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MULTI-DIMENSIONAL VOLUMES, SUPER-REFLEXIVITY AND NORMAL STRUCTURE IN BANACH SPACES[1]

BY

JAVIER BERNAL AND FRANCIS SULLIVAN

1. Introduction

The notion of the n-dimensional volume enclosed by $n + 1$ vectors in a Banach space, E, was introduced by Silverman [12], and some of the connections between higher dimensional volumes and geometric properties of E were studied in [13] and [6]. In particular, it was shown that a k-uniformly rotund Banach space is super-reflexive and has normal structure. (Definitions are given below.) The present paper is a more detailed study of the relationships between enclosed volumes, super-reflexivity and normal structure of Banach spaces.

James proved in [9] that if E is not super-reflexive, then for every $\delta > 0$ there are $\{x_1, x_2\} \in B$, the unit ball of E, such that

$$\left\|\frac{x_1 + x_2}{2}\right\| \geqslant 1 - \delta$$

while $A(x_1, x_2) = \|x_1 - x_2\| \geqslant 2 - \delta$. A consequence of Theorem 3.1 of [6] is that if E is not super-reflexive, then for every integer $k > 0$ there are vectors $\{x_1, x_2, \ldots, x_k\} \subset B$ such that

$$\left\|\frac{x_1 + x_2 + \cdots + x_k}{k}\right\| \geqslant 1 - \delta$$

with $A(x_1, x_2, \ldots, x_k) > 0$. In section 3 we generalize these results and show that if E is not super-reflexive, then, for every integer $n > 0$, there are vectors $\{x_1, x_2, \ldots, x_{n+1}\} \subset B$ such that

$$\left\|\frac{x_1 + \cdots + x_{n+1}}{n + 1}\right\| \geqslant 1 - \delta$$

while $A(x_1, x_2, \ldots, x_{n+1}) \geqslant 2^n - \delta$. This should be contrasted with the situation for l_2, where $A(x_1, x_2, \ldots, x_{n+1}) \geqslant \varepsilon$ implies that

$$\left\|\frac{x_1 + \cdots + x_{n+1}}{n + 1}\right\| \leqslant \left[1 - \frac{n}{n + 1}\left(\frac{\varepsilon^{2/n}}{(n + 1)^{1/n}}\right)\right]^{1/2}.$$

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[1] Some of the results of this paper are contained in the Ph. D. dissertation of the first author.

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(The preceding inequality is proved using the method of [6], Theorem 1.3.) Notice that this says that l_2 has a very strong "ergodic" property with respect to volumes: If (x_k) is any norm-1 sequence then, by passing to a subsequence, we may assume that either $\lim_n A(x_1, x_2, \ldots, x_{n+1}) = 0$ or that

$$\lim_n \left\| \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1} \right\| = 0.$$

A non super-reflexive space is at the other extreme; there are norm-1 vectors enclosing large volumes and having average norms close to 1. Also, Dixmier's Theorem which guarantees the existence of a line segment on the surface of the unit ball of a non-reflexive 4-th dual space generalizes. If E is not reflexive, then there is a non-trivial n-dimensional simplex on the surface of the ball of the $(2n + 2)$-th dual E [6], [7].

The result of James says that every non super-reflexive Banach space contains subspaces arbitrarily close to $l_1^{(2)}$. James also proved [10] that an irreflexive space need *not* contain subspaces close to $l_1^{(3)}$. Davis, Johnson and Lindenstrauss [3] and Davis and Lindenstrauss [4] have investigated the question of the "degree" of non-reflexivity of a space, E, and containment of subspaces close to $l_1^{(n)}$. Using their methods, we give a sufficient condition for a Banach space to contain n-tuples of norm-1 vectors which behave somewhat like the unit vector basis of $l_1^{(n)}$, i.e. have average close to 1 in norm and enclose a volume close to $n^{n/2}$. Combining this with a result of Bellenot [1] we get a sufficient condition, in terms of the four dimensional subspaces of E, for E to contain uniformly complemented $l_p^{(n)}$'s.

The occurrence of the value $n^{n/2}$ is related to Hadamard's inequality. This is discussed in Section 2, where we give upper and lower bounds on the enclosed volume in terms of the distances between the vectors. These inequalities are used throughout the rest of the paper.

In Section 4 we take up the subject of normal structure. Loosely described, this is a geometrical property which guarantees that every weakly compact convex set $K \subset E$ has a "center of mass" and, consequently, every non-expansive map from K to K has a fixed point [11]. We give a general sufficient condition in terms of average norms and enclosed volumes for E to be super reflexive and have normal structure. We also comment on the relation between our condition and properties implying normal structure which have been investigated by van Dulst [15] and Huff [8].

