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APPROXIMATION BY POLYNOMIALS OF GIVEN LENGTH

BY

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I. Introduction

The number of non-zero coefficients of a polynomial P is denoted by $l(P)$ and called the length of P. Accordingly, expressions of the form

$$\sum_{1 \leqslant k \leqslant m} c_k x^{n_k} \quad \text{and} \quad \sum_{1 \leqslant k \leqslant m} \gamma_k e^{is_k t},$$

where n_k, s_k are integers, $n_k \geqslant 0$, represent algebraic, resp. trigonometric polynomials of length $\leqslant m$.

We shall consider algebraic polynomials of length $\leqslant m$ on some interval $[a, b]$; by a change of variable $x = cx'$, $c \neq 0$, we may assume that the interval is of the form $[\delta, 1]$, where $-1 \leqslant \delta < 1$. (We cannot reduce all considerations to just one interval, like $[0, 1]$, since the length of an algebraic polynomial is not invariant under translation.)

The length of a polynomial is as natural a concept as the degree; however, the length is much less convenient to work with: polynomials of length $\leqslant m$ do not form a vector-space, nor even a convex set; in the standard functional spaces the set of polynomials of length $\leqslant m$ and norm less than or equal to 1 is not a compact set; polynomial leg-ngth, as we mentioned, is not invariant under translation; etc. There seems to be only one well-known positive statement on algebraic polynomials of length $\leqslant m$: they have at most $m - 1$ zeroes on $(0, +\infty)$.

To illustrate the curious consequences of that lack of previously established results, consider the statement: the polynomials of length $\leqslant m$ form a closed subset in $C[a, b]$. That is, of course, true. But even that quite simple statement can not be considered trivial, because in order to prove it, one needs to establish first some other, more basic result, like the estimate in the lemma below.

The main result of this paper is the existence of best approximation by polynomials of length $\leqslant m$ in $C[a, b]$, and, more generally, in $L^p[a, b]$, $1 \leqslant p \leqslant +\infty$ (Theorem 2).

The proof of Theorem 2 is based on (i) a generalization of the familiar fact that if V is a finite-dimensional subspace of the normed vector space B, then every element in B has a best approximation in V (that generalization

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is given in Theorem 1 and we believe is of independent interest); and (ii) an estimate for the polynomial coefficients, given in the lemma.

II. Union of countably many subspaces of bounded dimension

Let B be a normed vector-space, $\{x_n\}_{n=1}^{\infty}$ a linearly independent set in B. We shall consider any set S that can be represented as union of a family of subspaces of B, each subspace in the family satisfying the following two conditions:

(i) its dimension is less than or equal to m, and
(ii) it has a basis consisting of elements from $\{x_n\}$.

The question is: under what additional conditions on $\{x_n\}$ and S can we conclude that for every $z \in B$ there exists an element of best approximation in S? An answer to that question is given by Theorem 1. To state that theorem we need to introduce the (not necessarily continuous) linear functionals δ_k, $k = 1, 2, \ldots$, defined on the linear span of $\{x_n\}$ by $\delta_k(x_n) = 0$ if $k \neq n$, $\delta_k(x_k) = 1$. Let us note that if $s = \Sigma\, a_k x_k \in S$, then $\delta_n(s) = a_n$.

THEOREM I. *Suppose*

(i)
$$\|\delta_n\|_s = \sup_{x\in S} \frac{|\delta_n(x)|}{\|x\|} < +\infty \quad \textit{for every } n,$$

and

(ii) *there exists a family* Φ *of continuous linear functionals on* B *such that*

(a) Φ *is normalizing*, i.e.

$$\sup_{\varphi\in\Phi} \frac{|\varphi(z)|}{\|\varphi\|} = \|z\| \quad \textit{for every } z \in B,$$

and

(b) $\|\delta_n\|_s \varphi(x_n) \to 0$, $n \to \infty$ *for every* $\varphi \in \Phi$.

