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TENSOR INDUCTION OF GENERALIZED CHARACTERS AND PERMUTATION CHARACTERS

BY

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1. Introduction

Let H be a subgroup of a finite group G and let F be any field. There is a procedure called "tensor induction" which one can apply to an arbitrary FH-module W to obtain an FG-module $W^{\otimes G}$ of dimension equal to (dim $W)^{|G:H|}$. Roughly speaking, $W^{\otimes G}$ is the tensor product of $|G:H|$ copies of W, and the action of G permutes the factors in much the same way that it permutes the direct summands which make up the ordinary induced module W^G.

The construction of the tensor-induced module was introduced by Dade (Section 9 of [2]) and independently by Dress [3]. Tensor induction has been used by Dade, T. R. Berger, R. Knörr and others to study certain problems in representation theory. Expositions of tensor induction, including the details of the construction, can be found in [1] and [5].

Let us now limit attention to the case where $F = \mathbf{C}$, the complex numbers. If W affords the character $\theta \in \text{char}(H)$, it is not hard to compute the character afforded by $W^{\otimes G}$, which we denote by $\theta^{\otimes G}$. The formula thus obtained expresses the values of $\theta^{\otimes G}$ in terms of the values of θ, and this formula can be used to define $\varphi^{\otimes G}$ for any class function φ of H. What results is a class function of G which is not, of course, in general, a character.

THEOREM A. *Let $H \subseteq G$ and let φ be a class function of H.*

(a) *If φ is a character, then so is $\varphi^{\otimes G}$.*
(b) *If φ is a permutation character, then so is $\varphi^{\otimes G}$.*
(c) *If φ is a generalized character, then so is $\varphi^{\otimes G}$.*
(d) *If φ is a generalized permutation character, then so is $\varphi^{\otimes G}$.*

Part (a), of course, is no surprise since if φ is afforded by W, then $\varphi^{\otimes G}$ is afforded by $W^{\otimes G}$. Part (b) also is fairly routine, since if W is a permutation module, it is not hard to see that $W^{\otimes G}$ is also.

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By a "generalized character", we mean an element of $\mathbf{Z}[\mathrm{Irr}(G)]$, a $\mathbf{Z}$-linear combination of irreducible characters, or equivalently, a difference of characters. Since the tensor induction map is decidedly nonlinear, it is perhaps a surprise that part (c) of Theorem A, which is due to Knörr [6], should be true. Knörr's proof uses Brauer's characterization of characters to reduce the problem to the case where G is nilpotent and H is a maximal (hence normal) subgroup.

A "generalized permutation character" is a difference of permutation characters. Again, because of the nonlinearity of tensor induction, part (d) of the theorem might be somewhat unexpected. The authors are unaware of a reference to this fact in the literature.

The purpose of this paper is to give short and easy proofs of (c) and (d), assuming (a) and (b).

2. Definition of $\varphi^{\otimes G}$

Let $H \subseteq G$ and choose a right transversal T for H in G. Since G acts on the set of right cosets of H by right translation, this induces an action on T. We therefore write $t \cdot g \in T$ to be the representative of the coset Htg, so that $tg(t \cdot g)^{-1} \in H$.

Fix $g \in G$ and let $n(t)$ denote the size of the $\langle g \rangle$-orbit on T which contains t. Thus, by the same calculation as is used when developing the transfer map, we have

$$tg^{n(t)}t^{-1} \in H \quad \text{for } t \in T.$$

Furthermore, up to conjugacy in H, this element is independent of the choice of t in its $\langle g \rangle$-orbit. Let T_0 be a set of representatives for the $\langle g \rangle$-orbits on T.

DEFINITION 2.1. Let φ be a class function of H. The function $\varphi^{\otimes G}$ on G is given by the formula

$$\varphi^{\otimes G}(g) = \prod_{t \in T_0} \varphi(tg^{n(t)}t^{-1}).$$

It is clear from the fact that φ is a class function of H, that the value of $\varphi^{\otimes G}(g)$ given in Definition 2.1 is independent of the choice of T_0. It is also routine to check that it is independent of the choice of the transversal T, so that $\varphi^{\otimes G}$ is a well defined function on G. Using once again the fact that φ is a class function, it is easy to show that $\varphi^{\otimes G}$ is a class function of G.

