ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 3, Fall 1983

REMOVING INTERSECTIONS OF LAGRANGIAN IMMERSIONS

BY

ALAN WEINSTEIN[1]

Introduction

Critical points of functions can be identified with intersection points between lagrangian submanifolds. If $S : X \to \mathbf{R}$ is a function on a manifold X, the critical points of S correspond to points where the lagrangian submanifolds $dS(X)$ and the zero section meet in T^*X. Conversely, if $L \subseteq T^*X$ is lagrangian and projects diffeomorphically onto X (for instance, if L is C^1 close to the zero section), then $L = \omega(X)$ for some closed form ω. If ω happens to be exact (for instance, if $H^1(X; \mathbf{R}) = 0$), then L must intersect the zero section in as many points as a function on X has critical points.

Periodic solutions of hamiltonian dynamical systems can also be interpreted as lagrangian intersections, so that critical point theory can sometimes be used to prove the existence of periodic solutions. This idea goes back to Poincaré [19] and has been used recently by Arnol'd [1], Meyer [16], and the author [21], among others, but its application has been restricted to systems which are very close to ones with nice manifolds of periodic solutions.

Can one go further with lagrangian intersection theory? The theory is motivated by the idea that lagrangian submanifolds, being constrained by a differential condition, should intersect more often than freely chosen manifolds, just as area-preserving maps are forced to have more fixed points than arbitrary maps, Killing vector fields must have more zeros than arbitrary vector fields, and complex submanifolds intersect more often than real ones. In particular, one may conjecture that any lagrangian submanifold of T^*X, isotopic to the zero section by a deformation generated by globally hamiltonian vector fields, must intersect the zero section in as many points as a function on X has critical points.

We shall show in this paper that for any compact X satisfying the obvious necessary condition $\chi(X) = 0$, and $\dim X \neq 3$,[2] the zero section in T^*X can be deformed to be free of itself by a family of exact lagrangian immersions. Whether or not the immersions can be made to be embeddings is still

Received July 16, 1981.

[1] Research partially supported by grants from the National Science Foundation.

[2] Our result also applies to many 3-manifolds, including S^3 and T^3, and may in fact be true for all 3-manifolds.

unknown; the construction to be described here exchanges intersections for self-intersections in a fairly explicit way, and it remains to be seen whether the intersections can be removed by a further or different deformation.

Our construction can be carried out arbitrarily near (in the C^0 sense) to the zero section, and it can be transplanted to lagrangian intersections in general symplectic manifolds by the method of "cotangent bundle coordinates".

The proof for a general cotangent bundle T^*X proceeds by a reduction to the case of disc bundles over S^1, using the round handle decompositions of Asimov [4]; it is here that the topological hypotheses on X are used. There follows a reduction to S^1 itself, where the action really takes place, and the self-intersections are introduced. The exposition in this paper proceeds in the reverse order, beginning with the circle; some technical results needed along the way are given in two appendices.

A non-intersection theorem resembling ours can be derived from the lagrangian immersion theory of Gromov [10] and Lees [14], but our construction is simpler and more explicit, and the results are somewhat stronger (except for some cases in dimension three). This point is discussed in detail in Section 6, where we also consider the relation of our result to fixed-point theorems for area-preserving mappings.

1. Exact Deformations and Loose Manifolds

We recall that a *lagrangian immersion* in a symplectic manifold (P, Ω) is an immersion $f : L \to P$ with $\dim L = \frac{1}{2} \dim P$ and $f^*\Omega = 0$. A *lagrangian submanifold* is one whose inclusion map is lagrangian.

Each cotangent bundle T^*X carries a canonical 1-form ω_X and symplectic structure $\Omega_X = -d\omega_X$. A section $\sigma : X \to T^*X$ satisfies $\sigma^*\omega_X = \sigma$, so σ is lagrangian if and only if it is closed when considered as a 1-form on X. If $f : L \to T^*X$ is any lagrangian immersion, $f^*\omega_X$ is closed; if $f^*\omega_X$ is exact, we shall call f *exact*—this is consistent with the usual terminology if f is a section.

Given a smooth one parameter family $\{f_t\}$ of lagrangian immersions (we shall always take $t \in [0, 1]$), the derivative df_t/dt is a vector field ξ_t along f_t. By the homotopy formula ((12.1, in [13]),

$$\frac{d}{dt}(f_t^*\Omega) = d[f_t^*(\xi_t \lrcorner \Omega)] + f_t^*(\xi_t \lrcorner d\Omega).$$

Since $f_t^*\Omega$ and $d\Omega$ are zero, the form $f_t^*(\xi_t \lrcorner \Omega)$ is closed for all t; if it is exact for all t, we shall call $\{f_t\}$ an *exact deformation*.

In T^*X, the homotopy formula applied to ω_X shows that $\{f_t\}$ is an exact deformation if and only if $d(f_t^*\omega_X)/dt$ is exact for all t. In particular, if f_0 is exact, then f_t is an exact deformation if and only if f_t is exact for all t.

We need one more bit of notation before stating our main definition and theorem. If $A \to B$ is a vector bundle, we will denote the zero section from B to A by $z_{B,A}$, and the image of $z_{B,A}$ by $Z_{B,A}$.

DEFINITION. A compact manifold X is *loose* if there is an exact deformation $\{f_t\}$ from X to T^*X, arbitrarily C^0 close to z_{X,T^*X}, such that $f_0 = z_{X,T^*X}$ and $f_1(X) \cap Z_{X,T^*X} = \emptyset$.

MAIN THEOREM. *Let X be a compact manifold. If X is loose, then $\chi(X) = 0$. Conversely, if $\chi(X) = 0$, then X is loose if any of the following conditions holds:*

(i) $\dim X \neq 3$;
(ii) *X is a bundle whose fibre is S^1;*
(iii) *X is a bundle over S^1.*

The main theorem is proven as Theorem 5.5. Our result for general symplectic manifolds is Theorem 5.1; we refer the reader to §5 for its statement.

2. The Circle Is Loose

Every immersion $f : S^1 \to T^*S^1$ is lagrangian, so the only interesting deformations are the exact ones. (These are a special case of the lagrangian cobordisms studied by Arnol'd [3].) We can characterize the exact deformations by using the fundamental theorem of calculus.