Then for every $z \in B$ *there exists* $z^* \in S$ *such that*

$$\|z - z^*\| = \inf_{s\in S} \|z - s\|.$$

Proof. Let $z \in B$, $\inf_{s\in S}\|z - s\| = \gamma$; then there exists $s_n \in S$ such that

$$\|z - s_n\| \to \gamma, \quad n \to \infty.$$

This implies

$$\|s_n\| \leq M \quad \text{for all } n \text{ and some } M.$$

Write s_n as linear combination of elements from $\{x_n\}$; then

$$s_n = \alpha_1^{(n)} x_{k(n,1)} + \cdots + \alpha_m^{(n)} x_{k(n,m)}.$$

Let us assume that the sequence s_n is such that the set of indices (1, 2, ..., m) can be subdivided into two subsets I_1 and I_2 so that

$$i \in I_1 \text{ implies } k(n, i) \text{ is constant } = k(i),$$

and

$$i \in I_2 \text{ implies } k(n, i) \to \infty \text{ as } n \to \infty.$$

This assumption does not restrict generality, since if $\{s_n\}$ does not satisfy this condition, a suitably chosen subsequence of $\{s_n\}$ will satisfy it. We write

$$s_n = \sigma_n^{(1)} + \sigma_n^{(2)} \quad \text{where } \sigma_n^{(j)} = \sum_{i \in I_j} \alpha_i^{(n)} x_{k(n,i)} \text{ for } j = 1, 2.$$

The sequence $\sigma_n^{(1)}$ lies in a finite-dimensional subspace V of B (V is spanned by $x_{k(i)}$, $i \in I_1$). The coefficient of $x_{k(i)}$ in $\sigma_n^{(1)}$ is the same as in s_n and accordingly it is equal to $\delta_{k(i)}(s_n)$, so it is bounded by $\|\delta_{k(i)}\|_S \|s_n\| \leqslant M\|\delta_{k(i)}\|_S$. This implies that $\sigma_n^{(1)}$ has a subsequence which converges to some element $\sigma \in V \subset S$. We may and we shall assume that $\sigma_n^{(1)}$ itself converges to σ. We write $z - \sigma = y$, $\sigma_n^{(2)} = \sigma_n$ and obtain $\|y - \sigma_n\| \to \gamma$, $n \to \infty$. Let ε be an arbitrary positive number, and let $\varphi \in \Phi$ be such that $|\varphi(y)| \geqslant \|\varphi\|(\|y\| - \varepsilon)$, and $\varphi \neq 0$. For $i \in I_2$ the coefficient of $x_{k(n,i)}$ in σ_n is the same as in s_n and equals $\delta_{k(n,i)}(s_n)$, so it is bounded by

$$\|s_n\| \, \|\delta_{k(n,i)}\|_S \leqslant M\|\delta_{k(n,i)}\|_S.$$

Thus, by (ii) and by the definition of I_2,

$$|\varphi(\sigma_n)| \leqslant M \sum_{i \in I_2} \|\delta_{k(n,i)}\|_S \, |\varphi(x_{k(n,i)}| \to 0 \quad \text{as } n \to \infty$$

We then obtain

$$|\varphi(y - \sigma_n)| \leqslant \|\varphi\| \, \|y - \sigma_n\| \to \gamma\|\varphi\|$$

and

$$|\varphi(y - \sigma_n)| \to |\varphi(y)| \geqslant \|\varphi\|(\|y\| - \varepsilon).$$

This implies $\gamma \geqslant \|y\| - \varepsilon$ for every $\varepsilon > 0$, and so $\|z - \sigma\| = \|y\| \leqslant \gamma$, which, since $\sigma \in S$, proves the theorem.

III. An estimate for the coefficients

Given the L_p-norm of a polynomial P on some interval $[c, 1]$, $0 \leqslant c < 1$, $1 \leqslant p \leqslant \infty$, and given the length m of P, how large can be the coefficient of x^n in the polynomial P? Denoting that coefficient by a_n, we have the following result:

LEMMA. *For every r, $0 < r < 1$,*

(A) $$|a_n| \leq \frac{K}{r^n}\|P\|_p,$$

where $K = K(c, m, r, p)$ does not depend on n nor on the polynomial P.

(Recently, R. Bojanic and the author have proved the stronger inequality

(B) $$|a_n| \leq K(n + 1)^{m-1/q}\|P\|_p,$$

where $1/p + 1/q = 1$, and $K = K(c, m)$ does not depend on n nor on the polynomial P nor on p. The proof of that inequality will be published elsewhere.)

