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WEIGHTED SHIFTS WITH PERIODIC WEIGHT SEQUENCES

BY

JIM HARTMAN

Let $\{\beta(n) : n \in \mathbf{Z}\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $\sup\{\beta(n + 1)/\beta(n) : n \in \mathbf{Z}\} < \infty$. We then define the Hilbert space $L^2(\beta)$ by $L^2(\beta) = L^2(\mathbf{Z}, \beta^2)$. We denote $f \in L^2(\beta)$ by $f = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ with the summand $\hat{f}(n)z^n$ indicating that $\hat{f}(n)$ is the n-th term of the sequence. Thus we have $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2(\beta(n))^2$. We also use $z^0 = 1$. The set $\{z^n : n \in \mathbf{Z}\}$ can be thought of as an othogonal basis for $L^2(\beta)$.

For f, $g \in L^2(\beta)$ we define the formal product $h = fg$ by

$$h = \sum_{n=-\infty}^{\infty} \hat{h}(n)z^n \quad \text{where } \hat{h}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n-k)$$

if all the latter sums converge. Now let $L^\infty(\beta) = \{\varphi \in L^2(\beta) : \varphi f \in L^2(\beta)$ for all $f \in L^2(\beta)\}$. Then for $\varphi \in L^\infty(\beta)$ we can define the linear map $M_\varphi : L^2(\beta) \to L^2(\beta)$ by $M_\varphi(f) = \varphi f$. If we let $w_n = \beta(n + 1)/\beta(n)$, we have the following theorem.

THEOREM 1 (Shields, [3]). *For $\varphi \in L^\infty(\beta)$, M_φ is a bounded linear operator on $L^2(\beta)$ and M_z is unitarily equivalent to the bilateral weighted shift T with weight sequence $\{w_n : n \in \mathbf{Z}\}$. Furthermore, under this unitary equivalence, $\{M_\varphi : \varphi \in L^\infty(\beta)\}$ corresponds to the commutant of T.*

One could also start with a bilateral weighted shift with a weight sequence of positive terms and then define $\{\beta(n) : n \in \mathbf{Z}\}$ by:

(i) $\beta(n) = \prod_{k=0}^{n-1} w_k$ if $n > 0$;
(ii) $\beta(0) = 1$;
(iii) $\beta(n) = (\prod_{k=n}^{-1} w_k)^{-1}$ if $n < 0$.

In this paper when we say weighted shift, we are referring to a bilateral weighted shift with all terms of the weight sequence being positive. For the most part, we will be studying weighted shifts with periodic weight sequences. Theorem 2 provides some information about the adjoints of operators in a certain subalgebra of $L^\infty(\beta)$, and Theorem 3 is a spectral inclusion theorem for shifts with periodic weight sequences.

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Let

$$H^2(\beta) = \{f \in L^2(\beta) : \hat{f}(n) = 0 \text{ for all } n < 0\}$$

and

$$B(\beta) = (\varphi \in L^\infty(\beta) : M_\varphi^* = M_\Psi \text{ for some } \Psi \in L^\infty(\beta)\}.$$

For $T \in B(H)$ (the set of bounded linear operators on the Hilbert space H), we let $\sigma(T)$ denote the spectrum of T in $B(H)$ and $r(T) = \sup\{|z| : z \in \sigma(T)\}$. For $\varphi \in L^\infty(\beta)$, we let $\|\varphi\|_\infty = \|M_\varphi\|$. When $L^\infty(\beta)$ is endowed with this norm, it is a commutative Banach algebra (see Shields, [3]).

DEFINITION 1. For $f \in L^2(\beta)$ we define $\tilde{f} \in L^2(\beta)$ by

$$\hat{\tilde{f}}(n) = \overline{\hat{f}(-n)}\beta(-n)/\beta(n).$$

The following facts are easy to verify.

1. $H^2(\beta)$ is a closed subspace of $L^2(\beta)$.
2. $\mathbf{C} \subset B(\beta) \subset L^\infty(\beta)$.
3. For $f \in L^2(\beta)$, $\|f\|_2 = (f, f)^{1/2} = \|\tilde{f}\|_2$.

Also, we will let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto its closed subspace $H^2(\beta)$. This projection is described by the formula

$$P\left(\sum_{n=-\infty}^{\infty} \hat{f}(n)z^n\right) = \sum_{n=0}^{\infty} \hat{f}(n)z^n.$$

We now examine the problem of determining when equality holds at either end of the chain of inequalities $\mathbf{C} \subset B(\beta) \subset L^\infty(\beta)$. The solution of this problem is given in the following two lemmas and theorem.