We assume that the reader is familiar with the usual notions of Banach space theory. Notice that for $x, y \in E$ we have that

$$\|x - y\| = \sup \left\{ \begin{vmatrix} 1 & 1 \\ \langle g, x\rangle & \langle g, y\rangle \end{vmatrix} : g \in B^* \right\}.$$

Here, and throughout the sequel, the symbol $|\cdot|$ denotes the determinant. Generalizing this, we define the 2-dimensional "area" enclosed by vectors

$\{x, y, z\}$ as

$$A(x, y, z) \equiv \sup \left\{ \begin{vmatrix} 1 & 1 & 1 \\ \langle f, x\rangle & \langle f, y\rangle & \langle f, z\rangle \\ \langle g, x\rangle & \langle g, y\rangle & \langle g, z\rangle \end{vmatrix} : f, g \in B^* \right\}.$$

This idea is taken from the work of E. Silverman [12]. The n-dimensional volume enclosed by vectors $\{x_1, x_2, \ldots, x_{n+1}\}$ is defined in the obvious way, and is denoted by $A(x_1, x_2, \ldots, x_{n+1})$.

The authors should like to thank Professor Dick van Dulst for pointing out a defect in our initial attempt to prove Theorem 4.1, and for telling us how to use Ramsey's Theorem to repair it. The second author would also like to thank the Mathematical Institute of the Catholic University of Nijmegen, The Netherlands for the truly gracious hospitality they have shown him during the writing of this paper.

2. Bounds on the Volume

Recall that if $r_1, r_2, \ldots, r_k$ are the rows or columns of a $k \times k$ matrix, then from the Hadamard inequality we have

$$\det(r_1, r_2, \ldots, r_k) \leq \|r_1\|_2 \, \|r_2\|_2 \cdots \|r_k\|_2.$$

Here $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbf{R}^k$. A consequence is that for x_1, $x_2, \ldots, x_k$ norm-1 vectors in a Banach space E,

$$A(x_1, x_2, \ldots, x_k) \leq k^{k/2}.$$

For certain values of k there are matrices (having pairwise orthogonal rows) which actually give equality in the Hadamard inequality. An easily understood class of examples can be generated in the following way: Let

$$H_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and note that $\det(H_0) = 2$. Then, for each $k > 0$, let $H_k = H_0 \otimes H_{k-1}$, where "$\otimes$" denotes the tensor product of matrices. Using the fact that for A an $n \times n$ matrix and B an $m \times m$ matrix, $\det(A \otimes B) = \det(A)^m \cdot \det(B)^n$, it is not hard to see that, for $n = 2^{k+1}$, H_k is an $n \times n$ matrix with $\det(H_k) = n^{n/2}$. This shows that if $\{e_1, e_2, \ldots, e_n\}$ are the usual basis vectors in $l_1^{(n)}$, then $A(e_1, e_2, \ldots, e_n) = n^{n/2}$, the maximum possible value.

If $\{y, x_1, x_2, \ldots, x_n\}$ are vectors in an arbitrary E, then let

$$\operatorname{dist}(y, [x_1, x_2, \ldots, x_n])$$

denote the distance from y to $[x_1, x_2, \ldots, x_n]$, the affine span of $\{x_1, x_2, \ldots, x_n\}$.

THEOREM 1. *Suppose that* $\{x_1, x_2, \ldots, x_{n+1}\}$ *are norm*-1 *vectors in a*

Banach space, E. Let

$d_1 = \operatorname{dist}(x_1, [x_2, \ldots, x_{n+1}]), d_2 = \operatorname{dist}(x_2, [x_3, \ldots, x_{n+1}]), \ldots, d_n = \|x_n - x_{n+1}\|.$

Then

$$d_1 \cdot d_2 \cdots d_n \leqslant A(x_1, x_2, \ldots, x_{n+1}) \leqslant n^{n/2} d_1 \cdot d_2 \cdots d_n.$$