Proof of the lemma. We may assume $r > c$. Let

$$P(x) = \sum_{k=1}^{\infty} a_{n_k}x^{n_k}, \quad n_1 < n_2 < \cdots < n_m.$$

We shall first consider the case $p = \infty$. Let $M = \mathrm{Max}_{c \leq x \leq 1} |P(x)|$, $\alpha = r^{1/(m-1)}$. Evaluating the polynomial P at m points α^j, $j = 0, 1, \ldots, m - 1$, we obtain a system of m linear equations:

(1) $$\sum_{k=1}^{m} a_{n_k}\alpha^{jn_k} = P(\alpha^j), \quad j = 0, 1, \ldots, m - 1.$$

The determinant of this system is the Vandermonde determinant

(2) $$\Delta_m = \Delta_m(\alpha^{n_1}, \alpha^{n_2}, \ldots, \alpha^{n_m}) = \begin{vmatrix} 1 & 1 & \cdots 1 \\ \alpha^{n_1} & \alpha^{n_2} & \cdots \alpha^{n_m} \\ \alpha^{2n_1} & \alpha^{2n_2} & \cdots \alpha^{2n_m} \\ \cdots\cdots & \cdots\cdots & \cdot \\ \alpha^{(m-1)n_1} & \alpha^{(m-1)n_2} & \cdots \alpha^{(m-1)n_m} \end{vmatrix}$$

and so

(3) $$\Delta_m = \prod_{i,j=1,\ldots,m, i<j} (\alpha^{n_i} - \alpha^{n_j})$$

If $A_{j,k}$ denotes the minor of Δ_m corresponding to the entry in the j-th row and k-th column, it follows from (1) that

(4) $$a_{n_k} = \frac{1}{\Delta_m}\sum_{j=1}^{m}(-1)^{j+k}P(\alpha^{j-1})A_{j,k}.$$

On the other hand, $A_{j,k}$, being a determinant of order $m - 1$, is the sum of $(m-1)!$ terms. These terms are of the form

$$\alpha^d \text{ where } d = s_1n_1 + s_2n_2 + \cdots + s_{q-1}n_{q-1} + s_qn_{q+1} + \cdots + s_{m-1}n_m,$$

all s_j's being distinct, each s_j taking values $0, 1, \ldots, m-1$. It is easy to see that the largest term is obtained if $s_q = m - q - 1$. It follows that for every $j = 1, 2, \ldots, m$,

$$(5) \qquad \begin{aligned} |A_{j,k}| &\leqslant (m-1)!\, \alpha^{(m-2)n_1 + \cdots + (m-k)n_{k-1} + (m-k-1)n_{k+1} + \cdots + n_{m-1}} \\ &\leqslant (m-1)!\, \alpha^{\sum_{q=1}^{m-1}(m-q)n_q} \alpha^{-(m-1)n_k} \end{aligned}$$

Since $|P(\alpha^{j-1})| \leqslant M$, $\Delta_m > 0$, it follows from (4) and (5) that

$$(6) \qquad |a_{n_k}| \leqslant \frac{Mm!}{\Delta_m} \alpha^{-(m-1)n_k} \alpha^{\Sigma(m-q)n_q}$$

On the other hand it follows from (3) that

$$(7) \qquad \Delta_m = \prod_1^m \alpha^{n_i(m-i)} \prod_{j>i} (1 - \alpha^{n_j - n_i}),$$

Since $1 - \alpha^{n_j - n_i} \geqslant 1 - \alpha$ and $\Pi_{j>i}(1 - \alpha^{n_j - n_i}) \geqslant (1-\alpha)^{m-i}$, we deduce from (7) that

$$\Delta_m \geqslant \alpha^{\Sigma n_i(m-i)}(1-\alpha)^{m(m-1)/2}$$

From the last inequality and from (6) we obtain, since $\alpha^{m-1} = r$,

$$|a_{n_k}| \leqslant \frac{Mm!}{(1-\alpha)^{m(m-1)/2}} \alpha^{-(m-1)n_k} = \frac{K}{r^{n_k}} M,$$

which proves the lemma for the case $p = \infty$.