LEMMA 1. *If $\varphi \in B(\beta)$ and N is an integer such that $\hat{\varphi}(N) \neq 0$, then $\beta(N + k) = \beta(N)\beta(k)$ for every integer k.*

Proof.

$$\begin{aligned}\hat{\varphi}(N) &= (\varphi z^k, Z^{N+k})/\beta^2(N + k)\\ &= (z^k, M_\psi z^{N+k})/\beta^2(N + k) \quad \text{where } M_\varphi^* = M_\psi\\ &= \overline{\hat{\psi}(-N)}\beta^2(k)/\beta^2(N + k) \quad \text{for each } k.\end{aligned}$$

Thus

$$\beta^2(N + k)/\beta^2(k) = \overline{\hat{\psi}(-N)}/\hat{\varphi}(N) \quad \text{for all } k.$$

Letting $k = 0$, we get

$$\overline{\hat{\psi}(-N)}/\hat{\varphi}(N) = \beta^2(N).$$

Hence $\beta(N + k) = \beta(N)\beta(k)$ for every integer k since $\beta(i) > 0$ for all i.

Q.E.D.

We also note at this point that $\beta(N + k) = \beta(N)\beta(k)$ for all k implies $\beta(-mN) = 1/\beta(mN)$ for every integer m.

LEMMA 2. *Assume there exists an integer N such that $\beta(N + k) = \beta(N)\beta(k)$ for all k. Then $w_{N+k} = w_k$ for all k (i.e., the weight sequence for the weighted shift is periodic).*

Proof. We have the following string of equalities:

$$\begin{aligned} w_{N+k} &= \beta(N + k + 1)/\beta(N + k) \\ &= \beta(N)\beta(k + 1)/[\beta(N)\beta(k)] \\ &= \beta(k + 1)/\beta(k) = w_k. \end{aligned}$$

Q.E.D.

THEOREM 2. *Let T be a weighted shift with periodic weight sequence of least period N. Then $B(\beta) = \{\varphi \in L^\infty(\beta) : \hat{\varphi}(n) = 0$ for all n which are not integer multiples of $N\}$.*

Proof. Suppose $\varphi \in B(\beta)$ and $\hat{\varphi}(n) \neq 0$. Then Lemma 1 and Lemma 2 imply $w_{n+k} = w_k$ for every k. This implies $n = mN$ for some integer m since the least period of the weight sequence is N. Thus $B(\beta) \subset E \equiv \{\varphi \in L^\infty(\beta) : \hat{\varphi}(n) = 0$ for n not an integer multiple of $N\}$.

Now let $\psi \in E$. We will show that $M_\psi^* = M_{\bar{\psi}}$. This will be true if and only if $M_\psi^*(z^k) = \bar{\psi}z^k = M_{z^k}(\bar{\psi})$ for all k. The reason for writing this in such an awkward fashion is that it is unknown whether $\psi \in L^\infty(\beta)$ implies $\bar{\psi} \in L^\infty(\beta)$. Now we have

$$(M_\psi^* z^k, z^n)/\beta^2(n) = (z^k, \psi z^n)/\beta^2(n) = \overline{\hat{\psi}(k - n)}\beta^2(k)/\beta^2(n)$$

and

$$\begin{aligned} (M_{z^k}\bar{\psi}, z^n)/\beta^2(n) &= (z^k\bar{\psi}, z^n)/\beta^2(n) \\ &= \hat{\bar{\psi}}(n - k) \\ &= \overline{\hat{\psi}(k - n)}\beta(k - n)/\beta(n - k). \end{aligned}$$

If $\hat{\psi}(k - n) \neq 0$ then $k - n = mN$ for some integer m. Hence

$$\begin{aligned} \beta^2(k)/\beta^2(n) &= \beta^2(n + mN)/\beta^2(n) \\ &= \beta^2(mN) \\ &= \beta(mN)/\beta(-mN) \\ &= \beta(k - n)/\beta(n - k). \end{aligned}$$

So we have $(M_\psi^* z^k, z^n) = (z^k\overline{\psi}, z^n)$ for all integers n. This shows that for $\psi \in E$, $M_\psi^*(z^k) = \overline{\psi}z^k$ for every integer k. Q.E.D.

COROLLARY 1. *Under the involution $\psi \to \overline{\psi}$, $B(\beta)$ is a commutative C^*-subalgebra of $L^\infty(\beta)$.*

Proof. This is an easy consequence of Theorem 2.

