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CONVENIENT CATEGORIES FOR INTERNAL SINGULAR ALGEBRAIC TOPOLOGY

BY

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1. Introduction

Broadly speaking, classical algebraic topology may be viewed as the study of the category of topological spaces, or more recently, of kTop, the category of compactly generated spaces, by means of various constructs (e.g. (co)homology, homotopy) that arise from a certain family of objects; namely, the affine simplexes $\{\Delta_p\}$, $p = 0, 1, \ldots$. Due to the structure of both the category kTop and the family $\{\Delta_p\}$ there is defined a functor (the singular functor) from kTop to Simpl(Sets), the category of simplicial sets (upon which is based the construction of singular (co)homology) that has a biproduct preserving functor (the geometric realization functor) as a left adjoint (which, among other things, gives a simple description of the classifying spectrum for singular cohomology), while the simplex Δ_1 (the standard unit interval) gives rise to (singular) homotopy theory in kTop. Essential in this structure is the relationship of kTop to Sets, of both Simpl(Sets) and $\{\Delta_p\}$ to Δ, the category of finite ordinals, and of Δ to N, the set of natural numbers. In the threefold generalization of this situation in which kTop is replaced by an arbitrary category A, Sets is replaced by a topos B with a natural number object (this allows for the construction of both the internal category, Δ_B, of finite ordinals in B and the accompanying category, Simpl(B), of internal simplicial objects in B), and $\{\Delta_p\}$ is replaced by a "simplex structure" in A, A is said to be convenient for B-based internal singular topology if it is so structured, both internally and relative to B, that appropriate simplex structures in A induce functors $A \to$ Simpl(B) with binary product preserving left adjoints. The purpose of this paper is to give explicit conditions that render a category convenient (Section 4), to make precise the notion of a simplex structure (Section 4), and to describe several examples of convenient categories (Section 5). It will emerge that convenient categories need not be homotopically interesting. For example, the well known categories of Bornological spaces, Grill spaces, and Convergence spaces are shown to be Set-convenient but homotopically trivial. Section 2 deals with the necessary basic categorical results and Section 3 develops the category Simpl(B).

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2. Categorical Preliminaries

Most of our results depend upon a generalization of a special case of Diaconescu's theorem. Recall (Theorem 4.34[8]) that if C is an internal category in a topos B and $f:A \to B$ is a B-topos then there is an equivalence between the category of flat presheaves $f^*C^{\mathrm{op}} \to A$ (i.e., discrete fibrations $G \to f^*C$ with G filtered) and the category of geometric morphisms over B from A to B^C. The geometric morphism γ associated to the discrete fibration $g:G \to f^*C$ can be described in terms of the following commutative diagram:

(I)
$$\begin{tikzcd} B^C \arrow[r, "\bar{f}^*"] \arrow[d] & A^{f^*C} \arrow[r, "\bar{g}_o^*"] \arrow[d] & A^G \arrow[r, "\mathrm{colim}"] \arrow[d, "U"] & A \\ B/C_0 \arrow[r, "f^*"] & A/f^*C_0 \arrow[r, "g_o^*"] & A/G_0 \arrow[r, "\Sigma"] & A \end{tikzcd}$$

Here the vertical maps are the canonical forgetful functors, which are monadic by 2.21 [8], the functor $\bar{f}^*$ (resp. $\bar{g}_0^*$) is the lift, using lemma 1 [7], of the functor $B/C_0 \to A/f^*C_0$, also denoted by f^*, induced by the inverse image f^* of f (resp. of the map g_0^* induced by pullback along $g_0:G_0 \to f^*C_0$), (Note that the λ of lemma 1 [7] is, in both the f^* and g_0^* case, an isomorphism; f^* is left exact and g is a discrete fibration), colim is the colimit functor of 2.24 [8] and $\Sigma(F \to G_0) = F$. The inverse image γ^* of γ is the composite colim $\bar{g}_0^*\bar{f}^*$ while the direct image γ_* of γ is the right adjoint to γ^*, which exists since each of the functors making up γ^* has a right adjoint: colim has a right adjoint by 2.25 [8]; namely, the functor G^* that sends objects of A to the constant diagrams, and $\bar{f}^*$ (resp. $\bar{g}_0^*$) has a right adjoint by theorem 4 [7] since f^* (resp. g_0^*) has a right adjoint; namely, the functor that sends $(F \to f^*C)$ to the pullback of $f_*F \to f_*f^*C$ along the unit $C \to f_*f^*C$ (resp. the functor π_{g_0} of 1.43 [8]). Note that the only conditions on A needed to define γ^* are that A has finite limits (to get $\bar{g}_0^*$, $\bar{f}^*$) and reflexive coequalizers (to get colim). Further, in order for γ^* to have a right adjoint it is not necessary for each of the maps $\bar{f}^*$, $\bar{g}_0^*$ and colim to have a right adjoint, as happens above; it is sufficient for the composite $\Sigma g_0^* f^*:B/C_0 \to A$ to have a right adjoint. This is a consequence of the following generalization of Theorem 4 [7] applied to diagram I (i.e., with $L = g_0^* f^*$, $\overline{G} =$ colim, $\overline{H} = G^*$ etc.) since UG^* is right adjoint to Σ.

2.1 LEMMA. *Let T_1, T_2 be monads on A, B respectively with $\lambda:T_2L \approx LT_1$. If the functors $\overline{G}$, Σ, ΣL, in the following diagram, have right adjoints $\overline{H}$, $U\overline{H}$, R respectively, then $\overline{G}\overline{L}$ has a right adjoint, where $\overline{L}$ is the lift of*

L determined by λ:

$$\begin{tikzcd} A^{T_1} \arrow[r, "\overline{L}"] \arrow[d] & B^{T_2} \arrow[d, "U"'] \arrow[dr, "\overline{G}"] & \\ A \arrow[r, "L"] & B \arrow[r, "\Sigma"] & C \end{tikzcd}$$