From this it follows easily that the lemma holds also for $p < \infty$. Namely, if

$$P(x) = \sum_{k=1}^m a_{n_k} x^{n_k}$$

is a polynomial of length m, then

$$Q(x) = \int_x^1 P(t)dt = -\sum_{k=1}^m \frac{a_{n_k}}{n_k + 1} x^{n_k+1} + C$$

is a polynomial of length m or $m + 1$; thus, for every r, $c < r < 1$,

$$(8) \qquad \frac{a_{n_k}}{n_k + 1} \leqslant \frac{K}{r^{n_k+1}} \operatorname*{Max}_{c \leqslant x \leqslant 1} |Q(x)|$$

Since

$$\operatorname*{Max}_{c \leqslant x \leqslant 1} |Q(x)| = \operatorname*{Max}_{c \leqslant x \leqslant 1} \left| \int_x^1 P(t)dt \right| \leqslant \int_c^1 |P(t)|dt \leqslant \|P\|_p,$$

it follows from (8) that

$$|a_{n_k}| \leqslant K \frac{n_k + 1}{r^{n_k+1}} \|P\|_p \leqslant \frac{K_1}{r_1^{n_k}} \|P\|_p$$

for every r_1, $c < r_1 < 1$. This proves the lemma.

IV. An existence theorem

THEOREM 2. *Let* $1 \leqslant p \leqslant \infty$, $f \in L_p[a, b]$. *Then there exists a polynomial* P^* *of length* $\leqslant m$ *such that*

$$\|f - P^*\|_p = \inf\{\|f - P\|_p \colon P \text{ a polynomial, } l(P) \leqslant m\}.$$

Proof. We assume $a = \delta$, $b = 1$, and apply Theorem 1 with $B = L_p[\delta, 1]$, x_n the function x^n, and S the set of all polynomials of length $\leqslant m$. It then follows from the lemma that for every r, $c < r < 1$,

$$\|\delta_n\|_S \leqslant K/r^n. \tag{9}$$

To every continuous function $\tilde{\varphi}$, the support of which lies in the interior of $[\delta, 1]$ we associate the linear functional φ on $L_p[\delta, 1]$ defined by

$$\varphi(f) = \int_\delta^1 f(x)\tilde{\varphi}(x)dx.$$

The family Φ of all such linear functionals is normalizing. We need only to check whether

$$\|\delta_n\|_S\, \varphi(x_n) \to 0, \quad n \to \infty, \tag{10}$$

for every $\varphi \in \Phi$. Because of (9) and the choice of Φ, (10) will be established if we show that for every $\tilde{\varphi}$ continuous with the support in the interior of $[\delta, 1]$ there exists r, $c < r < 1$, such that

$$\frac{K}{r^n} \int_\delta^1 \tilde{\varphi}(x)x^n dx \to 0, \quad n \to \infty. \tag{11}$$

But, if the support of $\tilde{\varphi}$ is contained in $[-1 + \varepsilon, 1 - \varepsilon]$, and if

$$\operatorname*{Max}_{|x| \leqslant 1-\varepsilon} |\tilde{\varphi}(x)| = M,$$

we have

$$\left| \int_\delta^1 \tilde{\varphi}(x)x^n dx \right| \leqslant 2M(1 - \varepsilon)^n,$$

so that (11) obviously holds if we choose $r > 1 - \varepsilon$.

V. Notes and remarks

1. In case of trigonometric approximation on $[0, 2\pi]$, the existence of best approximation by polynomials of length $\leqslant m$, in any L_p-metric, $1 \leqslant p \leqslant \infty$, is easy to establish either directly, or as an immediate consequence of our Theorem 1.

Particularly transparent is the case of trigonometric L^2-approximation on $[0, 2\pi]$. Let

$$f(t) = \sum_{-\infty}^{+\infty} c_n e^{nti} \in L^2(0, 2\pi).$$

We write all non-zero Fourier coefficients $\{c_n\}_{n=-\infty}^{+\infty}$, $c_n \neq 0$, in order of decreasing absolute value $\{c_{\nu_k}\}_{k=1}^{+\infty}$, i.e., $|c_{\nu_1}| \geqslant |c_{\nu_2}| \geqslant \cdots$, and we set

(12) $e_m(f) = \inf\{\|f - T\|_2 : T$ trigonometric polynomials with $l(T) \leqslant m\}$.