Proof. The lower bound was proved in [6]. For the other inequality, let

$$y_i \in [x_{i+1}, x_{i+2}, \ldots, x_{n+1}]$$

be a vector such that $\|\bar{x}_i\| = \|x_i - y_i\| = d_i$. Since the determinant is a multi-linear form, we have

$$A(x_1, x_2, x_3, \ldots, x_n, x_{n+1}) = \sup \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ \langle f_1 \bar{x}_1\rangle & \langle f_1, \bar{x}_2\rangle & \cdots & \langle f_1, \bar{x}_n\rangle & \langle f_1, x_{n+1}\rangle \\ \langle f_2, \bar{x}_1\rangle & \langle f_2, \bar{x}_2\rangle & \cdots & \langle f_2, \bar{x}_n\rangle & \langle f_2, x_{n+1}\rangle \\ \vdots & \vdots & & & \vdots \\ \langle f_n, \bar{x}_1\rangle & \langle f_n, \bar{x}_2\rangle & \cdots & \langle f_n, \bar{x}_n\rangle & \langle f_n, x_{n+1}\rangle \end{vmatrix}$$

For each choice of norm-1 functionals $\{f_1, f_2, \ldots, f_n\}$ we can define $l_2^{(n)}$ vectors

$$r_i = (\langle f_1, \bar{x}_i\rangle, \langle f_2, \bar{x}_i\rangle, \ldots, \langle f_n, \bar{x}_i\rangle)$$

and expanding in minors and using the Hadamard inequality we get

$$\begin{aligned} A(x_1, x_2, \ldots, x_n, x_{n+1}) &\leqslant \sup\{\|r_1\|_2 \|r_2\|_2 \cdots \|r_n\|_2 : f_1, f_2, \ldots, f_n \in B^*\} \\ &\leqslant (\|f_1\|^2 + \|f_2\|^2 + \cdots + \|f_n\|^2)^{n/2} d_1 d_2 \cdots d_n \\ &\leqslant n^{n/2} d_1 d_2 \cdots d_n. \end{aligned}$$

Q.E.D.

A similar technique can be used to show that, for $\{x_1, x_2, \ldots, x_{n+1}\}$ norm-1 vectors in a Hilbert space, it is always the case that

$$d_1 d_2 \cdots d_n = A(x_1, x_2, \ldots, x_{n+1}).$$

3. Non-reflexive Spaces

The following result of James [9] is the fundamental tool for this section:

THEOREM 1. *Suppose that E is not reflexive and that* $0 < \theta < 1$. *Then there are sequences* $(z_k) \subset B^*$ *such that*

$$\langle f_j, z_k\rangle = \begin{cases} \theta & \text{for } j \leqslant k \\ 0 & \text{for } j > k. \end{cases}$$

Using this, James proves that if E is not reflexive and $\delta > 0$ is arbitrary, then there are vectors $x_1, x_2 \in B$ such that $\|x_1 + x_2\| \geqslant 2 - \delta$ and $\|x_1 - x_2\|$

$\geqslant 2 - \delta$. In other words, a non-reflexive E always contains subspaces arbitrarily close to $l_1^{(2)}$. It is clear that the same is true for any space which is not super-reflexive. The converse is also true [5]; namely, if E is super-reflexive, then E has an equivalent norm which is uniformly non-square, i.e., a norm such that for some fixed $\delta > 0$ and all $x_1, x_2 \in B$,

$$\min\{\|x_1 + x_2\|, \|x_1 - x_2\|\} \leqslant 2 - \delta.$$

Using the methods of James we can give a more general result in terms of volumes.

THEOREM 2. *If E is not super-reflexive then, for all $\delta > 0$ and all n, there are*

$$\{x_1, x_2, \ldots, x_{n+1}\} \subset B$$

such that

$$\|x_1 + x_2 + \ldots + x_{n+1}\| \geqslant (n + 1) - \delta \quad \textit{and} \quad A(x_1, x_2, \ldots, x_{n+1}) \geqslant 2^n - \delta.$$

Proof. We shall show that for every $\eta > 0$ and every integer, m, there are

$$\{x_1, x_2, \ldots, x_m\} \subset B$$

such that

$$\operatorname{dist}(x_l, [x_{l+1}, \ldots, x_m]) > 2 - \eta$$

for all $1 \leqslant l < m$, and $\|x_1 + x_2 + \cdots + x_m\| > m - \eta$. This, combined with Theorem 2.1, gives the result.