Proof. A direct diagram chase shows that if θ: $\Sigma LT_1R \to I$ is the $(\Sigma \dashv H)$ adjoint of

$$LT_1R \xrightarrow{\lambda^{-1}R} T_2LR \xrightarrow{T_2\varepsilon} T_2H \xrightarrow{\sigma} H,$$

where $\sigma: T_2H \to H = U\overline{H}$ is the T_2-action on H corresponding to the lift $\overline{H}$ of H and $\varepsilon: LR \to H$ is the $(\Sigma \dashv H)$-adjoint of the counit $\Sigma LR \to I$ of the adjunction $\Sigma L \dashv R$, then $\overline{\theta}$: $T_1R \to R$, the $(\Sigma L \dashv R)$-adjoint of θ, is an action of T_1 on R that defines a lift $\overline{R}: A \to B^{T_2}$ of R. Moreover, $\varepsilon: LR \to H$ is compatible with the actions

$$T_2LR \xrightarrow{\lambda R} LT_1R \xrightarrow{L\overline{\theta}} LR$$

(defining the lift $\overline{L}\overline{R}$) and $\sigma: T_1H \to H$ (defining the lift $\overline{H}$) and so induces a map $\overline{\varepsilon}: \overline{L}\overline{R} \to \overline{H}$ for which

$$\overline{G}\overline{L}\overline{R} \xrightarrow{\overline{G}\overline{\varepsilon}} \overline{G}\overline{H} \xrightarrow{\varepsilon'} I,$$

where ε' is the counit of $\overline{G} \dashv \overline{H}$, is universal and thus defines the counit of the desired adjunction $\overline{G}\overline{L} \dashv \overline{R}$.

The fact that γ^* preserves finite limits depends upon the fact that colim preserves such limits, since the functors $\overline{f}^*$, $\overline{g}_0^*$, by virtue of being lifts of functors that clearly preserve finite limits (f is geometric, g_0^* is a pullback), preserve such limits. In the proof (2.57 [8]) that colim preserves binary product, the comparison map ϕ: colim $(\alpha^1 \times \alpha^2) \to$ (colim α^1) $\times$ (colim α^2), where $(\alpha^i: F^i \to G) \in A^G$, $i = 1, 2$, is shown to be both an epi and a mono and thus, since A is a topos, an isomorphism. Under the weaker assumptions, i.e., that A has finite limits and reflexive coequalizers, ϕ can still be shown to be an isomorphism if we strengthen certain other assumptions as follows: Recall (p. 70 [8]) that $(\sigma^1 \times \sigma^2)\varepsilon$ factors through ϕ, where ε is the pullback of

$$d = \langle d_0\pi_1, d_0\pi_2\rangle : K(d_1) \to G_0 \times G_0$$

along

$$\alpha_0^1 \times \alpha_0^2 : F_0^1 \times F_0^2 \to G_0 \times G_0,$$

where $K(d_1)$ is the pullback of $d_1: G_1 \to G_0$ along itself and π_i is the canonical projection, $i = 1, 2$, and $\sigma^i: F_0^i \to$ colim α^i, $i = 1, 2$, is the defining map of colim α^i as a coequalizer (2.24 [8]). Since, in a topos, products of coequalizers and pullbacks of epis are epis, it follows that $\sigma^1 \times \sigma^2$ and ε (d is an epi as part of the filter assumption on G, 2.51 [8]), and consequently ϕ, are epis. However, if we replace the assumption that d is an epi by the assumption that the pullback of d along maps of the form $\alpha_0^1 \times \alpha_0^2$, where $\alpha = \bar{g}_0^* \bar{f}^*(k)$ for $k \varepsilon B^C$, is an extremal epi (i.e., factors through no mono, with same codomain, but an isomorphism) then, as above, ϕ is an extremal epi if binary products of coequalizers are extremal epis. (This last condition is equivalent to the condition that for each $a \varepsilon A$, the functor $a \times - : A \rightarrow A$ transforms coequalizer maps into extremal epis.) In this case ϕ is an isomorphism if it is a mono. On the other hand, if the direct image functor f_* of f preserves coequalizers then it easily follows that f_* commutes with colim in the sense that $f_*\phi$ coincides with the comparison map colim $(\bar{f}_*\alpha^1 \times \bar{f}_*\alpha^2) \rightarrow (\text{colim } \bar{f}_*\alpha^1) \times (\text{colim } \bar{f}_*\alpha^2)$, where $\bar{f}_*: A^G \rightarrow B^{f_*G}$ is the lift, as above, of the f_* induced map $A/G_0 \rightarrow B/f^*G_0$. Thus, if the internal category f_*G in B is filtered then, by 2.57 [8], $f_*\phi$ is a mono, and if f_* reflects monos, e.g., if f_* is faithful, then ϕ is a mono. We have, therefore, conditions which ensure that ϕ is an isomorphism and consequently that γ^* preserves binary products, even if A is not a topos. Similarly, γ^* preserves terminals if f_* reflects them. Further, if the unit of the adjunction $(f^* \dashv f_*)$ is an isomorphism then $f_*\gamma^*$ is the inverse image of the geometric morphism from B to B^C corresponding to the flat presheaf $f_*G \rightarrow f_*f^*C \approx C$. These considerations motivate Definition 2.2 and prove Theorem 2.4.

2.2 Definition. Let A be a category with finite limits, $f: A \rightarrow B$ a geometric morphism into a topos and C an internal category in B. An internal presheaf on f^*C with corresponding discrete fibration $g: G \rightarrow f^*C$ is said to be (*strongly*) *f-flat* if f_*G is filtered and the pullback of $d: K(d_1) \rightarrow G_0 \times G_0$ along any map of the form $(g_0 \times g_0)^* f^*(k)$, where $k \varepsilon B/C_0 \times C_0$ (of the form $(g_0 \times g_0)^*(k)$, where $k \varepsilon A/f^*C_0 \times f^*C_0$) is an extremal epi.

2.3 *Remark.* Note that the pullback condition on d in 2.2 holds if it holds for some map that factors through d, since if fg is an extremal epi then so is f.