COROLLARY 2. *For a weighted shift with a periodic weight sequence of least period N, M_{z^n} is normal if and only if n is an integer multiple of N.*

Proof. For $n = kN$, $z^n \in B(\beta)$. Thus $M_{z^n}^* = M_{\overline{z^n}}$ which implies M_{z^n} is normal.

For the converse we assume $M_{z^n}^* M_{z^n} = M_{z^n} M_{z^n}^*$. This implies that

$$\begin{aligned}(z^{n+k}, z^{n+m}) &= (M_{z^n}z^k, M_{z^n}z^m)\\ &= (M_{z^n}^* z^k, M_{z^n}^* z^m)\\ &= (z^{k-n}, z^{m-n})\beta^2(k)\beta^2(m)/[\beta^2(k-n)\beta^2(m-n)]\\ &= \begin{cases}\beta^4(m)/\beta^2(m-n) & \text{if } m = k\\ 0 & \text{if } m \neq k\end{cases}\end{aligned}$$

by a direct computation for $M_{z^n}^*(z^m)$. Also,

$$(z^{n+k}, z^{n+m}) = \begin{cases}\beta^2(m+n) & \text{if } m = k\\ 0 & \text{if } m \neq k\end{cases}$$

Thus we have $\beta(m+n)\beta(m-n) = \beta^2(m)$ for all m. By using $m+1$ instead of m we also get $\beta(m+n+1)\beta(m-n+1) = \beta^2(m+1)$. Now dividing the latter equality by the former one gets

$$[\beta(m+n+1)/\beta(m+n)][\beta(m-n+1)/\beta(m-n)] = [\beta(m+1)/\beta(m)]^2$$

or, equivalently, $w_{m+n} = w_m^2/w_{m-n}$. It can now be shown by induction that

$$w_{kn+m} = (w_{n+m}/w_m)^k w_m \quad \text{for all } k \geq 0.$$

Since a periodic weight sequence for a weighted shift whose weight sequence has all positive terms must be bounded above and bounded away from zero, we must have $w_{n+m} = w_m$ for every integer m. Hence $\{w_m\}$ is periodic with period $|n|$. This implies $n = kN$ for some integer k. Q.E.D.

COROLLARY 3. *For $\varphi \in B(\beta)$ and $f \in L^2(\beta)$ we have $\overline{\varphi}f = \overline{\varphi\overline{f}}$.*

Proof.

$$
\begin{aligned}
(\bar{\varphi} f)\hat{}\,(n) &= \sum_{k=-\infty}^{\infty} \hat{\bar{\varphi}}(k)\hat{f}(n-k) \\
&= \sum_{k=-\infty}^{\infty} \overline{\hat{\varphi}(-k)}\,\hat{f}\,(n-k)\beta(-k)/\beta(k) \\
&= [\beta(-n)/\beta(n)] \sum_{k=-\infty}^{\infty} \overline{\hat{\varphi}(-k)}\,\hat{f}\,(n-k)\beta(n)\beta(-k)/\beta(k)\beta(-n) \\
&= [\beta(-n)/\beta(n)] \sum_{k=-\infty}^{\infty} \overline{\hat{\varphi}(-k)}\,\hat{f}\,(n-k)\beta(n-k)/\beta(k-n) \\
&= [\beta(-n)/\beta(n)] \sum_{k=-\infty}^{\infty} \overline{\hat{\varphi}(-k)}\,\overline{\hat{\bar{f}}(k-n)} \\
&= [\overline{(\varphi \bar{f})}]\hat{}\,(n).
\end{aligned}
$$

This is true for all n, hence $\bar{\varphi} f = \overline{\varphi \bar{f}}$. Q.E.D.

COROLLARY 4. *The equality $B(\beta) = L^{\infty}(\beta)$ holds if and only if all of the weights are equal.*

Proof. If $B(\beta) = L^{\infty}(\beta)$, then $z \in B(\beta)$. This implies the weight sequence is periodic with period one.

The converse is immediate from the characterization of $B(\beta)$ given in Theorem 2. Q.E.D.

The following corollary follows immediately from Theorem 2.

COROLLARY 5. *The equality $B(\beta) = \mathbf{C}$ holds if and only if the weight sequence for T is not periodic.*

An important weighted shift is called the unweighted shift. The unweighted shift has $w_n = 1$, or equivalently $\beta(n) = 1$, for all n. This shift is known to be unitarily equivalent to M_z on $L^2(\partial D)$ where $\partial D = \{z \in \mathbf{C} : |z| = 1\}$. The measure is normalized arclength measure. We now discuss some properties of weighted shifts which have periodic weight sequences.