Following James, we use the sequences (f_j) and (z_k) given above to define, for every choice of integers $p_1 < p_2 < \cdots < p_{2n}$, a set

$$S = S(p_1, \ldots, p_{2n}) \equiv \{x : \langle f_j, x\rangle = (-1)^i\theta \text{ for } p_{2i-1} \leqslant j \leqslant p_{2i}\}$$

and a number

$$k_n \equiv \liminf_{p_1\to\infty} (\liminf_{p_2\to\infty} \ldots (\inf_{p_{2n}\to\infty} (\inf \|x\| : x \in S)) \ldots)).$$

It can be shown that, for all n, $k_{n+1} \geqslant k_n$ and $k_n \leqslant 2n$, while for $\eta > 0$ there exist n large enough and $\varepsilon > 0$ small enough so that

$$\frac{k_{n-1} - \varepsilon}{k_n + \varepsilon} > 1 - \eta \quad \text{and} \quad \frac{k_n - \varepsilon}{k_n + \varepsilon} > 1 - \eta.$$

For this n, there obviously is a $p > 0$ such that if $p \leqslant p_1 < p_2 < \cdots < p_{2n}$ and $z \in S(p_1, \ldots, p_{2n})$, then $\|z\| > k_n - \varepsilon$

We now choose sets of integers

$$\{p \leqslant p_1^{(l)} < p_2^{(l)} < \cdots < p_{2n}^{(l)}\}$$

and vectors

$$u^l \in S(p_1^{(l)}, p_2^{(l)}, \ldots, p_{2n}^{(l)})$$

for $1 \leqslant l \leqslant m$, so that $\frac{1}{2}(\Sigma_{k=1}^{m} \lambda_k u_k - u_l) \in S_{l+1}$, if $\Sigma_{l=1}^{m} \lambda_k = 1$ and $1 \leqslant l \leqslant m - 1$. The sets S_{l+1} are defined below. The required vectors are then given by

$$x_l = \frac{u_l}{k_n + \varepsilon} \quad \text{for all } 1 \leqslant l \leqslant m.$$

The sets S_{l+1} are as follows:

$$S_1 = S(p_1^{(m)}, p_2^{(1)}, p_3^{(m)}, p_4^{(1)}, \ldots, p_{2n-1}^{(m)}, p_{2n}^{(1)})$$

$$S_2 = S(p_3^{(1)}, p_2^{(2)}, p_5^{(1)}, p_4^{(2)}, \ldots, p_{2n-1}^{(1)}, p_{2n-2}^{(2)})$$

$$S_3 = S(p_3^{(2)}, p_2^{(3)}, p_5^{(2)}, p_4^{(3)}, \ldots, p_{2n-1}^{(2)}, p_{2n-2}^{(3)})$$

$$S_4 = S(p_3^{(3)}, p_2^{(4)}, p_5^{(3)}, p_4^{(4)}, \ldots, p_{2n-1}^{(3)}, p_{2n-2}^{(4)})$$

$$\vdots$$

$$S_{m-1} = S(p_3^{(m-2)}, p_2^{(m-1)}, p_5^{(m-2)}, p_4^{(m-1)}, \ldots, p_{2n-1}^{(m-2)}, p_{2n-2}^{(m-1)}$$

$$S_m = S(p_3^{(m-1)}, p_2^{(m)}, p_5^{(p-1)}, p_4^{(m)}, \ldots, p_{2n-1}^{(m-1)}, p_{2n-2}^{(m)}),$$

where

$$\begin{aligned} p \leqslant\ & [p_1^{(1)} < p_1^{(2)} < p_1^{(3)} < p_1^{(4)} < \cdots < p_1^{(m-1)} < p_1^{(m)}] \\ & < [p_2^{(1)} < p_3^{(1)} < p_2^{(2)} < p_3^{(2)} < \cdots < p_2^{(m-1)} < p_3^{(m-1)} < p_2^{(m)} < p_3^{(m)}] \\ & < [p_4^{(1)} < p_5^{(1)} < p_4^{(2)} < p_5^{(2)} < \cdots < p_4^{(m-1)} < p_5^{(m-1)} < p_4^{(m)} < p_5^{(m)}] \\ & < [p_6^{(1)} < p_7^{(1)} < p_6^{(2)} < p_7^{(2)} < \cdots < p_6^{(m-1)} < p_7^{(m-1)} < p_6^{(m)} < p_7^{(m)}] \\ & < [p_8^{(1)} < p_9^{(1)} < p_8^{(2)} < p_9^{(2)} < \cdots \\ & \qquad \cdots \\ & \qquad \cdots < p_{2n-4}^{(m)} < p_{2n-3}^{(m)}] \\ & < [p_{2n-2}^{(1)} < p_{2n-1}^{(1)} < p_{2n-2}^{(2)} < p_{2n-1}^{(2)} < \cdots < p_{2n-2}^{(m-1)} < p_{2n-1}^{(m-1)} < p_{2n-2}^{(m)} < p_{2n-1}^{(m)}] \\ & < [p_{2n}^{(1)} < p_{2n}^{(2)} < p_{2n}^{(3)} < \cdots < p_{2n}^{(m-1)} < p_{2n}^{(m)}]. \end{aligned}$$

Q.E.D.