Let $\theta(\text{mod } 2\pi)$ and p_θ be the canonical coordinates on T^*S^1, so that the fundamental 1-form is $p_\theta d\theta$ and the symplectic structure is $d\theta \wedge dp_\theta$. Then it is easy to prove:

LEMMA 2.1. *$f_t : S^1 \to T^*S^1$ is an exact deformation if and only if $\int_{S^1} f_t^*(p_\theta d\theta)$ is independent of t.*

From this lemma follows:

COROLLARY 2.2. *If $f : S^1 \to T^*S^1$ is an exact embedding which is homotopic to the zero section, then $f(S^1)$ meets the zero section in at least two points.*

Proof. Assume that there is at most one intersection point, and apply Stokes' theorem to the region bounded by $f(S^1)$ and the zero section. ■

The main point of this section is that Corollary 2.2 does not extend from embeddings to immersions. Figure 1 shows the image of an immersion f which does not meet the zero section. The integral of $f^*(p_\theta d\theta)$ is $A_1 + A_2 - A_3$, where A_k is the (positive) area of region k. Thus f is exact as long as we make $A_3 = A_1 + A_2$. To show that S^1 is loose, we must describe

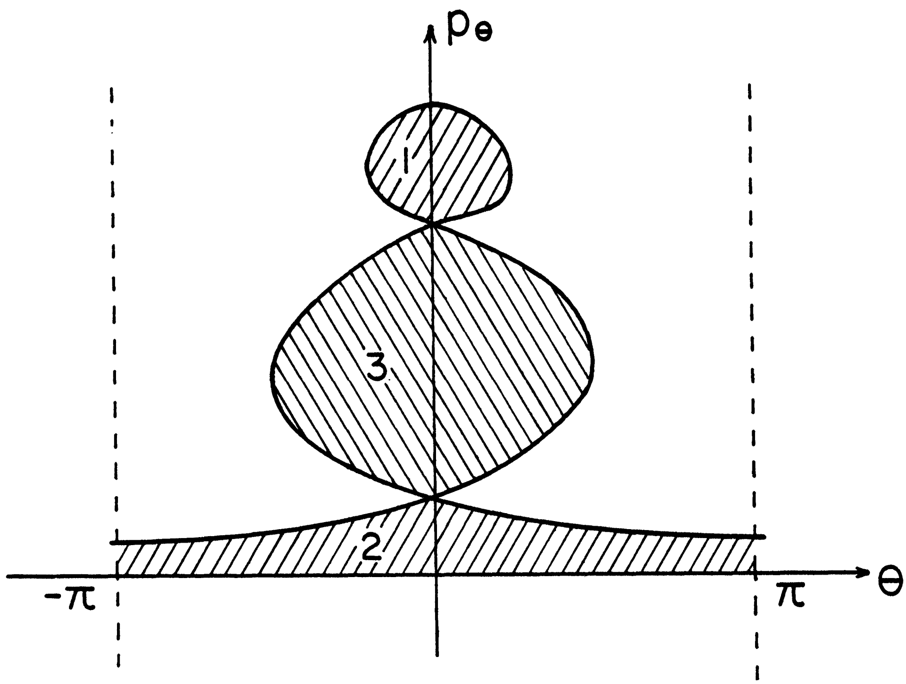

FIG. 1

an exact deformation from the zero section to f, which can be carried out in an arbitrarily small C^0 neighborhood of the zero section. Figure 2 shows such a deformation; specifically, it shows the image of f_t for a series of t values between 0 and 1. The immersions at t_2, t_4, t_6 are transitions between the "stable" types of t_1, t_3, t_5, and t_7: at t_2, f_t ceases to be transverse to the fibres of T^*S^1; at t_4, f_t stops being an embedding; at t_6, the two points of $f_t^{-1}(Z_{S^1,T^*S^1})$ merge and are about to disappear. For the deformation to be exact, it is necessary only to make sure that the areas of the regions determined by $f_t(S^1)$ add up as indicated in Figure 2.

By shrinking the horizontal size of the loops in Figure 1 and 2 and multiplying $p_\theta \circ f_t$ by a small constant, we can carry out the homotopy f_t in an arbitrarily small C^0 neighborhood of the zero section. This observation completes the proof of the following:

THEOREM 2.3. *S^1 is loose.*

Remarks. 1. As t goes from t_1 to t_7, the two points of intersection between $f_t(S^1)$ and Z_{S^1,T^*S^1} are replaced by two points of self intersection for $f_t(S^1)$. (At t_5, both types of intersection are present.) If we consider $f_t(S^1) \cup f_0(S^1)$ as the immersed image of the disjoint union of two circles, then all the intersections are self-intersections, and only their distribution

