ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 3, Fall 1983

EXTENSIONS OF CONTINUOUS FUNCTIONS ON DENSE SEMIGROUPS

BY

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1. Introduction

Let S be a dense subsemigroup of a semitopological semigroup T. In this paper we consider the following "extension problem": Given certain subalgebras A of $C(S)$ and B of $C(T)$ (say the algebras of weakly almost periodic functions), determine minimal conditions on S and T such that every function in A extends to a member of B; in symbols, $A \subset B|_S$.

A number of interesting results pertaining to this problem have appeared in the literature in recent years. For example, A. T. Lau, generalizing a result of S. J. Wiley [19], has shown that if T is a topological group, then $UC(S) \subset C(T)|_S$ [10]. P. Milnes, improving on Lau's result, showed that if T is a topological semigroup which is a group then $LMC(S) \subset C(T)|_S$ and, consequently, $AP(S) = AP(T)|_S$ and $WAP(S) = WAP(T)|_S$ [13]. The proofs of these results depend critically on both the group structure of T and the joint continuity of multiplication. In this paper we generalize and complement these results, requiring that multiplication in T be only separately continuous and that T satisfy some condition generally weaker than the group property. Some examples of the type of results we obtain are the following:

(A) If T is topologically right simple and contains a right identity, then $LUC(S) \subset C(T)|_S$.

(B) If T is topologically left and right simple (for example, if T is a semitopological group), then $WAP(S) \subset UC(S) \subset C(T)|_S$ and hence $WAP(S) = WAP(T)|_S$ and $AP(S) = AP(T)|_S$.

(C) If T is topologically simple, then $SAP(S) = SAP(T)|_S$.

The central theme of this paper is that of right topological compactification of a semigroup, and this notion is used systematically in the proofs of our theorems. Although we do not do so, many of our results may also be phrased in terms of these compactifications. For example, (C) may be restated as follows: If T is topologically simple, then S and T have the same SAP compactification (up to isomorphism).

Received June 9, 1981.

The necessary background in the theory of right topological compactifications is given in Section 2. Section 3 contains extension theorems for *LUC* functions and applications to invariant means. In Section 4 we treat the problem of extending *WAP* and *SAP* functions and give applications to semigroup representations. In the final section we apply the results of Sections 3 and 4 to the problem of enlarging the phase semigroup of a flow (S, Z) from S to T while preserving a specified property of the flow.

2. Preliminaries

All topological spaces are assumed to be Hausdorff. If X is a topological space, then $\mathrm{Int}(\cdot)$ and $\mathrm{Cl}(\cdot)$ denote, respectively, the interior and closure operations in X, and $C(X)$ denotes the Banach algebra of all bounded, real-valued, continuous functions on X.

A *right topological semigroup* is a semigroup S together with a topology relative to which the mappings $s \to st : S \to S$ are continuous $(t \in S)$. If, in addition, the mappings $s \to ts$ are continuous, then S is called a *semitopological semigroup*. The terminology *right topological group* and *semitopological group* will be used when S is a group. (Note that inversion is *not* assumed to be continuous in these definitions.)

For the remainder of this section, unless otherwise stated, S denotes a semitopological semigroup. A subspace A of $C(S)$ is *translation invariant* if for each $s \in S$,

$$R(s)A \cup L(s)A \subset A,$$

where $R(s)$ and $L(s)$ are the operators on $C(S)$ defined by

$$R(s)f(t) = f(ts),\ L(s)f(t) = f(st) \quad (t \in S, f \in C(S)).$$

For such a subspace A we shall not distinguish notationally between $R(s)$ and its restriction to A; the same convention applies to $L(s)$.

Let A be a translation invariant norm-closed subspace of $C(S)$ containing the constant function 1. A *mean* on A is a member of μ of A^*, the dual of A, such that $\mu(1) = 1$ and $\mu(f) \geqslant 0$ whenever $f \geqslant 0$. The set of all means on A is denoted by $M(A)$. Any $\mu \in M(A)$ with the property that $\mu(L(s)f) = \mu(f)$ for all $s \in S$ and $f \in A$ is called a *left invariant mean* (LIM).