It is somewhat more difficult to show, though we will not give the proof here, that if φ is the character of H afforded by some $\mathbf{C}H$-module W, then $\varphi^{\otimes G}$ is the character afforded by $W^{\otimes G}$. (This computation was first done by Dade and can be found in his unpublished preprint [2] where it appears as 9.20.)

Now suppose φ is the permutation character of the action of H on some set Ω. (We write α^h to denote the image of $\alpha \in \Omega$ under $h \in H$.) Let T be a right transversal for H in G as before, and let $\mathcal{F}$ be the set of all functions $T \to \Omega$. We describe an action of G on $\mathcal{F}$ as follows. If $f \in \mathcal{F}$ and $g \in G$, then f^g is defined by

$$(t)(f^g) = [(t \cdot g^{-1})f]^h \quad \text{for } t \in T$$

where $h = (t \cdot g^{-1})gt^{-1} \in H$. It is routine to check that this does define a permutation action of G on $\mathcal{F}$ and that the associated permutation character is $\varphi^{\otimes G}$.

In view of the above discussion, we shall assume parts (a) and (b) of Theorem A as known.

3. Generalized Characters

Let R denote the ring of all algebraic integers in $\mathbf{C}$. Note that if φ is an R-valued class function of $H \subseteq G$, then $\varphi^{\otimes G}$ also has values in R. Suppose $I \subseteq R$ is an ideal and that θ is another R-valued class function of H and that $\varphi(h) \equiv \theta(h) \bmod I$ for all $h \in H$. It is then clear that $\varphi^{\otimes G}(g) \equiv \theta^{\otimes G}(g)$ for all $g \in G$.

LEMMA 3.1. *Let $H \subseteq G$ and let $\varphi \in \mathbf{Z}[\mathrm{Irr}(H)]$. Then $\varphi^{\otimes G} \in R[\mathrm{Irr}(G)]$.*

Proof. Write $\varphi = \alpha - \beta$ where $\alpha, \beta \in \mathrm{Char}(H)$ and let $\theta = \alpha + (|G| - 1)\beta$ so that $\theta \in \mathrm{Char}(H)$. Then $\varphi(h) \equiv \theta(h) \bmod |G|R$ and so

$$\varphi^{\otimes G}(g) \equiv \theta^{\otimes G}(g) \mod |G|R.$$

It follows that

$$[\varphi^{\otimes G} - \theta^{\otimes G}, \chi] \in R$$

for all $\chi \in \mathrm{Irr}(G)$, since χ has values in R. Thus $\varphi^{\otimes G} - \theta^{\otimes G} \in R[\mathrm{Irr}(G)]$, and since $\theta^{\otimes G} \in \mathrm{Char}(G)$, the result follows. ■

We remark, in connection with Lemma 3.1, that it is not sufficient to assume that $\varphi \in R[\mathrm{Irr}(H)]$ in order to conclude that $\varphi^{\otimes G} \in R[\mathrm{Irr}(G)]$. In fact, taking $|G| = 2$ and H the identity subgroup, already yields a counterexample: If $a \in R$, one easily computes that

$$[(a1_H)^{\otimes G}, 1_G] = \frac{a + a^2}{2}$$

and this is not always integral.

To complete the proof of part (c) of Theorem A, it suffices to show that

$$\varphi^{\otimes G} \in \mathbf{Q}[\mathrm{Irr}(G)] \quad \text{when } \varphi \in \mathbf{Q}[\mathrm{Irr}(H)].$$

Since $\mathbf{Q} \cap R = \mathbf{Z}$, this will show that $\varphi^{\otimes G} \in \mathbf{Z}[\mathrm{Irr}(G)]$ if $\varphi \in \mathbf{Z}[\mathrm{Irr}(H)]$.

To this end we give a necessary and sufficient condition for a class function of G to lie in $\mathbf{Q}[\mathrm{Irr}(G)]$.

Let ε be a primitive $|G|$-th root of unity. For each $\sigma \in \mathrm{Aut}(\mathbf{C})$, we have that $\varepsilon^{\sigma} = \varepsilon^{m(\sigma)}$ for some uniquely determined integer $m(\sigma)$, with $1 \leqslant m(\sigma) \leqslant |G|$.