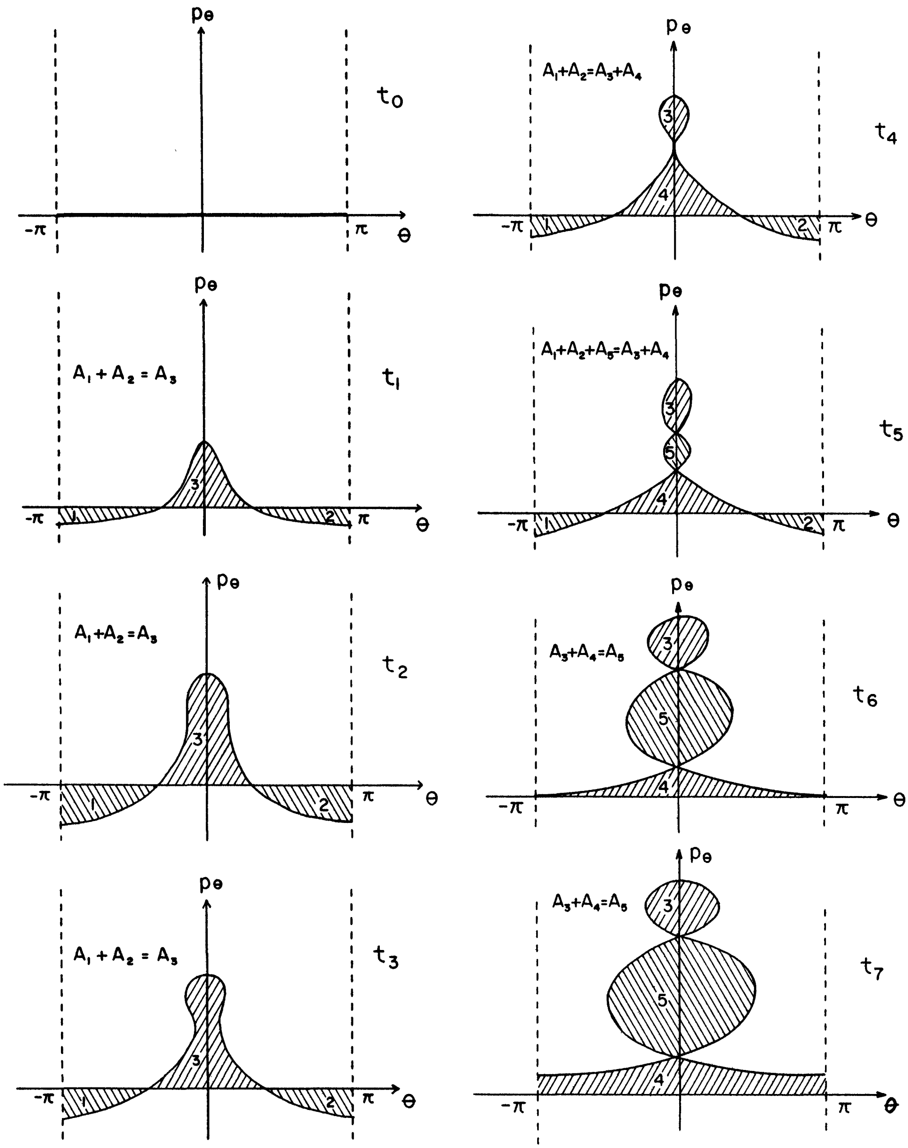

FIG. 2

has changed. Is there a reason for this "conservation of intersections" which extends beyond the one-dimensional case?

2. $f_1(S^1)$ can be thought of as $dS(S^1)$ for a *multiple-valued* function S on S^1. (As a function on the lagrangian submanifold $f_1(S^1)$, S is the primitive of $f_1^*(p_\theta d\theta)$. S is single-valued and smooth but becomes multiple-valued when pushed down to the base of T^*S^1.) The graph of S looks something like that in Figure 3 (cusps in Figure 3 correspond to vertical tangents in

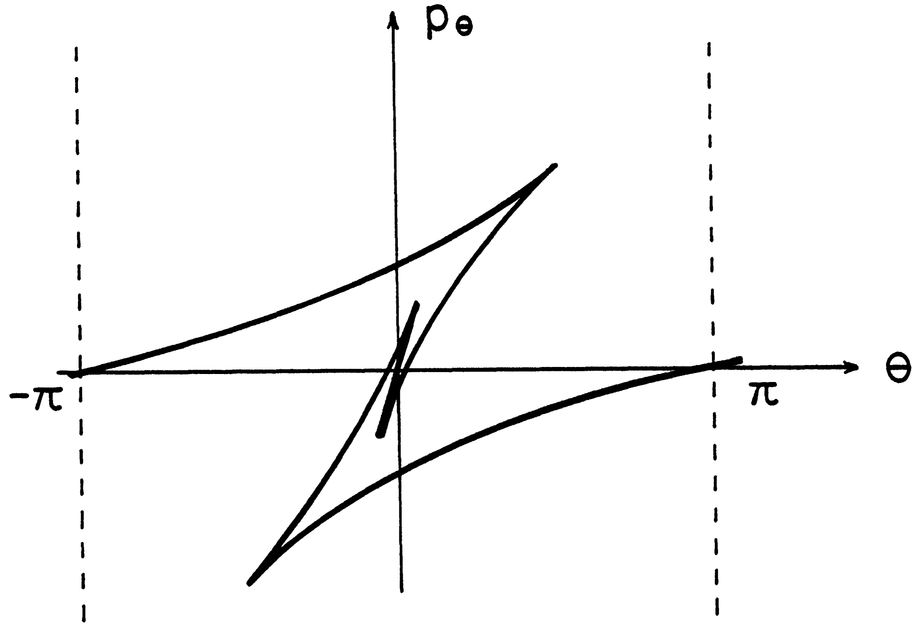

FIG. 3

Figure 1). Notice that, since $p_\theta \circ f_1 > 0$, S is "always increasing," even though it is periodic. This combination of monotonicity and periodicity brings to mind some of the prints of M. C. Escher [9], such as Ascending and Descending (N° 75), and Waterfall (N° 76).

The graph in Figure 3 may also be interpreted as the Cerf "graphic" [6] of a generating family (see [22]) for $f_1(S^1)$. Although this diagram plays no essential role in the present paper, we note that Cerf graphics are used by Eliashberg [8] in his work on fixed points of symplectic diffeomorphisms of surfaces. (See Section 6 for a further discussion of Eliashberg's paper.)

3. V. I. Arnol'd has pointed out that there is an allusion to a construction like our Figure 1 in his paper [1].

3. Bundles over Loose Manifolds Are Loose

If Y is a loose manifold and F is any manifold, then $Y \times F$ is also loose. In fact, given any exact deformation in T^*Y, we can take its product with the zero section in T^*F to get an exact deformation in

$$T^*(Y) \times T^*(F) \approx T^*(Y \times F).$$

We leave it to the reader to fill in details of this product construction, which is a special case of the pullback operation to be described next.

Let $\psi : X \to Y$ be a differentiable mapping and $f : L \to T^*Y$ a lagrangian immersion. When ψ and $\pi \circ f$ are transversal, π the projection from T^*Y

to Y, we can define a pulled back lagrangian immersion ψ^*f as follows. Let L_f be the fibre product $\{(l, x) \in L \times X | (\pi \circ f)(l) = \psi(x)\}$; by the transversality hypothesis, L_f is a submanifold of $L \times X$. Now define $\psi^*f : L_f \to T^*X$ by

$$(\psi^*f)(l, x) = (T_x\psi)^*(f(l));$$

since $f(l)$ belongs to $T^*_{\psi(x)}Y$, this is well defined and lies in T^*_xX. It turns out that ψ^*f is a lagrangian immersion; this is proven on p. 148 of [11] for the case where f is an inclusion map, and essentially the same proof works here. If $L = Y$ and f is a section, then L_f can be identified with X, and ψ^*f is just the usual pull-back of f as a 1-form. In general, if we write p for the projection of L_f onto L, then $(\psi^*f)^*(\omega_x) = p^*(f^*(\omega_y))$, so that ψ^*f is exact if f is.

We will now use the pull-back construction to prove the following result:

THEOREM 3.1. *If $\psi : X \to Y$ is a differentiable fibre bundle, and Y is loose, then X is loose.*

Proof. Let $f_t : Y \to T^*Y$ be an exact deformation as in the definition of looseness. Since ψ is a submersion, the transversality hypothesis is always satisfied, and we may define the pullbacks $\psi^*f_t : Y_{f_t} \to T^*X$. It follows from the discussion above that:

(i) Y_{f_0} may be identified with X, and ψ^*f is then the zero section;
(ii) ψ^*f_t is exact for each t.