Let A be a norm-closed subalgebra of $C(S)$ containing the constant functions. An A *compactification* of S is a pair (X, α), where X is a compact topological space and $\alpha : S \to X$ a continuous mapping with dense range such that $\alpha^*(C(X)) = A$, where $\alpha^* : C(X) \to C(S)$ is the dual mapping. The *canonical* A *compactification* of S is obtained by taking X to be spec (A) (the space of non-zero real homomorphisms on A) with the weak* topology, and $\alpha : S \to X$ the mapping defined by $\alpha(s)(f) = f(s)$ $(s \in S, f \in A)$.

A pair (X, α) is a *right topological compactification* of S if X is a compact right topological semigroup and $\alpha : S \to X$ is a continuous homomorphism with dense range such that the mappings $x \to \alpha(s)x : X \to X$ are continuous $(s \in S)$. It may be shown that if A is a norm-closed subalgebra of $C(S)$

containing the constant functions, then the canonical A compactification of S, (X, α), is a right topological compactification if and only if A is translation invariant and *left m-introverted,* i.e., the function $s \to x(L(s)f)$ is a member of A for each $f \in A$ and $x \in X$. In this case multiplication $(x, y) \to xy$ in X has the properties

$$[\alpha(s)x](f) = x(L(s)f) \text{ and } [x\alpha(s)](f) = x(R(s)f) \quad (x, y \in X;\ s \in S;\ f \in A).$$

We shall call a subalgebra A of $C(S)$ *admissible* if A is norm-closed, translation invariant, left m-introverted and contains the constant functions. Some well-known examples of admissible subalgebras of $C(S)$ are the following (see, for example, Chapter III of [3]):

$LMC(S) = \{f \in C(S) : \mu(L(\cdot)f) \text{ is continuous for each } \mu \in \text{spec}\,(C(S))\}$,
$LUC(S) = \{f \in C(S) : L(\cdot)f \text{ is norm continuous}\}$,
$WAP(S) = \{f \in C(S) : R(S)f \text{ is relatively weakly compact}\}$,
$AP(S) = \{f \in C(S) : R(S)f \text{ is relatively norm compact}\}$,
$SAP(S)$ = closed linear span in $C(S)$ of the coefficients of all continuous, finite dimensional, unitary representations of S.

(We shall occasionally suppress the letter S in the notation for these algebras.) All but the first algebra in this list are in fact *left introverted,* i.e., the function $\mu(L(\cdot)f)$ is a member of the algebra whenever f is in the algebra and $\mu \in C(S)^*$. We shall also need the following (not necessarily admissible) subalgebras of $C(S)$:

$RUC(S) = \{f \in C(S) : R(\cdot)f \text{ is norm continuous}\}$,
$UC(S) = LUC(S) \cap RUC(S)$.

A right topological compactification (X, α) of S is said to be *maximal with respect to a property P* if (X, α) possesses P, and whenever (X_0, α_0) is a right topological compactification of S with property P there exists a continuous homomorphism $\beta : X \to X_0$ such that $\beta \circ \alpha = \alpha_0$. The LMC, LUC, WAP, AP and SAP compactifications (X, α) of S are maximal, respectively, with respect to the following properties: the empty property, the mapping $(s, x) \to \alpha(s)x : S \times X \to X$ is continuous, X is a semitopological semigroup, X is a topological semigroup (i.e., multiplication in X is jointly continuous), and X is a topological group. (See, for example, [3, Chapter III].)

The following lemmas will be useful in the sequel. As these are standard results in the theory of extensions of functions on semigroups, we omit their straightforward proofs.

2.1. LEMMA. *Let S be a subsemigroup of a semitopological semigroup T and let F denote any one of the prefixes LMC, LUC, WAP, AP, SAP, RUC, UC. Then* $F(T)|_S \subset F(S)$.