2.4 Theorem. *If C is an internal category in a topos B, A is a category with finite limits and reflexive coequalizers, and $f: A \rightarrow B$ is a geometric morphism then each internal presheaf g on f^*C induces a functor $\gamma^*: B^C \rightarrow A$ such that:* (1) *If $\Sigma g_0^* f^*: B/C_0 \rightarrow A$ has a right adjoint then so does γ^*.* (2) *If, for each $a \varepsilon A$, $a \times -m: A \rightarrow A$ transforms coequalizer maps into extremal epis, f_* preserves reflexive coequalizers and reflects monos and terminals, and g is f-flat then γ^* preserves binary products and terminals. In addition, if the unit of $f^* \dashv f_*$ is an isomorphism then $f_*\gamma^*$ is the inverse image of the geometric morphism $B \rightarrow B^C$ corresponding to the flat presheaf f_*g on C.*

2.5 *Remark.* It readily follows from a slight extension of the previous considerations that if the map $\Sigma g_0^* : A/f^*C_0 \to A$ has a right adjoint then so does

$$\gamma_1^* = \operatorname{colim} \bar{g}_0^* : A^{f^*C} \to A$$

and that if g is strongly f-flat then γ_1^* preserves binary products and terminals.

3. Internal Simplicial Categories

If, for a topos, E, $\mathrm{Rel}(E)$ is the E-enriched category [2] of relations in E (an object of $\mathrm{Rel}(E)$ is a mono $S \to X^2$ (a relation on X) and, for objects $\alpha : S \to X^2$, $\beta : T \to Y^2$, $\mathrm{Rel}(E)(\alpha, \beta)$ is the object of E that, in the internal language of E (§5.4 [8]), is described by $\{f \mid \exists g(f^{2\ulcorner}\alpha^{\urcorner} = {}^{\ulcorner}\beta^{\urcorner}g)\}$, where f, g is a variable of type Y^X, T^S respectively. Internal composition in E induces the composition in $\mathrm{Rel}(E)$ rendering it an E-category) then, for a topos B, the correspondence

$$R : X \mapsto \text{the } (B/X)\text{-category } \mathrm{Rel}(B/X)$$

is readily seen to have the structure of a locally internal category over E (p. 335 [8]). An object $\alpha \in R(X) = \mathrm{Rel}(B/X)$ can be viewed as an internal X-indexed family F of objects of E with relations, and $\mathrm{Full}_R(\alpha)$, the internal category of E defined in [8], p. 340, plays the role of the subcategory of E with F as the family of objects and with all relation preserving maps between them as morphisms (compare [8], p. 58). In particular, if B is a topos with a natural number object $(N, +, s)$ (p. 165 [8]) and α^+ (resp. α) is the relation on

$$N^2 \xrightarrow{+} N \quad (\text{resp. on } N^2 \xrightarrow{s+} N)$$

in B/N induced by the relation $\{(y, z) \mid p_1(y) \leq p_1(z) \wedge +(y) = +(z)\}$ on N^2, where y, z are variables of type N^2 and $p_1 : N^2 \to N$ is projection on the first factor, then $\Delta_B^+ = \mathrm{Full}_R(\alpha^+)$ (resp. $\Delta_B = \mathrm{Full}_R(\alpha)$) is the internal category in B that, in view of the interpretation of the fibers of $+$ and $s+$ in [8], p. 173, plays the role, in the terminology of [9], p. 174, of the category Δ^+ (resp. Δ) when $B = \text{Sets}$. Thus $\mathrm{Simpl}(B) = B^{(\Delta_B^+)^{\mathrm{op}}}$ (resp. $\mathrm{AugSimpl}(B) = B^{\Delta_B{}^{\mathrm{op}}}$) is the category of internal simplicial (resp. augmented simplicial) objects of B.

4. Internal Singular Topology

In this section we expand upon the results of §2 when $C = (\Delta_B^+)^{\mathrm{op}}$.

4.1 Definition. Let B be a topos with a natural number object N. By a (*strongly*) *convenient category* for B-based singular algebraic topology, or simply by a (strongly) B-convenient category is meant a category A with finite limits, and reflexive coequalizers that the functors $a \times -$ transform into extremal epis, for all $a \in A$, together with a geometric morphism $f:A \to B$ with an isomorphism as unit and a direct image that preserves reflexive coequalizers and reflects monos and terminals and, for each map $g:Y \to f^*N$ in A,

$$\Sigma g^* f^* : B/N \to A \quad (\Sigma g^* : A/f^*N \to A)$$

is a left adjoint.

4.2 *Remark.* The conditions on f allow us (which we do) to view B as a left exactly embedded (via f^*), coreflective full subcategory of A. If the counit of f is a pointwise epi, i.e., if B is an epi-coreflective subcategory of A, then f_* is faithful and consequently reflects monos (see Prop. 0.3, p. 5 [2]).

4.3 Definition. Let $f:A \to B$ be a geometric morphism of a category A with finite limits into a topos with a natural number object. By a (*strong*) *simplex structure* in A is meant a (strong) f-flat presheaf (2.2) on $f^*\Delta_B^{+\mathrm{op}}$.

If, by a semi-geometric morphism we mean an adjunction ($f^* \dashv f_*$) for which f^* preserves binary products and terminals, then 2.4 gives:

4.4 Theorem. *Each* (*strong*) *simplex structure on a* (*strongly*) *B-convenient category $A \supset B$ induces a semi-geometric morphism $A \to$* Simpl(B) ($A \to$ Simpl(A)).

4.5 *Remarks.* (1) The category Top of topological spaces is not Set-convenient for

$$f = (\text{discrete} \dashv \text{forgetful}): \text{Top} \to \text{Set},$$

since the product of two coequalizers need not be an extremal epi, while the forgetful functor CGHaus $\to$ Sets from the category of compactly generated Hausdorff spaces (p. 181 [9]) does not preserve coequalizers. However, the category kTop of k-spaces in the sense of Vogt [17] is strongly Set-convenient.

(2) The direct (inverse) images of the semi-geometric morphisms $A \to$ Simpl(B) of 4.4 are a generalization of the usual singular (geometric realization) functor (p. 43 [3] or p. 55 [10]) which is the special case in which A is kTop (structured over Sets as in (1)) with the simplex structure corresponding to the affine simplex functor $\Delta^+ \to$ kTop of [9], p. 174.