PROPOSITION 1. *Let T be a weighted shift with periodic weight sequence of least period N. Then $\beta(n) = r^n \alpha(n)$ where $r = \beta(N)^{1/N}$ and $\alpha\,(n)$ is periodic.*

Proof. Suppose $n \geqslant 1$ and $n = tN + s$ where $0 \leqslant s < N$. Then

$$\begin{aligned}\beta(n) &= \prod_{k=0}^{n-1} w_k \\ &= \left(\prod_{k=0}^{N-1} w_k\right)^t \left(\prod_{k=0}^{s-1} w_k\right) \\ &= \beta(N)^t \left(\prod_{k=0}^{s-1} w_k\right) \\ &= r^n \left[r^{-s} \prod_{k=0}^{s-1} w_k \right].\end{aligned}$$

Since $0 \leqslant s < N$, the right half of the product above is a bounded sequence $\alpha(n)$ which is periodic and has $\alpha(kN) = 1$ for all $k \geqslant 0$.

Now suppose $n = tN + s$ where $t < 0$ and $-N < s \leqslant 0$. Then

$$\begin{aligned}\beta(n) &= \left(\prod_{k=-N}^{-1} w_k\right)^{-|t|} \left(\prod_{k=s}^{-1} w_k\right)^{-1} \\ &= \left(\prod_{k=0}^{N-1} w_k\right)^t \left(\prod_{k=s}^{-1} w_k\right)^{-1} \\ &= r^n \left[r^{-s} \left(\prod_{k=s}^{-1} w_k\right)^{-1} \right].\end{aligned}$$

Again since $-N < s \leqslant 0$, the right half of the product is a periodic sequence $\alpha(n)$ with $\alpha(-kN) = 1$ for all $k \geqslant 0$.

Thus for all n, $\beta(n) = r^n \alpha(n)$ where $\alpha(n)$ is bounded. To see that $\alpha(n)$ is periodic overall, we note that for every integer k,

$$\begin{aligned}\alpha(N + k)/\alpha(k) &= \beta(N + k) r^k / (\beta(k) r^{N+k}) \\ &= \beta(N)\beta(k)/(\beta(k) r^N) \\ &= \beta(N) r^{-N} \\ &= 1.\end{aligned}$$

Q.E.D.

PROPOSITION 2. *For a weighted shift T having a periodic weight sequence of least period N and $r = \beta(N)^{1/N}$ we have the following:*

(i) $r(T^{-1})^{-1} = r(T) = r$;

(ii) $\|T\| = \max\{w_0, \ldots, w_{N-1}\}$ *and* $\|T^{-1}\|^{-1} = \min\{w_0, \ldots, w_{N-1}\}$;

(iii) *if* $N \neq 1$, *then* $\|T^{-1}\|^{-1} < r(T^{-1})^{-1} = r(T) < \|T\|$;

(iv) *there exists a constant* $w > 0$ *such that*

$$w \sup\{\beta(k-n)/\beta(n) : k \geq 0\} \leq \inf\{\beta(k-n)/\beta(k) : k \geq 0\} \quad \text{for all } n;$$

(v) $f \in L^2(\beta)$ *if and only if* $\Sigma_{n=-\infty}^{\infty} \hat{f}(n) r^n z^n \in L^2(\partial D)$.

Proof. Parts (i) and (ii) follow from the corollary to Proposition 7 in Shields [3] and the theorem that $\|T^n\|^{1/n} \to r(T)$ as $n \to \infty$. Part (iii) then follows immediately from (i) and (ii). For part (iv) we consider

$$\begin{aligned} sup\{\beta(k-n)/\beta(k) : k \geq 0\} &= \sup\{r^{-n}\alpha(k-n)/\alpha(k) : k \geq 0\} \\ &= r^{-n} \sup\{\alpha(k-n)/\alpha(k) : k \geq 0\} \\ &= r^{-n} \max\{\alpha(k-n)/\alpha(k) : 0 \leq k \leq N\}. \end{aligned}$$

Similarly,

$$\inf\{\beta(k-n)/\beta(k) : k \geq 0\} = r^{-n} \min\{\alpha(k-n)/\alpha(k) : 0 \leq k \leq N\}.$$

If we let $w = \min\{\alpha(k-n)/\alpha(k) : 0 \leq k \leq N\}/\max\{\alpha(k-n)/\alpha(k) : 0 \leq k \leq N\}$ then w will satisfy the desired inequality. We note that $w > 0$ since $\alpha(n)$ is bounded away from zero.