Since $f_1(Y)$ does not meet the zero section and each $(T_x\psi)^*$ is injective, we also have:

(iii) $(\psi^*f_1)(Y_{f_1})$ does not meet the zero section.

To prove that X is loose, then, we must show that Y_{f_t} can be identified with X for *all* t, and that the resulting immersions of X in T^*X are C^0 close to the zero section if the f_t are.

Now, $Y_{f_t} = \{(y, x) | (\pi \circ f_t)(y) = \psi(x)\}$, considered as a bundle over Y, is the pullback of $\psi : X \to Y$ by the map $\pi \circ f_t$, which is C^0 close to the identity if f_t is C^0 close to the zero section. By introducing a metric on Y and a differentiable connection in the sense of Ehresmann [7] on $\psi : X \to Y$, we can use parallel translation along short geodesics in Y to identify nearby fibres of $\psi : X \to Y$ and thereby obtain a smooth family of maps $k_t : X \to Y_{f_t}$. Finally, the required exact deformation $g_t : X \to T^*X$ is defined by $g_t = (\psi^*f_t) \circ k_t$. ∎

Remark. If $Y = S^1$ and f_t is the deformation constructed in §1, the two self-intersection points of $f_1(S^1)$ become clean self-intersection manifolds for $g_1(X)$, each of them diffeomorphic to the fibre of $\psi : X \to Y$.

4. Vector Bundles over Loose Manifolds Are Very Loose

In this section $\psi : X \to Y$ will be a real vector bundle and $S : X \to \mathbf{R}$ a function which is a positive definite quadratic form on each fibre. The differential $dS : X \to T^*X$ is then an exact lagrangian embedding which intersects the zero section of the cotangent bundle $T^*X \to X$ cleanly along the zero section $Y_0 \subseteq X$ of the vector bundle $\psi : X \to Y$. (See Appendix A for the definition of clean intersection.) In this section, we shall prove the following relative version of Theorem 3.1.

THEOREM 4.1. (Notation as above). *Let $\varepsilon > 0$ and a C^0 neighborhood $\mathcal{U}$ of the map dS be given. If Y is loose, there is an exact deformation $h_t : X \to T^*X$ such that:*

(i) $h_0 = dS$;
(ii) $h_1(x)$ *does not meet the zero section in* T^*X;
(iii) $h_t(x) = dS(x)$ *whenever* $S(x) \geqslant \frac{1}{2}\varepsilon^2$
(iv) $h_t \in \mathcal{U}$ *for all* t.

The situation described in Theorem 4.1 is easiest to visualize if we consider the case where $Y = S^1$, $X = S^1 \times \mathbf{R}$, and $S(y, r) = r^2$, and think of $X_\varepsilon = \{x | S(x) \leqslant \varepsilon\}$ as an annular region in the plane. Then the graph of S on X_ε is a "trough" in $\mathbf{R}^3$ lying over X_ε. For any function on X_ε which agrees to first order with S along the boundary, there is a global minimum point in the interior of X_ε—the bottom of the distorted trough. There is also a second critical point—the saddle point encountered on a minimax circuit of the trough.[3] Our goal in Theorem 4.1 is to change the "trough" near the core circle X_0 so that it has no lowest point, and so that water can flow continually around it. (See Escher's waterfall [9] again.) The method of proof is to use the immersion f_1 of Section 2 to define the "profile" of the trough near X_0, and to pass continuously to the given boundary conditions by using the deformation f_t. This will give us h_1, and then we use the deformation f_t again to get the whole family h_t.

Proof of Theorem 4.1. We will carry out the construction first on the product $W = Y \times \mathbf{R}$ and then pull it back to X by the mapping $(\psi, \sqrt{2S}) : X \to Y \times \mathbf{R}$. We identify $T^*(Y \times \mathbf{R})$ with $T^*Y \times \mathbf{R} \times \mathbf{R}$.

Choose a smooth even function $k : (-\infty, \infty) \to [0, 1]$ such that $k(r) = 1$ when $|r| \leqslant \varepsilon/2$, $k(r) = 0$ when $|r| \geqslant \varepsilon$, and $rk'(r) < 0$ when $\varepsilon/2 < |r| < \varepsilon$. Let $f_t : Y \to T^*Y$ be an exact deformation of the zero section as in the definition of looseness. We will define the immersion $g_1 : W \to T^*W$ by

$$g_1(y, r) = (f_{k(r)}(y), r, \lambda(y, r)) \in T^*Y \times \mathbf{R} \times \mathbf{R},$$

[3] This argument is related to Poincaré's proof of a special case of his "last geometric theorem" [20]. We discuss this theorem in Section 6.

where $\lambda : Y \times \mathbf{R} \to \mathbf{R}$ will be chosen so as to make g_1 an exact lagrangian immersion satisfying the conditions on h_1 in Theorem 4.1 with $S(y, r) = \frac{1}{2}r^2$.

Denoting by Ω_Y and Ω_W the canonical 2-forms on the cotangent bundles T^*Y and T^*W, we have $\Omega_W = \Omega_Y + dr \wedge d\rho$, where (r, ρ) are the natural coordinates on $T^*\mathbf{R} \approx \mathbf{R} \times \mathbf{R}$. Then, $g_1^*\Omega_W = g_1^*\Omega_Y + dr \wedge d\lambda$. Let us denote by d_Y the operation of exterior differentiation on Y and write $\lambda_r(y)$ for $\lambda(r, y)$. Then $(dr \wedge d\lambda)(y, r) = dr \wedge d_Y\lambda_r(y)$.

To find $g_1^*\Omega_Y$, we consider its values on vector pairs of the form $(\partial/\partial r, v)$ and (ω_1, ω_2), where v, ω_1, and ω_2 are tangent to Y. The result is that, along $Y \times \{r\}$, $g_1^*\Omega_Y = f_{k(r)}^*\Omega_Y + dr \wedge k'(r) f_{k(r)}^*(\xi_{k(r)} \lrcorner\, \Omega_Y)$, where ξ_t is the deformation vector field df_t/dt (see §1). Since f_t is lagrangian, the term $f_{k(r)}^*\Omega_Y$ vanishes.