(3) The existence of the singular functors $A \to$ Simpl (B) allows the usual definitions of singular homology and cohomology to be extended to the category A.

(4) Since the geometric realization functors Simpl(B) $\to A$ preserve finite products and terminals they preserve homotopy as well as induce functors from any category of (external) universal algebras in Simpl(B) to the cor-

responding category of algebras in A. Thus, for example, many of the constructions (some of which depend upon a strong geometric realization $\mathrm{Simpl}(A) \to A$) of Boardman and Vogt [1] and Segal [13] can be extended to the category A.

We next consider some conditions under which a discrete fibration $g: Y \to (\Delta_B^+)^{op}$ is f-flat, where, as in 4.2, f^* is an inclusion $B \subset A$. Let $(d_0, d_1 : N_1 \rightrightarrows N_0 = N)$ represent the category Δ_B^+ in B. Define $(\delta_1 : E \to N^2) = (E \to E_1^2 \rightrightarrows N^2)$ as the equalizer of the restriction to E_1^2 of the maps $(\Delta)(+)(d_1^2): N_1^2 \to N^2 \to N \to N^2$ and d_0^2, where E_1, the subobject of "epis" of N_1, is described in internal language by

$$\{f \mid \forall_x \forall_y (d_1 x = d_1 y \wedge d_0 x = f \wedge d_0 y = f) \to (d_2 x = d_2 y)\}$$

where f (resp. x, y) is a variable of type N_1 (resp. $N_2 = N_1 \times_N N_1$). Since g is a discrete fibration, $d_0 g_1 = g_0 d_1$, and consequently $d_0^2 g_1^2 = g_0^2 d_1^2$, is a pullback. Thus, since δ_1 factors through d_0^2, there is induced a map $\delta_2: Z \to Y_1^2$ such that $d_1^2 \delta_2$ factors through the diagonal Δ of Y_0, where Z is obtained by pulling back δ_1 along $g_0^2 \Delta$. Moreover, since $K(d_1)$ is the pullback of Δ along d_1^2, δ_2 factors through $K(d_1)$ and $\delta = d_0^2 \delta_2 : Z \to Y_0^2$ factors through $d: K(d_1) \to Y_0^2$ of 2.2. For each pair $p, q: 1 \to N$ of natural numbers in B, the pullback of

$$Z \xrightarrow{\delta} Y_0^2 \xrightarrow{g_0^2} N^2$$

along $\langle p, q\rangle: 1 \to N^2$ has, as a direct diagram chase shows, the form

$$E_1(p + q, p) \times E_1(p + q, q) \times Y_0(p + q) \xrightarrow{\delta(p,\, q)} Y_0(p) \times Y_0(q) \to 1$$

where, in general, $E_1(p, q)$ (resp. $Y_0(p)$) is the fiber of

$$E_1 \to N_1 \xrightarrow{\langle d_0,\, d_1\rangle} N^2$$

(resp. of $g_0: Y_0 \to N$) over $\langle p, q\rangle$ (resp. p). Consequently each triple

$$(p, q, s) \in B(1, N)^2 \times B(1, E_1(p + q, p) \times E_1(p + q, q))$$

defines, for any map $k: K \to N^2$ with $\langle p, q\rangle$-fiber $K(p, q)$, the indicated commutative diagram, where all but the triangular subdiagram are pullbacks. From the diagram it readily follows that k^* is extremal if there is a universal (i.e., preserved by pullback) extremal sink (i.e., a jointly extremal epi family) $\pi \subset B(1, N)$ for which each of the families

$$\sigma(p, q) = \{\delta(p, q, s) \times \mathrm{id} \mid s \in B(1, E_1(p + q, p) \times E_1(p + q, q))\}$$

is an extremal sink, $p, q \in \pi$. For then k^* appears as the second factor in a factorization of the extremal sink $\{\langle p, q\rangle^* \sigma(p, q) \mid p, q \in \pi\}$ as is easily seen, and is therefore extremal. From these considerations we readily have:

4.6 THEOREM. *Let $\pi \subset B(1, N)$ be a universal extremal sink in a B-convenient category A for which $a \times - A \to A$ preserves extremal sinks for*

$$
\begin{tikzcd}
Y_0(p+q) \times K(p,q) \arrow[d, "s\times \mathrm{id}\times \mathrm{id}"] \arrow[dr, "\delta(p,q,s)\times \mathrm{id}"] & & & \\
E_1(p+q,p)\times E_1(p+q,q)\times Y_0(p+q)\times K(p,q) \arrow[r, "\delta(p,q)\times \mathrm{id}"'] \arrow[d] & Y_0(p)\times Y_0(q)\times K(p,q) \arrow[r] \arrow[d] & K(p,q) \arrow[r] \arrow[d] & 1 \arrow[dd, "\langle p,q\rangle"] \\
Z^* \arrow[r, "k^*"] \arrow[d] & (Y_0^2)^* \arrow[r, "\langle p,q\rangle^*"] \arrow[d] & K \arrow[dr, "k"] & \\
Z \arrow[r, "\delta"] & Y_0^2 \arrow[rr, "g_0^2"] & & N^2
\end{tikzcd}
$$

all $a \in B$ (resp. all $a \in A$). A discrete fibration $g: Y \to (\Delta_B^+)^{\mathrm{op}}$ is a simplex structure (resp. a strong simplex structure) in A if f_*Y is filtered and

$$\{\delta(p, q, s): Y_0(p + q) \to Y_0(p) \times Y_0(q) \mid s \in B(1, E_1(p + q, p) \times E_1(p + q, q))\}$$

is an extremal sink for all $p, q \in \pi$.