To prove (v), note that $f \in L^2(\beta)$ if and only if

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 r^{2n} \alpha^2(n) < \infty.$$

Using the boundedness of $\{\alpha(n) : n \in \mathbf{Z}\}$, we see that $f \in L^2(\beta)$ if and only if $\Sigma_{n=-\infty}^{\infty} |\hat{f}(n) r^n|^2 < \infty$. This is true if and only if $\Sigma_{n=-\infty}^{\infty} \hat{f}(n) r^n z^n \in L^2(\partial D)$. Q.E.D.

COROLLARY 6. *Let* $R : L^2(\beta) \to L^2(\partial D)$ *be given by*

$$R(f) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^n z^n.$$

Then R *is a similarity between* M_z *on* $L^2(\beta)$ *and a scalar multiple of the unweighted shift. (The number* r *is as in Proposition* 2.)

Proof. The fact that R is bounded and invertible follows easily from the boundedness of $\{\alpha(n) : n \in \mathbf{Z}\}$. For $f \in L^2(\partial D)$ we have

$$\begin{aligned} RM_z R^{-1}(f) &= RM_z \left(\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{-n} z^n \right) \\ &= R \left(\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{-n} z^{n+1} \right) \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{-n} r^{n+1} z^{n+1}$$

$$= r\left(\sum_{n=-\infty}^{\infty} \hat{f}(n) z^{n+1}\right). \qquad \text{Q.E.D.}$$

PROPOSITION 3. *If $\varphi \in L^\infty(\beta)$ and $f \in L^2(\beta)$, then $R(\varphi f) = R(\varphi)R(f)$.*

Proof.

$$\begin{aligned}(R(\varphi f))^\wedge(n) &= r^n(\varphi f)^\wedge(n) \\ &= r^n \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)\hat{f}(n-k) \\ &= \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) r^k \hat{f}(n-k) r^{n-k} \\ &= [R(\varphi)R(f)]^\wedge(n).\end{aligned}$$

This is true for all n. Hence $R(\varphi f) = R(\varphi)R(f)$. Q.E.D.

COROLLARY 7. *The map R is a Banach algebra isomorphism between $L^\infty(\beta)$ and $L^\infty(\partial D)$.*

Proof. By definition, $\varphi \in L^\infty(\beta)$ if and only if $\varphi f \in L^2(\beta)$ for all $f \in L^2(\beta)$. Thus, by Corollary 6 and Proposition 3, $\varphi \in L^\infty(\beta)$ if and only if $R(\varphi)\, g \in L^2(\partial D)$ for all $g \in L^2(\partial D)$. Hence $\varphi \in L^\infty(\beta)$ if and only if $R(\varphi) \in L^\infty(\partial D)$. We also have

$$\begin{aligned}\|R(\varphi)R(f)\|_{L^2(\partial D)} &\leq \|R(\varphi f)\|_{L^2(\partial D)} \\ &\leq \|R\|\, \|\varphi f\|_2 \\ &\leq \|R\|\, \|\varphi\|_\infty\, \|f\|_2 \\ &\leq \|R\|\, \|R^{-1}\|\, \|\varphi\|_\infty\, \|R(f)\|_{L^2(\partial D)}.\end{aligned}$$

Hence $\|R(\varphi)\|_{L^\infty(\partial D)} \leq \|R\|\, \|R^{-1}\|\, \|\varphi\|_\infty$. Thus $R : L^\infty(\beta) \to L^\infty(\partial D)$ is continuous and by the open mapping theorem so is R^{-1}. The one thing left to verify is that $R(\varphi\psi) = R(\varphi)R(\psi)$ for $\varphi, \psi \in L^\infty(\beta)$, but this follows from Proposition 3. Q.E.D.

COROLLARY 8. *Suppose T is a weighted shift with periodic weight sequence. If $\varphi \in L^\infty(\beta)$ and $0 \neq f \in H^2(\beta)$, then $\varphi f = 0$ implies $\varphi = 0$.*

Proof. If $\varphi f = 0$ then $R(\varphi f) = R(\varphi)R(f) = 0$ where R is as in Corollary 6. But $0 \neq f \in H^2(\beta)$ implies $0 \neq R(f) \in H^2(\partial D)$. However, $R(\varphi) \in L^\infty(\partial D)$ and $R(\varphi)R(f) = 0$ implies $R(\varphi) = 0$ by the F. and M. Riesz Theorem (Douglas, [1]). Finally $R(\varphi) = 0$ implies $\varphi = 0$. Q.E.D.