We conclude that g_1 is lagrangian if and only if

$$0 = dr \wedge [k'(r) f_{k(r)}^*(\xi_{k(r)} \lrcorner\, \Omega_Y) + d_Y\lambda_r],$$

which is true if and only if

$$d_Y\lambda_r = -k'(r) f_{k(r)}^*(\xi_{k(r)} \lrcorner\, \Omega_Y). \tag{4.1}$$

The form on the right-hand side of (4.1) is exact because the deformation $\{f_t\}$ is exact. In fact, we know that

$$f_t^*(\xi_t \lrcorner\, \Omega_Y) = d_Y(f_t^*(\xi_t \lrcorner\, \eta_Y)) - \frac{d}{dt} f_t^*\eta_Y, \tag{4.2}$$

where η_Y is the fundamental 1-form on T^*Y. Combining (4.1) and (4.2) gives

$$d_Y\lambda_r = -d_Y[k'(r) f_{k(r)}^*(\xi_{k(r)} \lrcorner\, \eta_Y)] + \frac{d}{dr} f_{k(r)}^*\eta_Y \tag{4.3}$$

which is exact because $f_{k(r)}^*\eta_Y$ is exact for each r.

We may solve (4.3) by choosing a base point y_0 in Y and writing

$$\lambda_r = -k'(r) f_{k(r)}^*(\xi_{k(r)} \lrcorner\, \eta_Y) + \frac{d}{dr}\left[\int_{y_0}^{\cdot} f_{k(r)}^*\eta_Y\right] + c(r), \tag{4.4}$$

where $\int_{y_0}^{\cdot}$ denotes the line integral from y_0, and $c(r)$ is a function of r alone which may still be freely chosen.[4]

The choice of $c(r)$ will be made so that $g_1(Y \times \mathbf{R})$ does not meet the zero section and so that the boundary conditions are satisfied. We must be sure that $\lambda_r(y)$ is not zero when $f_{k(r)}(y)$ meets the zero section in T^*Y; by the properties of f and k, we need only be concerned when $|r| > \varepsilon/2$. As for the boundary conditions, it is necessary and sufficient to have $c(r) = r$ for $|r| \geqslant \varepsilon$, since the other terms in (4.4) are zero in this range.

[4] This holds if Y is connected; otherwise the theorem can be proved component by component.

Define $M(r)$ to be the minimum value on $Y \times \{r\}$ of the sum of the first two terms on the right-hand side of (4.4). As we just observed, $M(r)$ is zero for $|r| \geq \varepsilon$; it is also zero for $|r| \leq \varepsilon/2$, since $k'(r)$ is zero there as well. Now we may easily choose $c(r)$ to be a smooth, odd function such that

(1) $c(r) > -M(r)$ when $\varepsilon/2 \leq r \leq \varepsilon$,
(2) $c(r) = r$ when $r \geq \varepsilon$.

This choice of $c(r)$ completes the construction of a g_1 satisfying all the requisite conditions, except that it might not be near to $d(\frac{1}{2}r^2)$.

To make g_1 close to $d(\frac{1}{2}r^2)$, we must have f_t close to the zero section, and λ must be close to r. Since Y is loose, we can choose f_t to be as close to the zero section as we wish. This is not quite enough, since (4.4) involves the derivative of f_t with respect to t. It is easy to make the required derivative small though: we need only replace the family $\{f_t\}$ by $\{\alpha f_t\}$ for a sufficiently small constant α. This has the effect of multiplying the first two terms in (4.4) by α, so that $M(r)$ becomes small; we may then choose $c(r)$ to be as close to r as we wish, and it will follow that λ will be close to $2r$.

Finally (within the special context of $Y \times \mathbf{R}$), we must construct the whole deformation g_t. To do this, we choose smooth functions $K(r, t)$ and $C(r, t)$ on $\mathbf{R} \times [0, 1]$ such that:

K is even in r	C is odd in r
$K(r, t) = t$ when $\|r\| \leq \varepsilon/2$	$C(r, t) = (1 - t)r$ when $\|r\| \leq \varepsilon/2$
$K(r, t) = 0$ when $\|r\| \geq \varepsilon$	$C(r, t) = r$ when $\|r\| \geq \varepsilon$
$K(r, 0) = 0$	$C(r, 0) = r$
$K(r, 1) = k(r)$	$C(r, 1) = c(r)$.

For each t, we now construct g_t as we did g_1 above, using $k_t(r) = K(r, t)$ and $c_t(r) = C(r, t)$ in place of $k(r)$ and $c(r)$.

To finish the proof of Theorem 4.1, we pull back g_t by the map

$$(\psi, \sqrt{2S}) : X \to Y \times \mathbf{R}$$

to get h_t. The non-differentiability of the square root when $S = 0$ causes no problem: on the set where $\sqrt{2S} < \varepsilon/2$, $(\psi, \sqrt{2S})^* g_t$ is just the pull-back of f_t by $\psi : X \to Y$, translated in each cotangent space $T_x X$ by $(1 - t)dS(x)$. The variation in t of the domain of $(\psi, \sqrt{2S})^* g_t$ can be handled just as in Theorem 3.1, and the proof is complete. ■

Remark. What do the self-intersections of h_1 look like? We can give a partial answer to this question. Let us look at the simplest case: $Y = S^1$, $X = S^1 \times \mathbf{R}$, with $g_1(y, r) = (f_{k(r)}(y), r, \lambda(y, r))$. If $g_1(y_0, r_0) = g_1(y', r')$, r' must equal r, and then we must have

$$f_{k(r)}(y) = f_{k(r)}(y') \tag{4.5}$$

and

$$\lambda_r(y) = \lambda_r(y'). \tag{4.6}$$

By (4.5), y and y' must correspond to a self-intersection of $f_{k(r)}$. To interpret (4.6), we note from (4.1) that

$$\lambda_r(y') - \lambda_r(y) = -k'(r)\int_y^{y'} f^*_{k(r)}(\xi_{k(r)} \lrcorner\, \Omega_Y). \tag{4.7}$$

We may interpret the integral (4.7) as the infinitesimal area in T^*Y swept out by the vector $\xi_{k(r)}$ as $f_{k(r)}$ goes around one of the loops in Figure 2, say, the one enclosing A_3 or the one enclosing A_1, A_2, and A_4. It appears (though this is not proven) that the deformation illustrated in Figure 2 can be constructed so that, between t_0 and t_7, the motion of curve segments is in a "constant sense," so that the integrand in (4.7) does not change sign along each loop. Now the factor $k'(r)$ in (4.7) is non-zero except when $|r| \leqslant \varepsilon/2$ or $|r| \geqslant \varepsilon$. In the latter case, $k(r) = 0$, and the curve $f_{k(r)}$ has no loops. It seems, therefore, that the self-intersection set of g_1 consists of two closed intervals (the product of the self-intersection set of f_1 by $[-\varepsilon/2, \varepsilon/2]$). It is likely that, by a further deformation, each of these intervals can be shrunk to an isolated point of self-intersection. To remove the self-intersections, if it is possible at all, would seem to require a construction of h_1 which is less symmetric. It also seems that self-intersections occur when the area of a loop has a critical point as a function of r, so that removing the self-intersections might require abandoning the simplifying ansatz that the middle component of $g_1(y, r)$ be just r.