LEMMA 3.2. *Let θ be a class function of G. Then $\theta \in \mathbf{Q}[\mathrm{Irr}(G)]$ iff $\theta(g)^{\sigma} = \theta(g^{m(\sigma)})$ for all $g \in G$ and $\sigma \in \mathrm{Aut}(\mathbf{C})$.*

Proof. It is well known that $\chi(g)^{\sigma} = \chi(g^{m(\sigma)})$ for $\chi \in \mathrm{Irr}(G)$ and therefore $\theta(g)^{\sigma} = \theta(g^{m(\sigma)})$ if θ is a rational linear combination of $\chi \in \mathrm{Irr}(G)$.

Conversely, let θ be a class function which satisfies the condition and write $\theta = \Sigma a_{\chi}\chi$ for some uniquely determined complex numbers a_{χ}. Then for $\sigma \in \mathrm{Aut}(\mathbf{C})$ and $g \in G$ we have

$$\sum a_{\chi}^{\sigma}\chi(g)^{\sigma} = \theta(g)^{\sigma} = \theta(g^{m(\sigma)}) = \sum a_{\chi}\chi(g^{m(\sigma)}) = \sum a_{\chi}\chi(g)^{\sigma}.$$

From the uniqueness of the a_{χ}, we conclude that $a_{\chi} = a_{\chi}^{\sigma}$ for all $\sigma \in \mathrm{Aut}(\mathbf{C})$, and thus all of the a_{χ} lie in $\mathbf{Q}$. ∎

Proof of Theorem A, *part* (c). As was remarked above, it suffices to show that

$$\varphi^{\otimes G} \in \mathbf{Q}[\mathrm{Irr}(G)] \quad \text{if } \varphi \in \mathbf{Q}[\mathrm{Irr}(H)],$$

and so we check the condition of Lemma 3.2. Let $\sigma \in \mathrm{Aut}(\mathbf{C})$ and write $m = m(\sigma)$. Let $g \in G$. Then

$$\varphi^{\otimes G}(g)^{\sigma} = \prod_{t \in T_0} \varphi(tg^{n(t)}t^{-1})^{\sigma} = \prod_{t \in T_0} \varphi(tg^{mn(t)}t^{-1}).$$

However, $(m, |G|) = 1$ and so $\langle g \rangle = \langle g^m \rangle$ and the orbit decomposition of T corresponding to g is the same as that for g^m. Therefore, we have the same exponents $n(t)$ for g as for g^m, and the same set T_0 works for both elements. It follows that

$$\varphi^{\otimes G}(g)^{\sigma} = \varphi^{\otimes G}(g^m)$$

and the proof is complete. ∎

4. Generalized Permutation Characters

Let $P(G)$ denote the ring of generalized permutation characters.

LEMMA 4.1. *Let θ be a generalized character with rational values divisible by $|G|^2$. Then $\theta \in P(G)$.*

Proof. We have $|G| \mid [\theta, \chi]$ for all $\chi \in \mathrm{Irr}(G)$ since χ has values in R. Thus $(1/|G|)\theta$ is a generalized character and we can write it in the form

$\alpha - \beta$ with $\alpha, \beta \in \mathrm{Char}(G)$ and $[\alpha, \beta] = 0$. Since α and β are uniquely determined by the rational character θ, they are fixed by all field automorphisms and so are themselves rational valued.

By a theorem of Artin (Theorem 5.21 of [4]), $|G|\alpha$ and $|G|\beta$ lie in $P(G)$ and hence $\theta = |G|(\alpha - \beta) \in P(G)$. ■

Proof of Theorem A, *part* (d). We have $\varphi \in P(H)$ and so we can write $\varphi = \alpha - \beta$ where α and β are permutation characters of H. Let $\theta = \alpha + (|G|^2 - 1)\beta$. Then θ is a permutation character and $\theta(h) \equiv \varphi(h) \bmod |G|^2$. Thus $\theta^{\otimes G}$ is a permutation character of G and $\varphi^{\otimes G}$ is at least a generalized character of G which is rational valued.

We have $\theta^{\otimes G}(g) \equiv \varphi^{\otimes G}(g) \bmod |G|^2$ and thus Lemma 4.1 applies to $\varphi^{\otimes G} - \theta^{\otimes G}$ which therefore lies in $P(G)$. Since $\theta^{\otimes G} \in P(G)$, the result follows. ■

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