2.2. LEMMA. *Let S be a dense subsemigroup of a semitopological semigroup T, and let A and B be admissible subalgebras of* $C(S)$ *and* $C(T)$, *respectively, such that* $B|_S \subset A$. *Let* $R : B \to A$ *denote the restriction mapping* ($Rf = f|_S$), $R^* : A^* \to B^*$ *its dual,* (X, α) *the canonical* A *compactification of* S, *and* (Y, β) *the canonical* B *compactification of* T. *Then* $\theta = R^*|_X$ *is a continuous homomorphism of* X *onto* Y *such that* $\theta \circ \alpha = \beta|_S$. *Also,* $R^*(M(A)) = M(B)$, *and*

$$\mu(L(s)Rf) = R^*\mu(L(s)f) \quad (s \in S, f \in B, \mu \in A^*).$$

Hence if B *has a LIM then so does* $R(B)$, *the converse holding if* B *is left introverted.*

2.3. Lemma. *Let* S *be a dense subset of a topological space* T, *and let* $f \in C(S)$. *Then* f *has a continuous extension to* T *if and only if whenever* (r_m) *and* (s_n) *are nets in* S *converging to* $t \in T$ *such that the limits* $a = \lim_m f(r_m)$ *and* $b = \lim_n f(s_n)$ *exist, then* $a = b$.

A right topological semigroup X is (*topologically*) *right simple* if X has no proper (closed) right ideals. *Left simple* and *topologically left simple* are defined analogously. X is (*topologically*) *simple* if it has no proper (closed) two-sided ideals.

Let S be a dense subsemigroup of a semitopological semigroup T. The following proposition demonstrates the relationship among various conditions we shall impose on S and T in the sequel. We omit the easy proof.

2.4. Proposition. *Let* S *be a dense subsemigroup of a semitopological semigroup* T.

(a) T *is topologically right simple if and only if* S *is topologically right simple and* $S \cap \mathrm{Cl}(tT) \neq \emptyset$ *for each* $t \in T$.

(b) T *is topologically simple if and only if* S *is topologically simple and* $S \cap \mathrm{Cl}(TtT) \neq \emptyset$ *for each* $t \in T$.

(c) *If* T *is topologically right simple and* $\mathrm{Int}(S) \neq \emptyset$, *then* $tS \cap S \neq \emptyset$ *for each* $t \in T$.

We conclude this section with a brief summary of that portion of the structure theory for compact right topological semigroups which will be needed in the sequel. *Every such semigroup* X contains a minimal two-sided ideal, $K(X)$, which is both the union of all of the minimal left ideals of X and the union of all of the minimal right ideals. The minimal left (right) ideals are precisely the sets Xe (eX), where $e^2 = e \in K(X)$. Each set eXe is a group with identity e, and $K(X)$ is the union of these groups. (For details and proofs see [3, Chapter II] or [17]).

3. Extensions of LUC Functions

Throughout this section, S denotes a dense subsemigroup of a semitopological semigroup T.

3.1. LEMMA. *Let T be topologically right simple and $tS \cap S \neq \emptyset$ for each $t \in T$. If $g \in C(T)$ and $f = g|_S \in LUC(S)$, then $g \in LUC(T)$.*

Proof. Note first that since T is topologically right simple, $\|L(r)h\| = \|h\|$ $(r \in T,\ h \in C(T))$. Let $t_0 \in T$ and define

$$d(t) = \|L(t)g - L(t_0)g\|.$$

Since S is dense in T,

$$\|L(s)g - L(s_0)g\| = \|L(s)f - L(s_0)f\| \quad (s,\ s_0 \in S),$$

hence d is continuous on S. Let $t \in T$ and (s_n) a net in S converging to t. Choose $r \in S$ such that $tr \in S$. Then

$$\begin{aligned} |d(s_n) - d(t)| &\leq \|L(s_n)g - L(t)g\| \\ &= \|L(r)(L(s_n)g - L(t)g)\| \\ &= \|L(s_nr)g - L(tr)g\| \\ &= \|L(s_nr)f - L(tr)f\| \\ &\to 0. \end{aligned}$$

By 2.3, then, d is continuous on T. In particular, if (t_m) is a net in T converging to t_0 then

$$\|L(t_m)g - L(t_0)g\| = d(t_m) \to d(t_0) = 0.$$

Therefore $g \in LUC(T)$.