From the work above several questions may be asked. First of all, we know that $\varphi \in B(\beta)$ implies $\overline{\varphi} \in B(\beta)$, and hence $\overline{\varphi} \in L^\infty(\beta)$. Does $\varphi \in L^\infty(\beta)$ imply $\overline{\varphi} \in L^\infty(\beta)$? One can answer affirmatively in special cases. For example if the weighted shift is rationally strictly cyclic the answer is yes. An affirmative answer is also obtained if $\beta(k) = r^k$ for all k or if $\beta(k) = \beta(-k)$ for all k. For shifts with periodic weight sequences the answer is determined by whether $\alpha(-n)/\alpha(n)$ is a multiplier on $L^\infty(\partial D)$.

The other question involves Corollary 8. Can one say that this corollary holds for all weighted shifts, not just for those with periodic weight sequences?

We now make the following definition.

DEFINITION 2. For $\varphi \in L^\infty(\beta)$, let $T_\varphi \in B(H^2(\beta))$ be given by $T_\varphi(f) = P(\varphi f)$.

The operator T_φ is called the Toeplitz operator with symbol φ. Toeplitz operators for the unweighted shift have been studied quite extensively. We may now ask which properties of Toeplitz operators for the unweighted shift carry over to Toeplitz operators for shifts with periodic weight sequences. Many of the same properties do hold, some with minor modifications. For example, it is known that $T_\varphi^* = T_{\overline{\varphi}}$ for all $\varphi \in L^\infty(\partial D)$ for the unweighted shift. For shifts with periodic weight sequences, we can only say that $T_\varphi^* = T_{\overline{\varphi}}$ for $\varphi \in B(\beta)$. This is an easy consequence of Theorem 2. We now examine further properties of Toeplitz operators for shifts with periodic weight sequences.

THEOREM 3. *Let T, a weighted shift with periodic weight sequence, be represented as M_z on $L^2(\beta)$. If $\varphi \in L^\infty(\beta)$ and T_φ is invertible then M_φ is invertible.*

Proof. Since T_φ is invertible there is a constant $c_1 > 0$ such that $\|\varphi f\|_2 \geqslant \|T_\varphi(f)\|_2 \geqslant c_1\|f\|_2$ for all $f \in H^2(\beta)$. Also, there is a constant $c_2 > 0$ such that $\|T_\varphi^*(f)\|_2 \geqslant c_2\|f\|_2$ for all $f \in L^2(\beta)$. Now let $n > 0$ and consider

$$\begin{aligned}\|z^{-n}f\|_2 &= \left(\sum_{k=0}^{\infty} |\hat{f}(k)|^2\beta^2(k-n)\right)^{1/2} \\ &= \left(\sum_{k=0}^{\infty} |\hat{f}(k)|^2\beta^2(k)\beta^2(k-n)/\beta^2(k)\right)^{1/2} \\ &\leqslant \left(\sum_{k=0}^{\infty} |\hat{f}(k)|^2\beta^2(k)\right)^{1/2} \sup\{\beta(k-n)/\beta(k) : k \geqslant 0\}.\end{aligned}$$

By Proposition 2 there exists $w > 0$ such that

$$sup\{\beta(k - n)/\beta(k) : k \geqslant 0\} \leqslant w^{-1} \inf\{\beta(k - n)/\beta(k) : k \geqslant 0\}.$$

Thus

$$\begin{aligned}
\|z^{-n}f\|_2 &\leqslant \|f\|_2\, w^{-1} \inf\{\beta(k - n)/\beta(k) : k \geqslant 0\} \\
&\leqslant \|T_\varphi f\|_2\, w^{-1} c_1^{-1} \inf\{\beta(k - n)/\beta(k) : k \geqslant 0\} \\
&\leqslant \|z^{-n}T_\varphi(f)\|_2\, (wc_1)^{-1} \\
&\leqslant \|z^{-n}M_\varphi(f)\|_2\, (wc_1)^{-1} \\
&\leqslant \|M_\varphi(z^{-n}f)\|_2\, (wc_1)^{-1} \quad \text{since } M_\varphi M_{z^{-n}} = M_{z^{-n}} M_\varphi.
\end{aligned}$$

Now since $\{z^{-n}f : f \in H^2(\beta),\ n > 0\}$ is dense in $L^2(\beta)$, we have M_φ is bounded below on $L^2(\beta)$. If we can show M_φ^* is bounded below on $L^2(\beta)$, we will have the desired result (Douglas, [1], p. 84).