If $Y = S^1$ and X is an $\mathbf{R}^n$ bundle, then the two closed intervals of the preceding paragraph become two closed n-discs. Again, one should be able to reduce each disc to an isolated self-intersection point.

5. Loose Manifolds and Circular Manifolds

The following theorem is interesting in its own right, as well as being a step in proving that most manifolds of Euler characteristic zero are loose.

THEOREM 5.1. *Let (P, Ω) be a symplectic manifold, $\Lambda \subseteq P$ a lagrangian submanifold, and $i{:}L \to P$ a lagrangian immersion which intersects Λ cleanly along $Y \subseteq L$. (See Appendix A for the definition.) If each component of Y is loose, then there is an exact deformation $\{i_t\}$ of i such that:*

(i) $i_0 = i$;
(ii) i_t is C^0 *close to* i_0 *for all* t;
(iii) $i_t = i$ *outside any preassigned neighborhood of* Y;
(iv) $i_1^{-1}(\Lambda) = i_0^{-1}(\Lambda)\setminus Y$.

In particular, if $Y = i_0^{-1}(\Lambda)$, then $i(L) \cap \Lambda = \emptyset$.

Proof. Use Theorem A in Appendix A to put the intersection into normal form, and Theorem 4.1 to remove it. Note that, if each component of Y is loose, then so is Y. This remark, as well as Theorem 4.1 and Theorem A, is true even if Y has components of different dimensions. (A vector bundle over a disconnected manifold is allowed to have different fibre dimensions over different components.) ■

COROLLARY 5.2. *Let X be a manifold which admits a function S whose critical point set is a non-degenerate critical manifold, each component of which is loose. Then X is loose.*

Proof. First deform the zero section in T^*X by $j_t = \varepsilon t dS$, where ε is small. Now j_1 intersects the zero section cleanly along the critical set of S, so we can apply Theorem 5.1 to get a further deformation which removes the intersection. ■

COROLLARY 5.3. *If $\psi: X \to Y$ is a differentiable fibre bundle whose fibres are loose, then X is loose.*

Proof. Pull back to X a Morse function on Y, and apply Corollary 5.2.

A manifold X will be called *circular* if it admits a function S whose critical point set is a non-degenerate critical manifold which is a disjoint union of circles. A special case of Corollary 5.2 is:

COROLLARY 5.4. *Circular manifolds are loose.*

Finally, we put together the results obtained so far as our main theorem.

THEOREM 5.5. *If X is loose, then $\chi(X) = 0$. Conversely, if $\chi(X) = 0$, then X is loose provided any one of the following conditions is satisfied:*

(i) $\dim X \neq 3$;
(ii) *X is a bundle whose fibre is S^1;*
(iii) *X is a bundle over S^1.*

Proof. The first statement follows from topological intersection theory. For part (i) of the converse, we refer to Appendix B, where theorems of Asimov and Meyer are used to show that every manifold with $\chi(X) = 0$ and $\dim X \neq 3$ is circular. Part (ii) follows from Corollary 5.3 and Theorem 2.3, and part (iii) follows from Theorems 2.3 and 3.1. ■

Remark. There exist 3-manifolds which are neither circular nor bundles over S^1 (see [18]). It would be interesting to see whether these manifolds are loose.

6. Discussion

Relation to Gromov-Lees. As we mentioned in the introduction, the lagrangian immersion theory of Gromov and Lees can be used to prove a result which is closely related to ours. Here is how it goes.

DEFINITION. Let (P, Ω) be a symplectic manifold, L any manifold. An almost-immersion from L to P is a bundle map $\hat{f}$:TL $\to$ TP, injective on fibres, which covers a map $f: L \to P$. The almost-immersion is lagrangian if the image of $\hat{f}$ takes the fibres of TL to lagrangian subspaces.

There is a natural inclusion from the space of lagrangian immersions to the space of lagrangian almost-immersions for which the underlying map f pulls back the de Rham cohomology class of Ω to zero. The immersion theorem of Gromov and Lees asserts that this inclusion is a weak homotopy equivalence. The theorem is stated by Gromov in a special case in [10] and proven by Lees in [14]. Using this immersion theorem, we can prove:

THEOREM 6.1. *If* $\chi(X) = 0$, *then there is a family* $f_t : X \to T^*X$ *of lagrangian immersions such that* $f_0 = z_{X,T^*X}$ *and* $f_1(X) \cap Z_{X,T^*X} = \emptyset$.

Proof. Since $\chi(X) = 0$, there is a nowhere-vanishing 1-form on X. Using this 1-form, it is easy to construct a smooth family $\{\hat{g}_t\}$ of lagrangian almost-immersions from X to T^*X such that $g_0 = z_{X,T^*X}$, $\hat{g}_0 = Tg_0$, and $g_1(X) \cap Z_{X,T^*X} = \emptyset$. Thinking of $\hat{g}_1$ as an almost-immersion from X to $\dot{T}^*X$ (T^*X with the zero section deleted), we may apply the Gromov-Lees theorem to construct a family h_t of lagrangian almost-immersions of X to $\dot{T}^*X$ such that $\hat{h}_0 = g_1$ and $\hat{h}_1 = Th_1$, i.e., h_1 is an honest lagrangian immersion. Finally, since there is a path from T_{g_0} to T_{h_1} in the lagrangian almost-immersions from X to T^*X we may apply the Gromov-Lees theorem again to obtain a family f_t of honest lagrangian immersions such that $f_0 = g_0 = Z_{X,T^*X}$, and $f_1 = h_1 : X \to \dot{T}^*X$. ■

Theorem 6.1 does not require the qualifying conditions in Theorem 5.5 (and so it applies, for example, to all 3-manifolds), but the conclusion is weaker in two ways. First of all, neither the deformation $\{f_t\}$ nor even the final map f_1 is shown to be exact. Second, although the deformation $\{f_t\}$ can be carried out in any neighborhood of $f_0(X)$, the maps f_t themselves might not be C^0 close to f_0. It is possible that these weaknesses could be removed by a closer look at the proofs of Gromov and Lees, in which case Theorem 5.5 would be true without the qualifying conditions. Nevertheless, our proof may still have the advantage of presenting f_t more explicitly, so that the self-intersections may be studied and perhaps eliminated.