If $\pi \subset B(1, N)$ defines N as a universal coproduct in A (and consequently is a universal extremal sink in A) and B has π-indexed coproducts then $\langle p^*\rangle: B/N \to \amalg_p B$, where $p^*: \mathrm{B/N} \to \mathrm{B}$ is induced by pullback along $p \in \pi$, is an equivalence of categories. Further, if A has π-indexed coproducts, $\Sigma g_0^*: B/N \to A$ has a right adjoint if $a \times - : B \to A$ has a right adjoint $(-)^a: A \to$ B for all $a \in A$ (i.e., if A is tensored over B, p. 20 [2]) since Σg_0^* corresponds, under the equivalence $\langle p^*\rangle$, to the functor

$$\coprod_p (p^*(Y_0) \times (-)_p) : \prod_p B \to A$$

which has the right adjoint $\{(-)^{p^*(Y_0)}\}_p$: $A \to \amalg_p B$. Similarly, if $\langle p^*\rangle$: $A/N \to \amalg_p A$ induces an equivalence then $\Sigma g_0^*: A/N \to A$ has a right adjoint if $ax^-: A \to A$ does (i.e., if A is cartesian closed). If B is a Set-topos then $N \approx \amalg_n 1$, $n = 0, 1, \ldots$, and $\pi = \{0, 1, \ldots\} \subset B(1, N)$ defines N as a universal coproduct in B. Thus, in view of 4.1 and 4.2 we have:

4.7 Theorem. *Let A be a category with finite limits, reflexive coequalizers and universal countable coproducts, that contains B as a left exactly embedded epi-coreflective subcategory such that, for all $a \in A$, $ax-: A \to A$ (resp. $a\times - : B \to A$) preserves coequalizers (resp. has a right adjoint). If B is a Set-topos with countable coproducts then $A \supset B$ is B-convenient. Moreover, if $\langle n^*\rangle: A/(\amalg_{n=0}^{\infty} 1) \to \amalg_{n=0}^{\infty} A$ is an equivalence and A is cartesian closed then A is strongly B-convenient.*

4.8 *Remark.* It is easy to see that $\langle n^*\rangle$ is an equivalence if A has universal countable corproducts and for each coproduct $\amalg_{m=0}^{\infty} Y_m$, the canonical map $Y_n \to \amalg_m Y_m$ is the pullback of $n: 1 \to \amalg_m 1$ along $\amalg_m t_m$, $t_m: Y_m \to 1$.

Since coequalizers are extremal, since sinks containing extremal subsinks are extremal, and since a sink $\{f_i: X_i \to X\}$ is extremal iff $\Sigma f_i: \amalg_i X_i \to X$ is extremal (if the coproduct exists) we obtain, as we did 4.6, an "external" analogue to 4.6 for suitable Set-toposes B.

4.9 Theorem. *Let $A \supset B$ be a B-convenient category where B is a Set-topos and A has universal countable coproducts. Let G: $\Delta = \Delta_{\mathrm{Set}}^+ \to A$ be the (external) functor associated ($G[n] = Y_0(n)$, etc.) to a discrete fibration $g: Y \to (\Delta_B^+)^{\mathrm{op}}$ with f_*Y filtered. If*

$$\Sigma\langle Gs_1, Gs_2\rangle: \amalg_s Y_0(m + n) \to Y_0(m) \times Y_0(n)$$

is a coequalizer, where $s = (s_1, s_2)$ *varies over the (finite) set of pairs of epis (i.e., degeneracies) in* $\Delta([m+n],[m]) \times \Delta([m+n],[n])$ *and* $a \times - : A \to A$ *is right exact for all* $a \in B$ *(resp. all* $a \in A$*) then* g *is a simplex structure (resp. a strong simplex structure) in* A.

4.10 *Remark.* Theorems 4.7 and 4.9 are useful in recognizing B-convenient categories and simplex structures in them when B is a Grothendieck topos, in particular, when B is a spatial topos. Note also that any topos B with a natural number object is B-convenient.

5. Applications and Examples

In this section we examine various topological categories for their suitability for singular algebraic topology. Recall [4] that an absolutely topological functor U: $A \to B$ is cocontinuous (§6 [4]) and defines B as an epi-coreflective subcategory of A (p. 135 [4]) (identifying B with the "discrete" objects of A). U need not reflect terminals, see 5.5 (that it does is often included in the definition of topological, see p. 1362 [12]) nor be geometric (e.g., the product of spaces with the discrete uniform structure (§3 [5]) need not be a discrete uniform space).

Recall (§3 [14]) that a *filter convergence space* $Y(*)$ consists of a set Y and, for each $a \in Y$, a set $Y(a)$ of filters on Y satisfying: (1) If F and G are filters on Y and $F \subset G$ then $G \in Y(a)$ if $F \in Y(a)$. (2) $[a] \in Y(a)$ (for a set α of subsets of Y, $[\alpha] = \{B \mid A \subset B \subset Y$, for some $A \in \alpha\}$). A morphism f: $Y(*) \to Z(*)$ of filter convergence spaces is a function f: $Y \to Z$ such that $[f(F)] \in Z(f(a))$ if $F \in Y(a)$. We denote the category so formed by FCO. Further, $Y(*) \in$ FCO is called a *convergence space* when $F \cap [a] \in Y(a)$ if $F \in Y(a)$, a *limit space* when $F \cap G \in Y(a)$ if $F, G \in Y(a)$, and a *pseudotopological space* when $F \in Y(a)$ if $G \in Y(a)$ for all ultrafilters $G \supset F$. These spaces are the objects of full subcategories Con, Lim, PsT of FCO. The full subcategory, ConsFCO, of FCO determined by the constant filter convergence spaces ($Y(a) = Y(b)$ for all $a, b \in Y$) is, by [14], isomorphic to both the category Grill of Grill spaces (p. 350 [14]), which, by p. 282 [6], contains both the contiguity spaces and the proximity spaces (§4 [5]), and the category Fil of filter merotopic spaces (p. 351 [14], p. 281 [6]). The functor U: $Y(*) \mapsto Y$ defines the categories FCO, ConsFCO, Con, Lim, PsT as absolutely topological over Sets; a source

$$\{f_i\colon Y(*) \to Y_i(*)\}_i$$

is initial precisely when

$$Y(a) = \{F | F \text{ is a filter on } Y \text{ and } [f_i(F)] \in Y_i(f_i(a)), \text{ all } i\}$$

for all $a \in Y$. Since the discrete structure on Y in all but ConsFCO is given by $Y(a) = \{[a], [\{\phi\}]\}$ and in ConsFCO by $Y(a) = \{[b], [\{\phi\}] \mid b \in Y\}$, products and subspaces of discrete spaces are readily seen to be discrete, thus U is geometric and obviously reflects terminals. In order to show that these categories are Set-convenient it remains, by 4.7, to prove that they have universal countable coproducts (5.5 shows that absolutely topological and geometric does not imply that coproducts are universal) since, by [14] and §3 [12], they are cartesian closed. The key notion for this and other results is that of coherence (cf. p. 2 [15]). An object $Y \in A$, where $U: A \to \mathrm{Set}$ is absolutely topological, is said to be α-coherent, where α is a cover of UY, if the family α^* of U-initial lifts of the inclusion maps of α is a U-final sink (§1 [12]), and a map $f: Y \to Z$ in A is said to be fiber coherent if Y is α-coherent where α is the family of fibers of Uf.