We attempt to imitate the proof that M_φ is bounded below. However, a new difficulty is encountered here since $M_{z^{-n}}$ does not necessarily commute with M_φ^*. We get around this difficulty by noting that it is sufficient to use $n = kN$ where N is the period of the weight sequence and k is a nonnegative integer because

$$\{z^{-kN}f : f \in H^2(\beta),\ k \geqslant 0\}$$

is also dense in $L^2(\beta)$. Now we have

$$\begin{aligned}
\|z^{-kN}f\|_2 &\leqslant (c_2 w)^{-1}\|z^{-kN}T_\varphi^* f\|_2 \\
&\leqslant (c_2 w)^{-1}\|z^{-kN}M_\varphi^* f\|_2 \\
&\leqslant (c_2 w)^{-1}\|M_\varphi^*(z^{-kN}f)\|_2.
\end{aligned}$$

Noting that $z^{-kN} \in B(\beta)$, the last inequality is a result of the following equation:

$$M_{z^{-kN}} M_\varphi^* = M_{\overline{z^{-kN}}}^* M_\varphi^* = M_\varphi^* M_{\overline{z^{-kN}}}^* = M_\varphi^* M_{z^{-kN}}.$$

Thus both M_φ and M_φ^* are bounded below. Q.E.D.

We have just shown that $\sigma(M_\varphi) \subset \sigma(T_\varphi)$ for shifts with periodic weight sequences. This is called a spectral inclusion theorem and was already known for the unweighted shift. However, a spectral inclusion theorem does not necessarily hold if the weight sequence is not periodic. The following example illustrates this point.

Example 1. Let T be the weighted shift with weight sequence given as below:

(i) $w_n = 1$ if $n \geqslant -1$;

(ii) $w_n = \frac{1}{2}$ if $n < -1$.

Then for $k \geqslant 0$, $\|M_{z^{-1}}^k\| = 2^k$. It is also not difficult to verify that $\|M_{z^{-1}}|_{H^2(\beta)}\| = 1$. Hence $r(M_{z^{-1}}) = 2$ from the first equality and

$$r(T_{z^{-1}}) \leqslant \|M_{z^{-1}}|_{H^2(\beta)}\| \leqslant 1.$$

Thus it is not possible that $\sigma(M_{z^{-1}}) \subset \sigma(T_{z^{-1}})$.

We now examine other conditions on T_φ which imply something about the invertibility of M_φ. The first result has been proven for the unweighted shift. Its proof can be found in Douglas [1].

PROPOSITION 4. *If T is a weighted shift with periodic weight sequence, then either* Ker $T_\varphi = \{0\}$ *or* Ker $T_\varphi^* = \{0\}$ *for all $\varphi \in L^\infty(\beta)$, $\varphi \neq 0$.*

Proof. Suppose both Ker $T_\varphi \neq \{0\}$ and Ker $T_\varphi^* \neq \{0\}$. Then there exist nonzero elements f, $g \in H^2(\beta)$ such that $T_\varphi g = 0 = T_\varphi^* f$. Since $T_\varphi g = 0$ we have

$$\overline{\varphi g} \in H_0^2(\beta) \quad \text{where } H_0^2(\beta) = \{h \in H^2(\beta) : \hat{h}(0) = 0\}.$$

Also $T_\varphi^* f = 0$ implies $(T_\varphi^* f, z^n) = \overline{(M_\varphi^* f, z^n)} = (f, \varphi z^n) = 0$ for all $n \geqslant 0$. This last equation says $\sum_{k=0}^\infty \hat{f}(k)\overline{\hat{\varphi}(k-n)}\beta^2(k) = 0$ for all $n \geqslant 0$.

Now let $h \in H^2(\beta)$ be given by $\hat{h}(k) = \hat{f}(k)\beta(-k)\beta(k)$. (We note that since the shift is periodic $\beta(k)\beta(-k)$ is bounded.) Now

$$\begin{aligned}(\varphi\bar{h})\hat{}(n) &= \sum_{k=-\infty}^{\infty} \hat{\bar{h}}(-k)\hat{\varphi}(n+k) \\ &= \sum_{k=-\infty}^{\infty} \overline{\hat{h}(k)}(\beta(k)/\beta(-k))\hat{\varphi}(n+k) \\ &= \sum_{k=0}^{\infty} \overline{\hat{f}(k)}\hat{\varphi}(n+k)\beta^2(k) \\ &= 0 \quad \text{for all } n \leqslant 0\end{aligned}$$

from the last line of the previous paragraph. Thus $\varphi\bar{h} \in H_0^2(\beta)$.