James's technique has been generalized in another way by Davis, Johnson and Lindenstrauss [3]. A space, E, is said to have a local k-structure if there is a constant, M, so that for each integer, n, there are n^k elements,

$$\{x_{i_1, i_2, \ldots, i_k}\}, \{f_{j_1, j_2, \ldots, j_k}\}, \quad 1 \leqslant i_1, i_2, \ldots, i_k \leqslant n, 1 \leqslant j_1, j_2, \ldots, j_k \leqslant n$$

such that

$$\|x_{i_1,i_2,\ldots,i_k}\| \leq M, \quad \|f_{j_1,j_2,\ldots,j_k}\| \leq M$$

and

$$\langle f_{j_1,j_2,\ldots,j_k}, x_{i_1,i_2,\ldots,i_k}\rangle = \begin{cases} 1 & \text{if all } j_p \leq i_p \text{ for } p = 1, \ldots, k \\ 0 & \text{otherwise.} \end{cases}$$

Define $R(E) \equiv E^{**}/E$ and $R^k(E) \equiv R(R^{k-1}(E))$. In [3] Davis, Johnson and Lindenstrauss prove:

THEOREM 3. *If $R(E)$ admits a local k-structure then E admits a local $(k + 1)$-structure.*

Because of James's Theorem, a non-super reflexive space is one which admits a local 1-structure. Thus, if for some k, $R^k(E)$ is not super-reflexive, then E admits a local $(k + 1)$-structure.

Another result of [3] is that if E admits a local k-structure then E contains subspaces arbitrary close to $l_1^{(k+1)}$. Hence, in particular, if $R(E)$ is not super-reflexive, E contains subspaces arbitrarily close to $l_1^{(3)}$. James gave an example of a space, F, which is not reflexive but has no subspace close to $l_1^{(3)}$. The preceding says that F^{**}/F is super-reflexive.

There is some evidence that a space with $R^{k-1}(E)$ not super-reflexive in fact contains subspaces arbitrarily close to $l_1^{(n)}$ for $n = 2^k$. In [6] it was proved that if $R(E)$ contains norm-1 vectors with

$$\left\|\frac{\hat{x}_1 + \cdots + \hat{x}_n}{n}\right\|$$

close to one while $A(\hat{x}_1, \ldots, \hat{x}_n) > a > 0$, then E contains norm-1 vectors with

$$\left\|\frac{x_1 + \cdots + x_{2n}}{2n}\right\|$$

close to 1 while $A(x_1, \ldots, x_{2n}) > a^2$. In [4] Davis and Lindenstrauss show that a space which admits a local 2-structure contains subspaces arbitrarily close to $l_1^{(4)}$. This fact follows from the following result of theirs:

THEOREM 4. *Suppose that E admits a local k-structure, m is a positive integer and $\varepsilon > 0$ is arbitrary. Then there are m^k norm-1 vectors $\{x_{i_1 i_2 \cdots i_k}\}$ where $1 \leq i_1, i_2, \ldots, i_k \leq m$ such that for every choice of indices $1 \leq r_1, r_2, \ldots, r_k \leq m$,*

$$\left\|\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\theta^{r_1}(i_1)\theta^{r_2}(i_2)\cdots\theta^{r_k}(i_k)x_{i_1 i_2\cdots i_k}\right\| \geq m^k - \varepsilon.$$

Here, for all j, $\theta^{r_j}(i_j) = 1$ *if* $i_j \leqslant r_j$ *and* $\theta^{r_j}(i_j) = -1$ *if* $i_j > r_j$.

LEMMA 5. *Suppose that E admits a k-structure and that* $n = 2^k$. *Then, for every* $\varepsilon > 0$, *there are norm-*1 *vectors* $\{x_1, x_2, \ldots, x_n\}$ *such that*

$$\|x_1 + x_2 + \cdots + x_n\| \geqslant n - \varepsilon \quad \text{and} \quad A(x_1, x_2, \ldots, x_n) \geqslant n^{n/2} - \varepsilon$$

Proof. We take $m = 2$ as in Theorem 4, so that we get $n = 2^k = m^k$ vectors $\{x_{i_1 i_2 \cdots i_k}\}$.