5.1 LEMMA. *If each map $f: Y \to N$ in A is fiber coherent, where N is the set of natural numbers with the discrete A-structure, and U reflects initial objects then countable coproducts in A are universal and $\langle n^* \rangle$ of* 4.7 *is an equivalence.*

Proof. Since the natural maps $i_n: Y_n \to Y = \amalg_m Y_m$ of a countable coproduct in A are embeddings, i.e., mono and U-initial (Y_n is, by assumption, an initial object if $UY_n = \phi$, and is a retract of Y if $UY_n \neq \phi$), it readily follows that i_n is the pullback of $n: 1 \to N = \amalg_m 1$ along $t = \amalg_m t_m$: $Y \to N$ for $t_m: Y_m \to 1$. Thus, for any map $g: Z \to Y$, the pullback $z_n \to Z$ of i_n along g coincides with the pullback of n along $tg: Z \to N$ which, in turn, coincides with the U-initial lift of the fiber of $U(tg)$ over n. The result now follows since $\{Z_n \to Z \mid n = 0, 1, \ldots\}$, being a final epi sink, is readily seen to be a coproduct. Recall, also, 4.8.

We next characterize α-coherence for the various spaces $Y(*)$.

5.2 LEMMA. *If $Y(*)$ is in* (1) FCO *or* Con, (2) ConsFCO, (3) Lim, (4) PsT, *then $Y(*)$ is α-coherent if and only if for each $a \in Y$ and for* (1) *each $F \in Y(a)$ there is an $A \in \alpha$ with $a \in A \in F$,* (2) *each $F \in Y(a)$ there is an $A \in \alpha$ with $A \in F$,* (3) *each $F \in Y(a)$ there are $A_i \in \alpha$, $F_i \in Y(a)$ with $a \in A_i \in F_i$, $i = 1, \ldots, n$ and $F_1 \cap \cdots \cap F_n \subset F$,* (4) *each ultrafilter $F \in Y(a)$* there is an A $\in \alpha$ with a $\in$ A $\in$ F, *respectively.*

Proof. By definition we must show that if α is a cover of Y then the family $\alpha^* = \{A(*) \to Y(*) \mid A \in \alpha\}$ is a final epi sink, where, due to the characterization (3.2.2 [12]) of initial sources, the structure $A(*)$ is given by $A(a) = \{G \mid G$ is a filter on A with $[G]_Y = \{B \mid C \subset B \subset Y$, for some $C \in G\} \in Y(a)\}$. Now α^* is a final epi sink iff for each $a \in Y$ and, in case (1) by 3.2.3 [12], for each $F \in Y(a)$ there are $A \in \alpha$ and $G \in A(a)$ with $[G]_Y \subset F$ and $a \in A$; in case (2), for each $F \in Y(a)$ there are $A \in \alpha$ and $G \in A(a)$ with $[G]_Y \subset F$; in case (3) by 3.2.6 [12], for each $F \in Y(a)$ there are $A_i \in \alpha$ and $G_i \in A_i(a)$ with $a \in A_i$, $i = 1, \ldots, n$ and $(\cap_{i=1}^n [G_i]_Y) \subset F$; in case (4) by 3.2.9 [12], for each ultrafilter $F \in Y(a)$ there are an $A \in \alpha$ and an ultrafilter $G \in A(a)$ with $[G]_Y = F$ and $a \in A$. The result now follows since these characterizations of α^* as a final epi sink are readily seen to be equivalent to the corresponding conditions of the lemma in view of the following:

5.3. LEMMA. *If $A \subset Y$ and F is a filter (ultrafilter) on Y then $F = [G]_Y$ for some filter (ultrafilter) G on A iff $A \in F$.*

Proof. If $A \in F$ let $G = \{C \mid A \supset C \in F\}$. For the converse note that $A \in G \subset [G]_Y$ for any filter G on A.

5.4 THEOREM. *The categories* FCO, ConsFCO, Con, Lim, PsT *are strongly convenient for* Set-*based algebraic topology.*

Proof. We have only to show, in view of the discussion preceding 5.1, that these categories have universal countable coproducts, or, by 5.1, since ϕ has a unique structure, that each morphism $f\colon Y(*) \to N(*)$ is fiber coherent where $N(n) = \{[m], [\{\phi\}] \mid m \in N\}$ or $\{[n], [\{\phi\}]\}$ as $N(*)$ is or is not in ConsFCO. If $\phi \notin F \in Y(a)$ and $[fF] \in N(f(a))$ then $[fF] = [m]$ for some $m \in N$ ($m = f(a)$ in all but the ConsFCO case), i.e., there is a $B \in F$ with $F(B) = m$, and consequently, since $B \subset f^{-1}(m)$, $f^{-1}(m) \in F$. Thus, by 5.2, $Y(*)$ is $\alpha = \{f^{-1}(m) \mid m \in N\}$-coherent in all cases and the result follows.