Relation to fixed point theorems. The original impetus for studying lagrangian intersections came from fixed point problems for symplectic maps and periodic orbit problems for hamiltonian systems. The perturbation theory of clean lagrangian intersections in [21] implies the existence of fixed points or periodic orbits only for problems which are "close" to nice ones. Nevertheless, there do exist fixed point and periodic orbit theorems for problems which are far away from nice ones, and these can be interpreted as intersection theorems for certain lagrangian submanifolds.

What is special about the lagrangian submanifolds which enter in the fixed point theorems? If $\mu: P \to P$ is a symplectomorphism, then the graph map $\gamma_\mu(p) = (p, \mu(p))$ is a lagrangian immersion from P to $(P, \Omega) \times (P, -\Omega)$ whose intersection points with the diagonal correspond to the fixed points of μ. The lagrangian immersions which occur as graph maps are special in two ways: they are embeddings, and they are transversal to the (symplectic) fibres of the projection of $P \times P$ onto either factor. Under these additional assumptions, exact lagrangian immersions near the diagonal may be forced to intersect the diagonal even if P is a circular manifold.

For example, the Poincaré-Birkhoff annulus theorem asserts that certain "twist" symplectic diffeomorphisms μ of $A = S^1 \times [-1, 1]$ have at least two fixed points. (See [5] for a nice clean proof of this result.) For μ which are C^1 close to the identity, this can be proved by lagrangian intersection theory, i.e., critical point theory, as was already observed by Poincaré [20]. The "twist" boundary condition on μ implies that the corresponding function on A has its gradient pointing outward on ∂A, and this implies the existence of at least two critical points (see the discussion following the statement of our Theorem 4.1). But Theorem 4.1 shows that some special properties of graphs must be used to get fixed points for a μ which is merely C^0 close to the identity. In fact, the proofs of the annulus theorem all use "dynamical" ideas—especially *iteration* of μ.

For closed manifolds, V. I. Arnol'd has proposed the following conjecture. Let μ_t be a 1-parameter family of symplectic diffeomorphisms of a compact surface P such that $\{\gamma_{\mu_t}\}$ is an exact deformation and μ_0 is the identity. Then μ_1 has at least as many fixed points as a function on P has critical points. For P a torus, there are at least three fixed points, and this turns out to imply the annulus theorem (see Appendix 9 of [2]). Our Theorem 5.5 shows that the extension of Arnol'd's conjecture to lagrangian immersions of T^2 in $T^2 \times T^2$ is false. (An unpublished manuscript of Eliashberg [8] contains a proof of Arnol'd's conjecture, but this author at least has some doubt as to the completeness of that proof which uses an intricate combination of "dynamical" and "topological" constructions.)

Appendix A. Clean Intersections

Let (P, Ω) be a symplectic manifold, $\Lambda \subseteq P$ a lagrangian submanifold, and $i: L \to P$ a lagrangian immersion. We say that *i intersects Λ cleanly along a submanifold $Y \subseteq L$* if:

(i) $i(Y) \subseteq \Lambda$;

(ii) for each $l \in Y$, the inclusion $T_lY \subseteq (T_li)^{-1}(T_{i(l)}\Lambda)$ is an equality;
(iii) $i|Y$ is an embedding.

In this appendix, we shall prove a normal form theorem for clean intersections which is global in Y; it is closely related to Theorem 4.3 of [21]. A purely local normal form was derived by Guillemin and Uhlmann in [12].

THEOREM A. *Let $i:L \to P$ intersect Λ cleanly along Y. Let $X \to Y$ be the normal bundle of Y in L, and let $S:X \to \mathbf{R}$ be a function which is positive definite quadratic on fibres. Then the intersection along Y of $i:L \to P$ with $\Lambda \subseteq P$ is equivalent to the intersection along $Z_{Y,X}$ of $dS:X \to T^*X$ with Z_{X,T^*X}. More precisely, there exist:*

(i) *a diffeomorphism ψ from a neighborhood $\mathcal{U}$ of y in L to a neighborhood $\mathcal{V}$ of $Z_{Y,X}$ in X, with $\psi|Y = z_{Y,X}$;*
(ii) *a symplectic diffeomorphism Ψ from a neighborhood $\mathcal{A}$ of $z_{X,T^*X}(Z_{Y,X})$ in T^*X to a neighborhood $\mathcal{B}$ of $i(Y)$ in P, with $\Psi(Z_{X,T^*X} \cap \mathcal{A}) = \Lambda \cap \mathcal{B}$, such that $i = \Psi \circ dS \circ \psi$.*

Proof. We use cotangent coordinates as constructed in Section 4 of [21]. It follows from those constructions that there is a symplectic diffeomorphism ϕ from a neighborhood of Λ in P to a neighborhood of $Z_{\Lambda,T^*\Lambda}$ in $T^*\Lambda$ such that $\phi|_\Lambda = Z_{\Lambda,T^*\Lambda}$, and such that $(\phi \circ i)(L)$ is the graph of a section near Y; i.e., there is a 1-form σ defined on a neighborhood of $i(Y)$ in Λ such that $\phi \circ i = \sigma \circ \pi \circ (\phi \circ i)$ near Y. ($\pi_\Lambda:T^*\Lambda \to \Lambda$ is the projection.)

The clean intersection of i with Λ along Y implies that $\sigma = dS_2$ for a function S_2 defined near $i(Y)$ on Λ and having $i(Y)$ as a non-degenerate critical manifold; we may assume that $S_2 \equiv 0$ on $i(Y)$.

Next we observe that $\pi \circ \phi \circ i$ gives a local diffeomorphism between L near Y and Λ near $i(Y)$. Using this diffeomorphism and the previously constructed maps, we obtain a symplectic diffeomorphism μ from P near $i(Y)$ to T^*L near $Z_{L,T^*L}(Y)$ such that, on the natural domains of the maps:

(i) μ takes Λ to Z_{L,T^*L};
(ii) $\mu \circ i = dS_1$, where S_1 is a function on L near Y having Y as a non-degenerate critical manifold and vanishing identically on Y.

Now we apply the Morse-Bott-Meyer lemma (see Section 4 of [17]) for non-degenerate critical manifolds to find a diffeomorphism δ from a neighborhood of Y in L to a neighborhood of $Z_{Y,X}$ in the normal bundle X such that $S_1 \circ \delta^{-1} = S_0$ is non-degenerate and quadratic on fibres.

The proof of Theorem A will be complete if we know that S_0 is positive definite and not just non-degenerate. In fact, a careful look at the constructions in Section 4 of [21] (particularly the proof of Proposition 4.1) shows that they can be carried out in such a way that S_0 does become positive definite.