3.2. THEOREM. *If T is topologically right simple and contains a right identity, then $LUC(S) \subset C(T)|_S$. If, in addition, $tS \cap S \neq \emptyset$ for each $t \in T$ (for example, if* Int$(S) \neq \emptyset$*), then $LUC(S) = LUC(T)|_S$.*

Proof. The second part of the theorem follows from the first part, 3.1, and 2.1. For the first part, let (X, α) denote the canonical LUC compactification of S and (r_m), (s_n) nets in S both converging to $t \in T$ such that the limits $x = \lim_m \alpha(r_m)$ and $y = \lim_n \alpha(s_n)$ exist in X. By 2.3, the proof of the theorem will be complete once we show that $x = y$.

Choose any $s_0 \in S$. We construct subnets $(r_{m(i)})$ and $(s_{n(i)})$, and a net (p_i) in S, all indexed by the same directed set $\{i\}$, such that $r_{m(i)}p_i \to s_0$ and $s_{n(i)}p_i \to s_0$: Since $s_0 \in T = \mathrm{Cl}(tT) = \mathrm{Cl}(tS)$, we may choose for each open neighborhood N of s_0 and each pair of indices m', n' a point p in S and indices $m = m(N, m', n') \geqslant m'$, $n = n(N, m', n') \geqslant n'$ such that $r_mp \in N$ and $s_np \in N$. Set $i = (N, m', n')$ and $p_i = p$, and direct $\{i\}$ as follows: $i_2 \geqslant i_1$ if and only if $N_2 \subset N_1$, $m_2' \geqslant m_1'$ and $n_2' \geqslant n_1'$. Then $(r_{m(i)})$, $(s_{n(i)})$ and p_i have the stated properties.

Since $\mathrm{Cl}(p_iS) = T$ we may choose for each i a net (q_k) in S such that $\lim_k p_iq_k = e$, where e denotes a right identity for T. We may assume that the limit $z_i = \lim_k \alpha(q_k)$ exists in X. Then

$$\alpha(r_{m(i)}) = \lim_k \alpha(r_{m(i)}p_iq_k) = \alpha(r_{m(i)}p_i)z_i. \tag{1}$$

Taking a subnet if necessary, we may suppose that $z = \lim_i z_i$ exists in X. Taking limits on $\{i\}$ in (1) and recalling that the mapping

$$(r, u) \to \alpha(r)u : S \times X \to X$$

is continuous (see Section 2), we see that $x = \alpha(s_0)z$. Similarly, $y = \alpha(s_0)z$. Therefore $x = y$, as required.

3.3. COROLLARY. *If T is a semitopological group and S has the finite intersection property (f.i.p.) for right ideals (respectively, S has the f.i.p. for right ideals and the f.i.p. for left ideals), then $LUC(S) = LUC(T)|_S$ (respectively, $UC(S) = UC(T)|_S$).*

Proof. The LUC part follows from the first part of 3.2 and from Lemma 2.7 of [10] (the proof of which works also in the present setting). The UC part follows by considering RUC analogs.

3.4. THEOREM. *If T is topologically left and right simple then $UC(S) \subset C(T)|_S$. If, in addition, $tS \cap S \neq \emptyset$ and $St \cap S \neq \emptyset$ for all $t \in T$ (for example, if $\mathrm{Int}(S) \neq \emptyset$), then $UC(S) = UC(T)|_S$.*

Proof. By 2.1, and 3.1 and its RUC analog, it suffices to prove the first inclusion. Let $A = UC(S)$, and let (X, α) denote canonical (not necessarily right topological) A compactification of S. Let (r_m) and (s_n) be nets in S converging to $t \in T$ such that the limits $x = \lim_m \alpha(r_m)$ and $y = \lim_n \alpha(s_n)$ exist in X, and let $s_0 \in S$. As in the proof of 3.2 we may construct subnets $(r_{m(i)})$, $(s_{n(i)})$, and a net (p_i), all indexed by the same directed set, such that $(r_{m(i)}p_i)$ and $(s_{n(i)}p_i)$ both converge to s_0. Let $s \in S$ be arbitrary, and for fixed i choose a net (q_k) in S such that $\lim_k p_iq_k = s$ and $z_i = \lim_k \alpha(q_k)$ exists in X. Then for any $f \in A$ we have

$$\alpha(r_{m(i)})(R(s)f) = \lim_k f(r_{m(i)}p_iq_k) = \lim_k \alpha(q_k)(L(r_{m(i)}p_i)f) = z_i(L(r_{m(i)}p_i)f).$$