A prebornological space is a pair $(Y, \mathcal{B})$, where $\mathcal{B}$ is a family of subsets of Y that is closed under finite union and contains all finite nonempty subsets of Y. Further, if $\mathcal{B} \neq \emptyset$ and $\mathcal{B}$ contains all subsets of each element of $\mathcal{B}$ then $(Y, \mathcal{B})$ is called a bornological space (3.3 [12]). A morphism $f\colon (Y, \mathcal{B}_1) \to (Z, \mathcal{B}_2)$ of such spaces is a function $f\colon Y \to Z$ such that $f(B) \in \mathcal{B}_2$ if $B \in \mathcal{B}_1$. We denote by PBorn and Born, respectively, the categories so formed and by PBorn* the full subcategory of PBorn determined by those spaces $(Y, \mathcal{B})$ with $\phi \notin \mathcal{B}$. The forgetful functor $U\colon (Y, \mathcal{B}) \mapsto Y$ defines these categories as absolutely topological over sets with a source

$$f_i\colon (Y, \mathcal{B}) \to (Y_i, \mathcal{B}_i)$$

initial iff

$$\mathcal{B} = \{B \mid B \subset Y, f_i(B) \in \mathcal{B}_i, \text{ all } i\}$$

and an epi sink $f_i\colon (Y_i, \mathcal{B}_i) \to (Y, \mathcal{B})$ final in PBorn or PBorn* (resp. Born) iff $\mathcal{B} = \{B \mid B \subset Y$ and B is (resp. is contained in) a finite union of sets $f_i(C)$ with $C \in \mathcal{B}_i\}$. Since the discrete PBorn, PBorn* (Born) structure on Y is the set of nonempty finite (finite) subsets of Y, U is readily seen to be geometric. However, U reflects initial and terminal objects only in the PBorn* and Born cases since ϕ (resp. 1) has the two PBorn structures $\{\ \}$ and $\{\phi\}$ (resp $\{\{1\}\}$ and $\{\phi, \{1\}\}$). Furthermore, since the projection on the first factor $Y = \{0, 1\} \times N \to N$ is a morphism in PBorn (PBorn*), when N is discrete and Y has the structure $\mathcal{B} = \{A \mid A$ is a nonempty finite (finite) subset of Y or is $Y\}$, that is not fiber coherent, it easily follows that:

5.5 *Remark.* PBorn and PBorn* are absolutely topological and geometric over Sets but in neither are coproducts universal and only in PBorn* are initial and terminal objects reflected.

In Born, however, any morphism f of $(Y, \mathcal{B})$ into the discrete space N is fiber coherent since each $B \in \mathcal{B}$ lies in the union of a finite number of fibers ($f(B)$ is finite) and, as is readily shown, $(Y, \mathcal{B})$ is, in general, α-coherent iff each $B \in \mathcal{B}$ lies in the union of a finite number of elements of α. It follows then, as in the proof of 5.4, in view of 5.1 and the fact that Born is cartesian closed (3.3.4 [12]), that Born is strongly Set-convenient.

The category PrOrd of preordered spaces $(y, \leqslant)$ and order preserving functions is, by 3.1 [12], absolutely topological over Sets and is cartesian closed. Furthermore, since $(Y, \leqslant)$ is α-coherent iff for all $a, b \in Y$, $a \leqslant b$ iff there is a sequence $a = a_1 \leqslant \cdots \leqslant a_n = b$ with each pair $a_i, a_{i+1} \in A$, for some $A \in \alpha$, $i = 1, \ldots, n - 1$, and since the discrete structure on Y is given by $a \leqslant b$ iff $a = b$ for all $a, b \in Y$, it readily follows that PrOrd is geometric over Sets, has universal coproducts and is, consequently, Set-convenient.

In summary, recalling that kTop, the category of compactly generated spaces in the sense of Vogt (p. 228 [1], [17]), is the classical example of a Set-convenient category, we have:

5.6 THEOREM *The categories* Born, PrOrd *and* kTop *are strongly* Set-*convenient.*

For our last example, let $A = \text{kTop}(B)$ be the category of k-spaces over the Grothendieck topos B of sheaves over a locally compact Hausdorff space Y. An object of B is a local homeomorphism $p: E \to Y$, while an object of A is a pair $(p: E \to Y, \tau)$ where $p \in B$ and τ is a topology on E that is coarser than the sheaf topology on E and relative to which E is a k-space (p. 228 [1]) and p is continuous. A morphism f: $(p_1, \tau_1) \to (p_2, \tau_2)$ of A is a morphism of B that is continuous relative to τ_1 and τ_2. It is readily checked that A has universal countable coproducts and that the forgetful functor

$$U : (p, \tau) \mapsto p : A \to B$$

is absolutely topological and reflects initial and terminal objects. A source $f_i : (p, \tau) \to (p_i, \tau_i)$ is initial iff τ is the k-ification [1, Prop. 1.2, p. 229] of the topology induced by the f_i's from the τ_i's. That τ is coarser than the sheaf topology on E follows from the fact that E is a k-space (each point of E has a compact, Hausdorff neighborhood, induced from Y by the local homeomorphism p). An epi sink $f_i : (p_i, \tau_i) \to (p, \tau)$ is final iff τ is the topology coinduced by the f_i's from the τ_i's; i.e., $V \in \tau$ iff $f_i^{-1}(V) \in \tau_i$, all i. Clearly τ is a k-topology on E and is coarser than the sheaf topology on E since each $f_i : E_i \to E$ is, relative to the sheaf topology, an open map and $\{f_i\}_i$ is epi. Moreover, since (p, τ) is discrete iff τ is the sheaf topology on E, U is easily seen to be geometric. Thus, by 4.7, A is B-convenient if, for each $(p, \tau) \in A$, the functor $(p, \tau) \times - : A \to A$ (resp. its restriction $B \to A$, where B is identified with the category of discrete objects of A) preserves coequalizers (resp. has a right adjoint). To show this we need:

5.7 LEMMA. *If* $\{f_i : (p_i, \tau_i) \to (p, \tau)\}$ *is a final epi sink in* $A = \mathrm{kTop}(B)$ *then so is* $\{f_i \times 1 : (p_i, \tau_i) \times (q, \sigma) \to (p, \tau) \times (q, \sigma)\}$, *for* $(q : F \to Y, \sigma) \in A$.