It is clear that taking all $r_j = 2$ gives all plus signs in the sums, so that

$$\|x_1 + x_2 + \cdots + x_n\| \geqslant n - \varepsilon.$$

To complete the proof, we verify that there are choices for the values of the k tuples $(r_1, r_2, \ldots, r_k)$ so that all of the sign combinations appearing in the rows of the matrix H_{k-1}, discussed in Section 2, can be obtained. For each of the 2^k sign combinations there must be a norm-1 linear functional which, when evaluated on the corresponding sum, give a value close to n. These can be used to give a value close to $n^{n/2}$ for $A(x_1, x_2, \ldots, x_n)$.

Because of the inductive definition of the H_k's, it is not hard to see how to specify the r_j's. We simply note that

$$H_0 = \begin{bmatrix} \theta^2(2) & \theta^2(1) \\ \theta^1(2) & \theta^1(1) \end{bmatrix}$$

and then, that for $k \geqslant 1$,

$$H_k = H_0 \otimes H_{k-1} = \begin{bmatrix} \theta^2(2)H_{k-1} & \theta^2(1)H_{k-1} \\ \theta^1(2)H_{k-1} & \theta^1(1)H_{k-1} \end{bmatrix}. \qquad \text{Q.E.D.}$$

In [1] Bellenot proved that if (E) is reflexive and RE is not, then E contains uniformly complemented $l_p^{(n)}$'s. We say that E has no large tetrahedra if there is an $\varepsilon > 0$ such that

$$4 - \varepsilon \geqslant \min\left\{\frac{1}{4}A(x_1, x_2, x_3, x_4), \|x_1 + x_2 + x_3 + x_4\|\right\}.$$

COROLLARY 6. *If E is non-reflexive and has no large tetrahedra, then E contains uniformly complemented* $l_p^{(n)}$*'s.*

4. Normal Structure

Let $C \subset E$ be a closed bounded convex set. We say C has normal structure if, for every non-empty closed convex $H \subset C$, either H is a singleton or else there is an $x \in H$ such that

$$\sup\{\|x - y\| : y \in H\} < \operatorname{diam}(H) \equiv \sup\{\|y_1 - y_2\| : y_1, y_2 \in H\}.$$

We say that the space, E, has normal structure in case every closed bounded convex subset of E has normal structure. It is known [11] that if C is a weakly compact convex set with normal structure and $T : C \to C$ is a map such that for all $x, y \in C$, $\|Tx - Ty\| \leqslant \|x - y\|$, then T has a fixed point in C.

If E is a Banach space lacking normal structure, then there must be a closed bounded convex $K \subset E$ which is "abnormal", i.e., for every $x \in K$,

$$\sup\{\|x - y\| : y \in K\} = \operatorname{diam}(K).$$

In [13] it was proved that an abnormal set cannot exist in a *k-UR* space. A more careful examination of the structure of an abnormal set yields the following:

THEOREM 1. *Suppose that for some $\delta > 0$ and some $0 < \varepsilon < 1$ there is an integer m such that, for all $x_1, x_2, \ldots, x_m \in B$ if*

$$\left\|\frac{x_1 + x_2 + \cdots + x_m}{m}\right\| > 1 - \delta$$

then

$$A(x_1, x_2, \ldots, x_m) < \varepsilon.$$

Then E is super-reflexive and has normal structure.

The proof requires some preliminary observations and results. Notice, first, that the super-reflexivity is immediate from the results of Section 3. We assume that E contains an abnormal closed bounded convex K. It is no loss of generality to assume, further, that $\operatorname{diam}(K) = 1$ and $\operatorname{dist}(0, K) > 0$.

LEMMA 1. *Suppose that (β_k) is a positive sequence decreasing to zero. Then there is a sequence $(x_k) \subset K$ such that*

(i) $\|z - x_{L+i}\| \geqslant 1 - \beta_{L+i} > 1 - \beta_L$, *for all L, all i and all*

$$z \in \operatorname{co}(x_{L+1}, x_{L+2}, \ldots, x_{L+i-1})$$

(ii) $\|z - x_{L+i}\| \geqslant 1 - N(i - 1)\beta_{L+i}$, *for all L, all i and all $z = \sum_{j=1}^{i-1} \lambda_j x_{L+j}$ where $\sum_{j=1}^{i-1} \lambda_j = 1$ and $\lambda_j \leqslant N$ for all positive λ_j.*

