i+1}^n = \inf\{t \geqslant T_i^n : |M(t) - M(T_i^n)| \geqslant 2^{-n}\}.$$

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Let

$$X_n(t, \cdot) = \sum_{i=0}^{\infty} (M(t \wedge T_{i+1}^n) - M(t \wedge T_i^n))^2,$$
$$Y_n(t, \cdot) = M^2(t) - X_n(t).$$

Observe that $K_n(t, w) = j$ *iff* $T_j^n \leq t < T_{j+1}^n$, so that

$$\frac{K_n(t, w)}{2^{2n}} \leq X_n(t, w) < \frac{K_n(t, w) + 1}{2^{2n}},$$

and hence $X'(t, w) = \limsup_n X_n(t, w)$.

It is easy to see that $X'(t, \omega)$ is an increasing process and hence $X'(t-, \omega)$ and $X'(t+, \omega)$ are well defined. Thus U is well defined. Also, it follows easily that $X(t, \omega)$ is a continuous process.

Let $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$ be defined by $\mathscr{F}_t = \mathscr{G}_{t+}$. Then T_i^n are $\mathscr{F}$ stop times and hence X_n is $\mathscr{F}$ adapted for all n, so that X' is $\mathscr{F}$-adapted. Right continuity of $(\mathscr{F}_t)$ implies that U is an $\mathscr{F}$-stop time so that X is also $\mathscr{F}$ adapted. Since X is a continuous process, this implies that X is $\mathscr{G}$-adapted.

In order to show that $M^2 - X$ is a local martingale, it suffices to show that

(1) Y_n converges a.s. to a continuous local martingale Y (in the u.c.c. topology on $C[0, \infty)$).

This will imply that X' is continuous a.s. so that $X = X'$ a.s. and hence that $Y = M^2 - X$.

To prove (1) suffices to consider the special case when M is bounded. Since in general we can get stop times $S_k \uparrow \infty$ such that M^{S_n} (defined by $M^{S_n}(t) = M(t \wedge S_n)$) is bounded, the general case will follow.

Now, if M is bounded (by K say), then $(M(t), \mathscr{G}_t, P)$ is a martingale. Writing

$$M^2(t) = \sum_{i=0}^{\infty} (M^2(t \wedge T_{i+1}^n) - M^2(t \wedge T_i^n))$$

we get

$$\begin{aligned} Y_n(t) &= \sum_{i=0}^{\infty} 2M(t \wedge T_i^n)(M(t \wedge T_{i+1}^n) - M(t \wedge T_i^n)). \\ &= \sum_{i=0}^{\infty} Z_{n,i}(t) \quad \text{(say)}. \end{aligned}$$

(Observe that for each (t, w), these are actually finite sums.) The fact that $(M(t), \mathscr{G}_t, P)$ is a bounded martingale implies $(Z_{n,i}(t), \mathscr{G}_t, P)$ is a martingale for all n, i.

Also, for fixed t, n, $\{Z_{n,i}(t) : i \geq 1\}$ is a centered sequence, so that

$$E\left(\sum_{i=r}^{s} Z_{n,i}(t)\right)^2 = \sum_{i=r}^{s} E\,Z_{n,i}^2(t)$$
$$\leq 4K^2 \sum_{i=r}^{s} E(M(t \wedge T_{i+1}^n) - M(t \wedge T_i^n))^2$$
$$= 4K^2\, E(M^2(t \wedge T_{s+1}^n) - M^2(t \wedge T_r^n))$$
$$\to 0 \quad \text{as } r, s \to \infty.$$

Thus $\Sigma_{i=0}^{\infty} Z_{n,i}(t)$ converges in L^2 so that, for all n, $(Y_n(t), \mathcal{G}_t, P)$ is a martingale.

For each n, let M_n be the process defined by

$$M_n(t) = M(T_i^n) \quad \text{if} \quad T_i^n \leq t < T_{i+1}^n$$

It is not difficult to verify that for all w, n

$$\{T_i^n(w) : i \geq 1\} \subset \{T_j^{n+1}(w) : j \geq 1\}.$$

Thus

$$Y_{n-1}(t) = \sum_{j=0}^{\infty} 2\, M_{n-1}(t \wedge T_j^n)(M(t \wedge T_{j+1}^n) - M(t \wedge T_j^n)).$$

Hence

$$E(Y_n(t) - Y_{n-1}(t))^2$$
$$= E\left[2 \sum_{j=0}^{\infty} (M(t \wedge T_j^n) - M_{n-1}(t \wedge T_j^n))(M(t \wedge T_{j+1}^n) - M(t \wedge T_j^n))\right]^2$$
$$\leq 4 \sum_{j=0}^{\infty} E(M(t \wedge T_j^n) - M_{n-1}(t \wedge T_j^n))^2(M(t \wedge T_{j+1}^n) - M(t \wedge T_j^n))^2$$

(as the summands form a centered sequence)

$$\leq \frac{4}{2^{2(n-1)}} \sum_{j=0}^{\infty} E(M^2(t \wedge T_{j+1}^n) - M^2(t \wedge T_j^n))$$
$$= \frac{16}{2^{2n}} E\, M^2(t).$$

Now by Doob's maximal inequality,

$$E \sup_{s \leq t} |Y_n(s) - Y_{n-1}(s)|^2 \leq \frac{64}{2^{2n}} E\, M^2(t).$$

By Borel-cantelli lemma, this implies that $Y_n(\cdot)$ converges a.s in the u.c.c. topology to some process Y (say). Further $Y_n(t)$ converges to $Y(t)$ in L^2 for each t. Thus $(Y(t), \mathcal{G}_t, P)$ is a continuous martingale.

As remarked earlier, this completes the proof.

Remark 1. If M is a continuous process of bounded variation and $M(0)$

$= 0$ then observe that

$$|X_n(t, w)| \leq \frac{1}{2^n} \operatorname{Var}(M(u, w) : 0 \leq u \leq t)$$

so that $X \equiv 0$. If moreover M is a P-local martingale then, by the theorem, M^2 is also a P-local martingale so that $M \equiv 0$ a.s. P.

Remark 2. The quadratic variation process X is usually denoted by $\langle M \rangle$. If A is a continuous increasing process such that M and $M^2 - A$ are P-local martingales, then $A - \langle M \rangle$ is a P-Local Martingale and hence (by Remark 1) $A = \langle M \rangle$ a.s. P. Existence and uniqueness of $\langle M \rangle$, for right continuous martingales M, was first proved by P. A. Meyer [3], [4].

Remark 3. Kunita Watanabe proved in Theorem 1.3 of [2] that if $\{T_i^n, i \geq 1\}$ is a $1/2^n$ partition for M, $\langle M \rangle$, t and if moreover these partitions form a chain then X_n defined as above converges a.s. to $\langle M \rangle$. Thus the existence of $\langle M \rangle$ is assumed in their proof.

Remark 4. In [1] we had arrived at exactly the same (pathwise) formula for $\langle M \rangle$ as given here, but again that proof assumed the existence of $\langle M \rangle$.

Remark 5. Observe that we have defined $X(t, w)$ explicitly in terms of $\{M(u, w) : 0 \leq u \leq t\}$ so that $\langle M \rangle$ neither depends upon the underlying probability measure P nor on the underlying σ fields $\mathcal{G}$.

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