So $R(\varphi\bar{h}) = R(\varphi)R(\bar{h}) \in H_0^2(\partial D)$ and $\overline{R(\varphi g)} = \overline{R(\varphi)}\,\overline{R(g)} \in H_0^2(\partial D)$. This says

$$R(\varphi)R(\bar{h})R(g) \in H_0^1(\partial D) \quad \text{and} \quad \overline{R(\varphi)R(\bar{h})R(g)} \in H_0^1(\partial D)$$

since $\overline{R(\bar{h})} \in H^2(\partial D)$ and $R(g) \in H^2(\partial D)$. Now, by Douglas [1, Corollary

6.7], we have $R(\varphi)R(\bar{h})R(g) = 0$ which implies $R(\varphi g)R(\bar{h}) = 0$. By the F. and M. Riesz Theorem if $R(\bar{h}) \neq 0$, then $R(\varphi g) = 0$. Corollary 8 of this paper then implies that $\varphi = 0$. This is a contradiction, and so we are done.
Q.E.D.

The next two corollaries have also been proven in the unweighted case. Their proofs are also found in Douglas [1].

Corollary 9. *If $0 \neq \varphi \in B(\beta)$ and T_φ has closed range, then M_φ is invertible.*

Proof. We may assume without loss of generality that the weighted shift has periodic weight sequence. If not, $B(\beta) = \mathbf{C}$ in which case the result is trivial.

Now if $\varphi \in B(\beta)$ then $T_\varphi^* = T_{\bar{\varphi}}$. Also by Proposition 4, we may assume Ker $T_\varphi = \{0\}$. Then, since T_φ has closed range and Ker $T_\varphi = \{0\}$, we have T_φ is bounded below on $H^2(\beta)$. One can then show, as before, that M_φ is bounded below on $L^2(\beta)$. That is, there exists a constant $c > 0$ such that $\|\varphi f\|_2 \geqslant c\|f\|_2$ for all $f \in L^2(\beta)$. Then, for $f \in L^2(\beta)$, we have

$$\|\bar{\varphi} f\|_2 = \|\overline{\varphi \bar{f}}\|_2 = \|\varphi \bar{f}\|_2 \geqslant c\|\bar{f}\|_2 = c\|f\|_2.$$

The second equality holds by Corollary 3. The above chain shows that $M_\varphi^* = M_{\bar{\varphi}}$ is bounded below on $L^2(\beta)$. As before, we conclude that M_φ is invertible.
Q.E.D.

We recall now that $S \in B(H)$ is said to be Fredholm if the range of S is closed and if both the kernel of S and S^* are finite dimensional. If S is Fredholm, we define the index $i(S)$ of S by $i(S) = \dim(\text{Ker } S) - \dim(\text{Ker } S^*)$.

Corollary 10. *If T is a weighted shift with periodic weight sequence and $\varphi \in L^\infty(\beta)$, then T_φ is invertible if and only if T_φ is Fredholm and $i(T_\varphi) = 0$.*

Proof. It is easy to verify that if T_φ is invertible then T_φ is Fredholm and $i(T_\varphi) = 0$. So if T_φ is Fredholm and $i(T_\varphi) = 0$, then Ker $T_\varphi = \{0\}$ and Ker $T_\varphi^* = \{0\}$ by Proposition 4. This implies both T_φ and T_φ^* are bounded below on $H^2(\beta)$ since they both have closed range. Thus T_φ is invertible.
Q.E.D.

We now present the last result concerning $B(\beta)$.

Proposition 5. *The map $\varphi \mapsto T_\varphi$ from $B(\beta)$ to $B(H^2(\beta))$ is *-linear and isometric.*

Proof. The proof that the map is *-linear follows easily from Theorem

2 which says that $M_\varphi^* = M_{\bar{\varphi}}$ for $\varphi \in B(\beta)$. We also have

$$\|T_\varphi(f)\|_2 = \|P(\varphi f)\|_2 \leqslant \|\varphi f\|_2 \leqslant \|\varphi\|_\infty \|f\|_2 \quad \text{for all } f \in H^2(\beta).$$

Hence $\|T_\varphi\| \leqslant \|\varphi\|_\infty$. However, since $B(\beta)$ is a commutative C^*-algebra we have $r(M_\varphi) = \|M_\varphi\|$ for all $\varphi \in B(\beta)$. Thus

$$\|\varphi\|_\infty = \|M_\varphi\| = r(M_\varphi) \leqslant r(T_\varphi) \leqslant \|T_\varphi\| \leqslant \|\varphi\|_\infty \quad \text{for } \varphi \in B(\beta).$$

The first inequality holds from the spectral inclusion theorem. The result is achieved since equality must hold throughout. Q.E.D.

References

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2. W. Rudin, *Functional analysis,* McGraw-Hill, New York, 1974.
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College of Wooster
Wooster, Ohio