Remark. An alternate way of getting S_0 to be positive definite is by using the following:

LEMMA. *Let $X \to Y$ be a vector bundle, Q_1 and Q_2 functions on X which are non-degenerate quadratic on fibres. Then there is a symplectomorphism $\psi: T^*X \to T^*X$, fixing the zero section, which maps $dQ_1(X)$ onto $dQ_2(X)$.*

Proof. If Q_1 and Q_2 are both positive definite, then there is a bundle map $\phi: X \to X$ with $Q_2 \circ \phi = Q_1$, so we may take ψ to be the natural lift of ϕ to T^*X. To prove the lemma, then, it suffices to construct ψ for a given Q_1 and a suitably chosen positive definite Q_2.

Given Q_1, we may decompose X into an orthogonal direct sum $X_+ \oplus X_-$ of positive and negative definite subspaces for Q_1; define Q_2 by changing the sign of Q_1 on X_-.

We now define a hamiltonian function H on T^*X as follows. Let Q_1^* and Q_2^* be functions on the dual bundle X^* which are dual to Q_1 and Q_2. Given an element $\xi \in T^*X$, restrict it to T^*(fibre) to get an element of X^*; evaluate $Q_1^* - Q_2^*$ on this element and call the result $H(\xi)$.

Now define ψ to be the time-1 map of the hamiltonian flow generated by H. A simple calculation in vector bundle coordinates (y, x_1, x_2) for which

$$2Q_1 = \|x_1\|^2 - \|x_2\|^2 \quad \text{and} \quad 2Q_2 = \|x_1\|^2 + \|x_2\|^2$$

shows that ψ maps $dQ_1(X)$ to $dQ_2(X)$.

In fact, if (η, ξ_1, ξ_2) are the dual variables, then $H = \|\xi_2\|^2$, and so

$$\psi(y, x_1, x_2, \eta, \xi_1, \xi_2) = (y, x_1, x_2 + 2\xi_2, \eta, \xi_1, \xi_2).$$

Then

$$\begin{aligned}\psi \circ dQ_1(y, x_1, x_2) &= (y, x_1, x_2, 0, x_1, -x_2)\\ &= (y_1, x_1, -x_2, 0, x_1, -x_2) = dQ_2(y, x_1, -x_2). \quad \blacksquare\end{aligned}$$

Appendix B. Circular manifolds

D. Asimov [4] has shown that every closed manifold with $\chi(X) = 0$ and dim $X \neq 3$, except the Klein bottle, admits a so-called round handle decomposition. A "round handle" is the product of S^1 with an ordinary handle, which is just a cell. Using this decomposition, he constructs a non-singular Morse-Smale vector field with one periodic orbit for each round handle.

Given such a vector field, Meyer [15] constructs a "Liapunov" function S whose critical point set consists of one non-degenerate circle for each periodic orbit. One could bypass Meyer's result, as well as some aspects of Asimov's construction of the vector field, by proceeding directly from the round handle decomposition to S, one handle at a time.

The Klein bottle is a circle bundle over the circle, so it carries a function whose critical point set consists of two non-degenerate circles. (This function does not come from a round handle decomposition because the normal bundles of the critical circles are non-orientable.)

Morgan [18] has determined those 3-manifolds which admit a round handle decomposition; there are many which do not.

References

1. V. I. Arnol'd, *Sur une propriété topologique des applications globalement canoniques de la mécanique classique,* C. R. Acad. Sci. Paris, vol. 261 (1965), pp. 3719–3722.
2. ———, *Mathematical methods of classical mechanics,* Graduate Texts in Math., vol. 60, Springer-Verlag, New York, 1978.
3. ———, *Lagrange and Legendre cobordisms,* Funkcional Anal. i Prilŏzen, vol. 14 (1980), pp. 1–13.
4. D. Asimov, *Round handles and non-singular Morse-Smale flows,* Ann. of Math., vol. 102 (1975), pp. 41–54.
5. M. Brown and W. D. Neumann, *Proof of the Poincaré-Birkhoff fixed point theorem,* Michigan Math., J., vol. 24 (1977), pp. 21–31.
6. J. Cerf, *La stratification naturelle des espaces de fonctions différentiables réeles et la théorème de la pseudoisotopie,* Publ. Math. IHES, vol. 39 (1970), pp. 5–173.
7. C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable,* Colloque de Topologie, Centre Belge des Recherches Mathematiques, 1951, pp. 29–55.
8. Ya. M. Eliashberg, *Estimates of the number of fixed points of area-preserving mappings,* Syktyvkar University, Syktyvkar, 1978 (reviewed in Referatyvnyi Zhurnal, 5A, 1979, #491).
9. M. C. Escher, *The graphic work of M. C. Escher,* Ballantine Books, New York, 1967.
10. M. L. Gromov, *A topological technique for the construction of solutions of differential equations and inequalities,* Actes Congrès Intern. Math., 1970, T.2, pp. 221–225.
11. V. W. Guillemin and S. Sternberg, *Geometric asymptotics,* Amer. Math. Soc., Providence, R.I., 1976.
12. ———, *Oscillatory integrals with singular symbols,* MIT preprint, 1980.
13. R. Hermann, *Differential geometry and the calculus of variations,* Academic Press, New York, 1968.
14. J. A. Lees, *On the classification of lagrange immersions,* Duke Math. J., vol. 43 (1976), pp. 217–224.
15. K. R. Meyer, *Energy functions for Morse-Smale systems,* Amer. J. Math., vol. 90 (1968), pp. 1031–1040.
16. ———, *Generic bifurcation of periodic points,* Trans. Amer. Math. Soc., vol. 149 (1970), pp. 95–107.
17. W. Meyer, *Kritische Mannigfaltigkeiten in Hilbertmannigfaltigkeiten,* Math. Ann., vol. 170 (1967), pp. 45–66.
18. J. W. Morgan, *Non-singular Morse-Smale flows on 3-dimensional manifolds,* Topology, vol. 18 (1979), pp. 41–53.
19. H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste,* Gauthier-Villars, Paris, 1892, vol. 3.
20. ———, *Sur un théorème de géométrie,* Rend. Circ. Math. Palermo, vol. 33 (1912), pp. 375–407.
21. A. Weinstein, *Lagrangian submanifolds and hamiltonian systems,* Ann. of Math., vol. 98 (1973), pp. 377–410.
22. ———, *Lectures on symplectic manifolds,* Regional Conferences in Mathematics, vol. 29, Amer. Math. Soc. Providence, R.I., 1977.

University of California
Berkeley, California