Then:

(i) $e_m(f) = (\Sigma_{k=m+1}^{\infty} |c_{\nu_k}|^2)^{1/2}$;

(ii) $\Sigma_{k=1}^{m} c_{\nu_k} e^{-\nu_k t i}$ is a trigonometric polynomial of length m of best L^2-approximation to f;

(iii) the best approximation of length $\leqslant m$ is generally not unique, but the number of such best approximations is always finite, equal to $\binom{r}{s}$, where r is the number of indices k such that $|c_{\nu_k}| = |c_{\nu_m}|$, and s is the number of indices k such that $k \leqslant m$ and $|c_{\nu_k}| = |c_{\nu_m}|$.

It is worth mentioning that trigonometric polynomials of length $\leqslant m$ of best L^2-approximation to f have appeared in some important contexts: first, in Stechkin's theorem (see [12] or [6]) that the Fourier series of f is absolutely convergent if and only if

$$\sum_{1}^{\infty} \frac{1}{\sqrt{n}} e_n(f) < +\infty,$$

(where $e_n(f)$ is defined by (12)), and, second, in Carleson's proof [4, Section 7] of convergence almost everywhere of Fourier series of square-integrable functions, when the trigonometric polynomial $\Sigma_{|c_n| \geqslant \gamma} c_n e^{nti}$ is formed from a Fourier expansion $\Sigma\, c_n e^{nti}$.

2. A polynomial of length $\leqslant m$ can be viewed as a special case of a generalized polynomial $\Sigma_{\nu=1}^{m} c_\nu x^{\mu_\nu}$, $0 \leqslant \mu_1 < \mu_2 < \cdots < \mu_\nu$, or of an exponential or Dirichlet polynomial $\Sigma_{\nu=1}^{m} c_\nu e^{\lambda_\nu x}$, $\lambda_1 < \lambda_2 < \cdots < \lambda_m$. One can go a step further and consider generalized exponential polynomials of order less than or equal to m. These are expressions of the form $\Sigma_{\nu=1}^{r} P_j(x)\, e^{\lambda_j x}$, where $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, P_j are polynomials, and $\Sigma_{j=1}^{r}(\deg P_j + 1) \leqslant m$. Equivalently, a generalized exponential polynomial of order less than or equal to m is a solution of some differential equation $P(D)y = 0$, where $P(x)$ is a polynomial of degree m with real zeroes.

It is, of course, not true that a continuous function on $[a, b]$ has best approximation by exponential polynomials of length $\leq m$ (the sequence

$$\frac{e^{\lambda_n x} - e^{\lambda x}}{\lambda_n - \lambda}, \quad \lambda_n \to \lambda$$

shows that the set of exponential polynomials of length $\leq m$ is not even closed in $C[a, b]$).

However, every continuous function on $[a, b]$ has a best approximation by generalized exponential polynomials of length $\leq m$. That was discovered by Rice [9]. Several proofs of that result exist; for example, see [14], [5, pp. 154–161] and [7].

With a slight change of notation, the proofs of the lemma and Theorem 2 yield the following result.

THEOREM 2′. *Let the set S of real numbers satisfy the condition:*

(13) *There exists $\alpha > 0$ such that if $s_1, s_2 \in S$, $s_1 \neq s_2$, then $|s_1 - s_2| \geq \alpha$.*

Let $\Pi_m(S) = \{\sum_{j=1}^m c_j e^{s_j x} | c_j \in R,\ s_j \in S\}$ and let $-\infty \leq a < b < +\infty$. Then, for every $f \in L_p(a, b)$, there exists a best approximation in $\Pi_m(S)$.

We do not know whether the condition (13) can be weakened to:

(14) S is a discrete set of real numbers.

3. Dirichlet polynomials have been extensively studied, and various estimates for their coefficients are known. For particular values of p or for $c = 0$, one can deduce the inequality (A) from these known estimates for the coefficients of Dirichlet polynomials.

For example, Turan [13, p. 56] has the following estimate: If

$$f(x) = \sum_{j=1}^{k} b_j e^{-\lambda_j x}, \quad 0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_k,$$

then

$$|b_j| \leq e^{(m+1)\lambda_j} \prod_{1 \leq l \leq k, l \neq j} \frac{1 + e^{-\lambda_l}}{|e^{-\lambda_j} - e^{-\lambda_l}|} \underset{\substack{m+1 \leq x \leq m+k \\ x \text{ an integer}}}{\mathrm{Max}} |P(x)|$$

for $j = 1, 2, \ldots, k$ and for any non-negative integer m.