Taking limits on $\{i\}$, and assuming, as we may, that $\lim_i z_i = z$ exists in X, we obtain $x(R(s)f) = z(L(s_0)f)$. Similarly, $y(R(s)f) = z(L(s_0)f)$. Thus

$$x(R(s)f) = y(R(s)f) \quad (s \in S, f \in A). \tag{2}$$

Now choose any $s_1 \in S$. Since T is topologically left simple so is S (Proposition 2.4), hence $\alpha(Ss_1)$ is dense in X. Therefore we may choose a net (t_j) in S such that $x = \lim_j \alpha(t_js_1)$ and such that the limit $u = \lim_j \alpha(t_j)$ exists in X. Then, for each $f \in A$,

$$x(f) = \lim_j \alpha(t_j)(R(s_1)f) = u(R(s_1)f).$$

Similarly, there exists $v \in X$ such that $y(f) = v(R(s_1)f)$. By (2),

$$u(R(s_1s)f) = x(R(s)f) = y(R(s)f) = v(R(s_1s)f) \quad (s \in S).$$

Since S is topologically right simple it follows from the definition of $RUC(S)$ that $u(R(s)f) = v(R(s)f)$ for all $s \in S$. In particular,

$$x(f) = u(R(s_1)f) = v(R(s_1)f) = y(f) \quad (f \in A).$$

By Lemma 2.3, the proof is complete.

3.5. *Remarks.* (a) Some of the above results may be extended to include the case where S is not necessarily dense in T. For example, using Tietze's extension theorem and 3.2 one can easily prove the following: If T is a semitopological semigroup whose topology is normal, and if S is a subsemigroup of T whose closure in T is a group, then $LUC(S) \subset C(T)|_S$. This result is a generalization of Theorem 2.3 of [10], where the conclusion $UC(S) \subset C(T)|_S$ was obtained under the additional assumption that T is a topological group. For other theorems of this type see [19].

(b) The LMC analog of 3.3 is false, as is seen by the following result of P. Milnes and J. Pym: If T is the additive group of real numbers and S the subgroup of rationals, then $LMC(S) \not\subset LMC(T)|_S$ [14].

(c) Theorems 3.2 and 3.4 are generally false if T is not topologically right simple. As an example, let T be the interval $[0, \infty)$ with the usual topology and ordinary addition, and let $S = (0, \infty)$. The function $f(s) = \sin(s^{-1})$ is easily seen to be in $UC(S)$, but f has no continuous extension to T. (Note that $UC(S)$ is *not* the space of functions which are uniformly continuous on S relative to the usual uniformity.)

The following applications to invariant means are easily proved by combining the last part of Lemma 2.2 with 3.2, 3.3, and 3.4, respectively.

3.6. THEOREM. *Let T be topologically right simple and contain a right identity. Then if $C(T)$ has a LIM, so does $LUC(S)$.*

3.7. THEOREM. *Let T be a semitopological group and suppose that S has the f.i.p. for right ideals (respectively, S has the f.i.p. for right ideals and the f.i.p. for left ideals). Then if $LUC(T)$ (respectively, $UC(T)$) has a LIM, so does $LUC(S)$ (respectively, $UC(S)$).*

3.8. THEOREM. *Let T be topologically left and right simple. Then if $C(T)$ has a LIM, so does $UC(S)$.*

3.9. *Remarks.* (a) Theorem 3.8 and a somewhat weaker version of Theorem 3.7 were proved by A. T. Lau for the case T a topological group [10, Theorem 3.1]. See also [12, Theorem 9] for an LMC version of these results.

(b) An example due to T. Mitchell [16, p. 640] shows that 3.6–3.8 are false if T is not topologically right simple.

4. Extensions of WAP Functions and SAP Functions

Throughout this section, S denotes a dense subsemigroup of a semi-topological semigroup T. The main results of this section are Theorems 4.3 and 4.6, which give sufficient conditions for the equalities $WAP(S) = WAP(T)|_S$ and $SAP(S) = SAP(T)|_S$ to hold. The proof of 4.3 requires two lemmas, the first one of which is due to J. F. Berglund [1, Proposition 4].