Proof. The first statement is a well known property of abnormal sets (see [2] for a proof). For the second statement, write $z = \sum \eta_j x_j - \sum \gamma_j x_j$

where all coefficients are positive and $\Sigma\,\eta_j - \Sigma\,\gamma_j = 1$. Then

$$\begin{aligned} \|z - x_{L+i}\| &= \left\| \sum \eta_j x_j - \left(\sum \eta_j\right) x_{L+i} - \left(\sum \gamma_j x_j - \left(\sum \gamma_j\right) x_{L+i}\right) \right\| \\ &\geqslant \left(\sum \eta_j\right)(1 - \beta_{L+i}) - \sum \gamma_j \\ &= 1 - \left(\sum \eta_j\right)\beta_{L+i} \\ &\geqslant 1 - N(i-1)\beta_{L+i}. \end{aligned}$$

Q.E.D.

The aim, now, is to show that there is a subsequence of the sequence (x_k) contradicting the hypothesis for every m. As we shall see, the main idea is to show that a property similar to (i) in the lemma holds even if

$$\mathrm{co}(x_{L+1}, \ldots, x_{L+m-1})$$

is replaced by

$$[x_{L+1}, \ldots, x_{L+m-1}].$$

By passing to a subsequence, we may assume that for some $x \in K$, $x_i \to x$ weakly. Furthermore, using Ramsey's Theorem as in [16] we may also assume that $\lim_{L\to\infty} \|\Sigma_{i=1}^k a_i x_{L+i}\|$ exists, for all integers k and all real $\{a_1, a_2, \ldots, a_k\}$. The limit is a continuous function on $\mathbf{R}^k$.

Proof of the theorem. Let m be given and for each L define

$$y_1^L = x_{L+1} - x_{L+m+1},\ y_2^L = x_{L+2} - x_{L+m+1}, \ldots, y_m^L = x_{L+m} - x_{L+m+1}.$$

Notice that, for any $\delta > 0$ and for L large enough,

$$\left\| \frac{y_1^L + \cdots + y_m^L}{m} \right\| > 1 - \delta.$$

We get a contradiction by using Theorem 1.1 and showing that, for each $1 \leqslant i < m$,

$$\begin{aligned} &\liminf_{L\to\infty} \mathrm{dist}(y_{L+i+1}, [y_{L+i}, \ldots, y_{L+1}]) \\ (*) \qquad &= \liminf_{L\to\infty} \mathrm{dist}(x_{L+i+1}, [x_{L+i}, \ldots, x_{L+1}]) \\ &= 1. \end{aligned}$$

Let

$$U = \left\{ (a_1, a_2, \ldots, a_i\} \in \mathbf{R}^i : \|(a_1, \ldots, a_i)\|_\infty = 1 \text{ and } \sum_{j=1}^i a_j \geqslant 0 \right\}.$$

We claim that, for all $(a_1, a_2, \ldots, a_i) \in U$, $\lim_L \|\Sigma_{j=1}^i a_j x_{L+j}\| > 0$. If not, then there is an i-tuple $(a_1, a_2, \ldots, a_i) \in U$ with $\lim_L\|\Sigma_{j=1}^i a_j x_{L+j}\| = 0$. If $\Sigma_{j=1}^i a_j > 0$ then, since $x_{K+i} \to x \in K$ and $\text{dist}(0, K) > 0$, we get a contradiction. Hence, we may suppose that $\Sigma_{j=1}^i a_j = 0$ and, without loss of generality, that $a_i \neq 0$.

Notice that

$$\sum_{j=1}^{i-1} \frac{a_j}{-a_i} = 1.$$

Now, since the limit of the norms is zero, for $\beta > 0$ arbitrary and L sufficiently large,

$$\left\| \frac{a_1}{-a_i} x_{L+1} + \cdots + \frac{a_{i-1}}{-a_i} x_{L+i-1} - x_{L+i} \right\| < \frac{\beta}{|a_i|}$$

so that, by Lemma 1 part (ii) and the fact that $\|(a_1, \ldots, a_i)\|_\infty = 1$,

$$\min(1 - \beta_{L+i}, 1 - N(i-1)\beta_{L+i}) < \frac{\beta}{|a_i|} \quad \text{for any fixed } N \geqslant \frac{1}{|a_i|}.$$

Since $\beta_{L+i} \to 0$, we have that $|a_i| < \beta$ which contradicts the fact that $\beta > 0$ was arbitrary. Thus the claim is established.