To obtain our lemma for the case $p = \infty$ from this inequality with $m = 0$, it suffices to make the substitution $t = e^{-\gamma x}$, $\mu_j \gamma = \lambda_j$ in the polynomial $P(t) = \sum_{j=1}^k b_j t^{\mu_j}$, where γ is defined by $e^{-k\gamma} = r$, and to observe that $1 + e^{-\lambda_l} \leq 2$, and that

$$|e^{-\lambda_j} - e^{-\lambda_l}| = e^{-\mu_j \gamma} |1 - e^{(\mu_j - \mu_l)\gamma}| \geq e^{-\mu_j \gamma} |1 - e^{-\alpha}|$$

$$\text{where } \alpha = \min_{j=2,3,\ldots,k} (\lambda_j - \lambda_{j-1}).$$

Another inequality of interest here is due to L. Schwartz [11, Théorème fondamental II, p. 36]. In case of a Dirichlet polynomial

$$f(x) = \sum_{j=1}^{l} c_j x^{\mu_j}, \quad 0 \leqslant \mu_1 < \mu_2 < \cdots < \mu_l,$$

that inequality becomes

$$\sum_{j=1}^{l} |c_j| r^{\mu_j - \mu_1} \leqslant C(r) \|f\|_{L_p(0,1)}, \quad 0 < r < 1, \tag{15}$$

where $C(r)$ depends on r *and* on the exponents $\mu_1, \mu_2, \ldots, \mu_l$. However, if the proof is analyzed, it is not difficult to show that when $\mu_{j+1} - \mu_j \geqslant a > 0$, the inequality (15) remains valid with $C(r)$ dependent only on r, m and a. Obviously, that strengthened form of (15) implies our lemma in the special case when the L_p-norm is taken over the interval [0, 1].

4. Since

$$\frac{|c_k|}{\left\| \sum c_j x^{\mu_j} \right\|_p} = \frac{1}{\left\| x^{\mu_k} - \sum_{j \neq k} d_j x^{\mu_j} \right\|_p}, \quad d_j = -\frac{c_j}{c_k},$$

any estimate for the coefficient of x^n in polynomials P with $l(P) \leqslant m + 1$, $\|P\|_{L_p(c,1)} = 1$, gives an estimate of how well can x^n be approximated in $L_p(c, 1)$ by polynomials of length $\leqslant m$, which do not contain the term with x^n. For example, the inequality (B), proved in [1], can be stated as follows:

There exists $A = A(c, m, p) > 0$ such that for every polynomial P with $l(P) \leqslant m$ which does not contain the term with x^n,

$$\|x^n - P(x)\|_{L_p(c,1)} \geqslant A n^{-m-1/p}. \tag{C}$$

Special cases of (C) were obtained previously by Borosh, Chui and Smith [3]. Their results have been significantly sharpened by Saff and Varga in [10].

5. Very interesting results have been obtained by G. G. Lorentz and by others [8] about "incomplete polynomials", primarily polynomials of the form $\Sigma_{k=s}^{n} a_k x^k$, where $s > 0$ may be large, and, more generally, polynomials P such that $\deg(P) \geqslant l(P)$.

Particularly worth mentioning here is a simple and beautiful result of Borosh, Chui, and Smith [2]. To state that result, let l and k be integers such that $k > 0$ and $0 \leqslant l \leqslant k$; let

$$\Lambda_{k,l} = \{\lambda = (\lambda_1, \ldots, \lambda_k); 0 \leqslant \lambda_1 < \cdots < \lambda_l < N < \lambda_{l+1} < \cdots < \lambda_k; \lambda_1, \ldots, \lambda_k \text{ integers}\},$$

$S(\lambda)$ the linear span of $x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_k}$,

$$d(x^N, S(\lambda)) = \inf_{f \in S(\lambda)} \|x^n - f\|_{L_\infty(c,1)},$$

and $\hat{\lambda} = (N - l, \ldots, N - 1, N + 1, \ldots, N + k - l)$. Borosh, Chui and Smith have proved that if $\lambda \in \Lambda_{k,l}$ and $\lambda \neq \hat{\lambda}$, then $d(x^N, S(\lambda)) > d(x^N, S(\hat{\lambda}))$.

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