4.1. LEMMA. *If $g \varepsilon C(T)$ and $g|_S \varepsilon WAP(S)$, then $g \varepsilon WAP(T)$.*

4.2. LEMMA. *If S is topologically left and right simple, then $WAP(S) \subset UC(S)$.*

Proof. Let (X, α) denote the canonical WAP compactification of S. Then $\alpha(s)X = X = X\alpha(s)$ for all $s \varepsilon S$. Let $r \varepsilon S$ and choose $e \varepsilon X$ such that $\alpha(r)e = \alpha(r)$. If $s \varepsilon S$, and $u \varepsilon X$ is such that $\alpha(s) = u\alpha(r)$, then $\alpha(s)e = u\alpha(r)e = u\alpha(r) = \alpha(s)$. Similarly, there exists $d \varepsilon X$ such that $d\alpha(s) = \alpha(s)$ for all $s \varepsilon S$. Since multiplication in X is separately continuous (see Section 2), d and e must be left and right identities, respectively, hence $d = e$ is the identity for X. Given $s \varepsilon S$ choose $x, y \varepsilon X$ such that $\alpha(s)x = e = y\alpha(s)$. Then $y = ye = y\alpha(s)x = ex = x$, so $\alpha(s)$ has an inverse in X. Thus $\alpha(S)$ is contained in the group of units of X. By a result of J. D. Lawson [11, Proposition 6.1], multiplication restricted to $\alpha(S) \times X \cup X \times \alpha(S)$ is jointly continuous, hence $WAP(S) \subset UC(S)$.

4.3. THEOREM. *If T is topologically left and right simple, then*

$$WAP(S) = WAP(T)|_S.$$

Proof. Since T is topologically left and right simple so is S (Proposition 2.4). By 3.4 and 4.2, $WAP(S) \subset C(T)|_S$. The desired conclusion now follows from 4.1 and 2.1.

4.4. *Remarks.* (a) Theorem 4.3, for the case T a topological semigroup and a group, was proved by P. Milnes [13]. See also [6] which treats the commutative topological group case.

(b) If T is not both topologically left and right simple, then the conclusion of 4.3 may be false, as the following example shows: Let T be the interval $[0, 1]$ with the usual topology and with multiplication $st = s$, and let $S = (0, 1]$. Then T is left simple and the function $f(s) = \sin(s^{-1})$ is in $WAP(S)$ (in fact in $AP(S)$), but f has no continuous extension to T.

(c) Lemma 4.2 is false if S is not both topologically left and right simple. As an example, let S be the space $[0, 1] \times R$ with the usual topology and with multiplication $(a, s)(b, t) = (a, s + t)$. Then S is left simple (and even right cancellative), but, as shown by Berglund and Milnes, $WAP(S) \not\subset LUC(S)$ [2].

(d) The proof of Lemma 4.2 shows that if S is topologically left and right simple, then X has an identity, where (X, α) is the canonical WAP

compactification of S. This fact may be used to show that for such a semigroup S, $WAP(S)$ has an invariant mean. (Sketch of proof: Let $f \varepsilon WAP(S)$. Since S is topologically left simple, $R(s)$ is an isometry on $WAP(S)$, hence by Ryll-Nardzewski's fixed point theorem [18], the closure $K(f)$ of the convex hull of $R(S)f$ contains a function g such that $R(s)g = g$ for all $s \varepsilon S$. Thus $xy(g) = x(g)$ for all $x, y \varepsilon X$, and taking x to be the identity of X shows that $y(g) = x(g)$ for all $y \varepsilon X$. Therefore $K(f)$ contains a constant function for each $f \varepsilon WAP(S)$. As is well known, this is sufficient to guarantee that $WAP(S)$ have a LIM (see, for example, [4], [8], or [15]). Similarly, $WAP(S)$ has a right invariant mean and therefore an invariant mean.)

(e) The example constructed in (c) shows that if S is not both topologically left and right simple then $WAP(S)$ need not have an invariant mean. (If μ were a LIM and $g \varepsilon C[0, 1]$ then $\mu(f) = \mu(L(a, s)f) = g(a)$ for all $a \varepsilon [0, 1]$, where f is defined by $f(a, s) = g(a)$.)