An immediate consequence of the fact that the limit is continuous on $\mathbf{R}^i$ is that there is an $\varepsilon > 0$ such that for L large enough and $\lambda \equiv (\lambda_1, \ldots, \lambda_i)$,

$$\left\| \sum_{j=1}^{i} \lambda_j x_{L+j} \right\| > t\,\|\lambda\|_\infty \quad \text{for all} \sum_{j=1}^{i} \lambda_j = 1.$$

Let $M = \sup\{\|x\| : x \in K\}$. Clearly, for all L large enough,

$$\left\| \sum_{j=1}^{i} \lambda_j x_{L+j} \right\| > M + 1 \quad \text{if } \|\lambda\|_\infty > \frac{M+1}{t} = N_0.$$

This shows that if $z = \Sigma_{j=1}^i \lambda_j x_{L+j}$ is such that

$$\|z - x_{L+i+1}\| = \text{dist}(z, [x_{L+i}, \ldots, x_{L+1}]),$$

then $\sup|\lambda_j| \leqslant N_0$ because, otherwise,

$$1 \geqslant \|z - x_{L+i+1}\| > M + 1 - M = 1.$$

Hence, using Lemma 1 again,

$$\|z - x_{L+i+1}\| \geqslant \min(1 - \beta_{L+i+1}, 1 - N_0(i)\beta_{L+i+1}),$$

and the result follows from the fact that $\beta_{L+i+1} \to 0$. Q.E.D.

The sequence (x_k) constructed above has some additional interesting properties. It is not hard to extend the method above to show that, for every $i > 0$, there is an $\eta(i) > 0$ such that for all $(\alpha_1, \alpha_2, \ldots, \alpha_i) \in \mathbf{R}^i$,

$$\eta(i)\left(\sum|\alpha_i|\right) \leqslant \lim_L \left\| \sum_{j=1}^{i} \alpha_j x_{L+j} \right\| \leqslant M\left(\sum|\alpha_i|\right).$$

But, since E is super-reflexive, lim sup $\eta(i) = 0$. If we consider an ultrapower of E over a free ultrafilter and define a norm-1 sequence $\hat{y}_i$ by

$$\hat{y}_i = (x_i - x_1, x_{i+1} - x_2, x_{i+2} - x_4, \ldots, x_{i+k} - x_{2^k}, \ldots),$$

then the following hold:

(i) if $\Sigma_{j=1}^{i} \alpha_j = 1$ and $\alpha_j \geqslant 0$, then $\|\Sigma_{j=1}^{i} \alpha_j \hat{y}_j\| = 1$;
(ii) if $\hat{y}$ is a weak-cluster point of $(\hat{y}_i)$, then $\|\hat{y}\| = 1$;
(iii) if $\hat{y}$ is a weak-cluster point of $(\hat{y}_i)$, then $\lim_i \|\hat{y}_i - \hat{y}\| = 1$;
(iv) for all L and all i, $A(\hat{y}_{L+1}, \ldots, \hat{y}_{L+i}) \geqslant 1$.

Generalizing the work of Huff [8], van Dulst [15] defines property (P) for a Banach space as follows: There exist $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that if $(x_n) \subset B$, $x_n \to x$ weakly, and $\operatorname{sep}(x_n) \geqslant \varepsilon$ then $\|x\| \leqslant \delta$. Here

$$\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : m \neq n\}.$$

van Dulst proves that a reflexive space satisfying (P) has normal structure. It is immediate that for each $0 < \varepsilon < 1$ there is a subsequence, $(\hat{y}_j)$, of the sequence, $(\hat{y}_i)$, such that $\sup(\hat{y}_j) > \varepsilon$ and, still, every weak cluster point has norm 1. Thus, the ultrapower of E is a super-reflexive space failing property (P).

Finally, it is worth noticing that if Theorem 3.2 is combined with the methods used above one can prove:

THEOREM 2. *If E is not super-reflexive, then there is a Banach space F, finitely representable in E, and a closed convex set $K \subset F$ such that*

(i) *$K \subset S$, the unit sphere of f,*
(ii) $\operatorname{diam}(K) = 2$,
(iii) *K is abnormal.*

In [14] van Dulst shows that every separable space has an equivalent norm where a set like K exists. Theorem 2 guarantees that, for a non super-reflexive E, such a K can always be finitely represented in E in the *given* norm.

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CENTER FOR APPLIED MATHEMATICS, NATIONAL BUREAU OF STANDARDS
WASHINGTON, D.C.
KATHOLIEKE UNIVERSITEIT
TOERNOOIVELD, THE NETHERLANDS