Proof. Note that, in general, in A,

$$(p, \tau) \times (q, \sigma) = (p \times_Y q, \tau \times_Y \sigma),$$

where $\tau \times_Y \sigma$ is the topology on $E \times_Y F = \{(e, f) \mid p(e) = q(f)\}$ induced from the product topology $\tau \times \sigma$ (in kTop) on $E \times F$, and that, since p and q are continuous relative to τ and σ, respectively, into the Hausdorff space Y, $(E \times F - E \times {}_Y F) \in \tau \times \sigma$. Since kTop is cartesian closed, it follows that

$$\{f_i \times 1 : (E_i \times E, \tau_i \times \sigma) \to (E \times F, \tau \times \sigma)\}$$

is a final epi sink in kTop. Thus, for $V \subset E \times {}_Y F$, if $(f_i \times {}_Y 1)^{-1}(V) \in \tau_i \times_Y \sigma$, all i, then, since

$$\begin{aligned}(f_i \times 1)^{-1}&(V \cup (E \times F - E \times {}_Y F))\\ &= ((f_i \times {}_Y 1)^{-1}(V) \cup (E_i \times F - E_i \times {}_Y F)) \in \tau_i \times \sigma, \quad \text{all } i,\end{aligned}$$

$V \cup (E \times F - E \times {}_Y F) \in \tau \times \sigma$ and thus $V \in \tau \times_Y \sigma$, i.e., $\{f_i \times {}_Y 1\}$ is a final epi sink in kTop and the result follows.

It follows at once, from 5.7, that coequalizers in A (being final epi sinks) are preserved by $(p, \tau) \times -$. Furthermore, the functor

$$(q, \sigma) \mapsto (q^p)_A : \mathrm{A} \to \mathrm{B},$$

where $(q^p)_A$ is the sheaf of germs of A morphisms $(p, \tau) \to (q, \sigma)$, is right adjoint to $(p, \tau) \times - : B \to A$. To see this, note that since the family $\{s : U \to (q^p)_A\}$, where s varies over all sections of $(q^p)_A$ over all open sets $U \subset Y$, is, with the discrete A-structure, a final epi sink in A, by 5.7,

$$\{s \times 1 : U \times (p, \tau) \to (q^p)_A \times (p, \tau)\}$$

is a final epi sink in A. Since, by definition of $(q^p)_A$, the composition (in B) $e_A(s \times 1)$, where e_A is the restriction to $(q^p)_A \times p$ of the evaluation map $q^p \times p \to q$, is a map $U \times (p, \tau) \to (q, \sigma)$ in A, $e_A : (q^p)_A \times (p, \tau) \to (q, \sigma)$ is a morphism in A that serves as the counit of the desired adjunction. We have therefore:

5.8 THEOREM. *The category of k-spaces over the Grothendieck topos B of sheaves over a locally compact Hausdorff space is B-convenient.*

5.9 *Remark.* If the k-space condition in the definition of kTop(B) is omitted then one obtains the category Top(B) of topological spaces over B, which, by [16], coincides with the category of internal topological spaces in the topos B. Also, if Y is a point then B = Sets and kTop(B) reduces to the category kTop of 5.6.

We end this section with the observation that, while all the categories in 5.4 and 5.6 are convenient for Set-based algebraic topology, some are, as the following lemmas will show, homotopically uninteresting.

By an interval in a convenient category we mean the object $Y_0(1)$ of a simplex structure.

5.10. LEMMA. *If Y is an interval in a cartesian closed category A, where U: $A \to$* Sets *is absolutely topological, then $Y \times Y$ is $\{R, R^{-1}\}$-coherent for some linear order $R \subset UY \times UY$ on UY.*

Proof. This is a consequence of Theorem 4.1 [11] ($R = UY_0(2)$) in case A is kTop, but it is readily seen that the proof of 4.1 [11] (note that T of 4.1 [11] coincides with G of 4.9) extends to any category A as above, so the lemma follows.

An object $Y \in A$, where U: $A \to$ Sets is absolutely topological, is said to be totally disconnected if any two distinct points of Y can be separated, i.e., if $y_1, y_2 \in UY$, $y_1 \neq y_2$, then $Y \approx Y_1 \amalg Y_2$, where $y_i \in UY_i$, $i = 1, 2$.

5.11. LEMMA. (a) *All objects of* Born *are totally disconnected.*

(b) *For $Y(*) \in$* Con *or* ConsFCO, *if $Y(*) \times Y(*)$ is $\{R, R^{-1}\}$-coherent, where $R \subset Y \times Y$ is a linear order on Y, then $Y(*)$ is discrete and, consequently, totally disconnected.*

Proof. (a) As already noted, $(Y, \mathscr{B}) \in$ Born is α-coherent iff each $B \in \mathscr{B}$ lies in the union of a finite number of elements of α. The result now follows since each finite partition of Y induces a coproduct decomposition of $(Y, \mathscr{B})$.

(b) Note that, for $a \in Y$, $F \in Y(a)$, the filter

$$G = \{B \mid A \times A \subset B \subset Y \times Y, \text{ some } A \in F\}$$

is, by definition, in $(Y(*) \times Y(*))(a, a)$, since $[p_iG] = F$, where p_i: $Y^2 \to Y$ is the i^{th} projection, $i = 1, 2$. Hence, by 5.2 (1), (2), if $Y(*) \times Y(*)$ is $\{R, R^{-1}\}$-coherent then G contains either R or R^{-1}, i.e., there is an $A \in F$ with $A \times A \subset R$ or R^{-1}. In either case, since R is antisymmetric, if $\phi \notin F$ then $A = \{b\}$ for some $b \in Y$, and consequently $F = [b]$. Thus, in the ConsFCO case, $Y(*)$ is discrete. The Con case follows from the further observation that, since $[a] \cap F = [a] \cap [b] \in Y(a)$ and thus, as above, $[a] \cap [b] = [c]$ for some $c \in Y$, $a = b = c$ and thus $F = [a]$.

As in the classical kTop case (§3 [11]), each interval Y in a convenient category A defines a notion of $(Y-)$homotopy in A. The category A is said to be homotopically trivial if, for each nonterminal interval Y in A, every pair of parallel maps in A are Y-homotopic. From 5.10 and 5.11, we have the following result:

5.12. THEOREM. *The categories* Born, Con, *and* ConsFCO $\simeq$ Grill $\simeq$ Fil *are homotopically trivial.*

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