(f) In [6] deLeeuw and Glicksberg proved that if G is a locally compact abelian topological group, then $WAP(H) = WAP(G)|_H$ for any closed subgroup H of G. Ching Chou has given an example of a locally compact group G which is solvable and contains an abelian normal subgroup H such that $WAP(H) \not\subset UC(G)|_H$ and therefore $WAP(H) \not\subset WAP(G)|_H$ [20, p. 192].

In contrast to the AP case (see 4.4(b)), SAP functions on S may be extended continuously to T if T is only topologically simple. To prove this we shall need the following lemma due to J. F. Berglund [1, Proposition 4].

4.5. LEMMA. *If $g \varepsilon C(T)$ and $g|_S \varepsilon SAP(S)$ then $g \varepsilon SAP(T)$.*

4.6. THEOREM. *If, for each $t \varepsilon T$,*

$$S \cap \mathrm{Cl}(TtT) \neq \emptyset \tag{3}$$

then $SAP(S) = SAP(T)|_S$.

Proof. By 2.1 and 4.5 it suffices to show that $SAP(S) \subset C(T)|_S$. Let (r_m) and (s_n) be nets in S converging to $t \varepsilon T$ such that the limits

$$x = \lim_m \alpha(r_m) \quad \text{and} \quad y = \lim_n \alpha(s_n)$$

exist in X, where (X, α) is the canonical SAP compactification of S. Choose any $s \varepsilon S \cap \mathrm{Cl}(TtT) = S \cap \mathrm{Cl}(StS)$. An obvious modification of the construction carried out in the proof of 3.2 yields nets (p_i), (q_i) in S and subnets $(r_{m(i)})$, $(s_{n(i)})$, all indexed by the same directed set $\{i\}$, such that $(p_i r_{m(i)} q_i)$ and $(p_i s_{n(i)} q_i)$ converge to s and the limits $u = \lim_i \alpha(p_i)$ and $v = \lim_i \alpha(q_i)$ exist in X. By the joint continuity of multiplication in X,

$$uxv = \lim \alpha(p_i r_{m(i)} q_i) = \alpha(s).$$

Similarly, $uyv = \alpha(s)$. Therefore, since X is a group, $x = y$. The desired conclusion now follows from 2.3.

4.7. COROLLARY. *If T is topologically simple, then*

$$SAP(S) = SAP(T)|_S.$$

4.8. *Remark.* Theorem 4.6 generalizes the main result of [9], which asserts that $SAP(S) = SAP(T)|_S$ provided that T is a topological semigroup and $(tT \cup Tt) \cap S \neq \emptyset$ for each $t \in T$. (See also [1, Corollary 10], which treats the case T and S topological groups, and also [3, Theorem 15.17], which treats the commutative case.) Examples given in [9] show that the conclusion of 4.6 may be false if condition (3) does not hold for each $t \varepsilon T$.

It seems appropriate at this point to comment on the absence in this section of any extension theorem for the AP case. As the example of 4.4(b) shows, the AP analogs of 4.6 and 4.7 are false. On the other hand, the AP analog of 4.3 is true, this being a consequence of 4.3, 2.1, and the AP analog of 4.1. However, if T is topologically left and right simple, then $AP(T) = SAP(T)$ and $AP(S) = SAP(S)$ (see Proposition 4.9, below), hence the AP analog of 4.3 is, in fact, included in 4.7. We have been unable to find any AP extension theorem which is *not* covered by 4.6 or 4.7. The following proposition (which will be useful again in Section 5) may provide some hint as to what class of semigroups admits an AP extension theorem which is not included in 4.7.

4.9. PROPOSITION. *Let S be a topologically simple semitopological semigroup such that $\mathrm{Cl}(sS) = \mathrm{Cl}(Ss)$ for some $s \varepsilon S$. If Y is a compact topological semigroup and $\beta : S \to Y$ a continuous homomorphism with dense range, then Y is a group. In particular, for such a semigroup S, $AP(S) = SAP(S)$.*

Proof. Since $\mathrm{Cl}(sS) = \mathrm{Cl}(Ss)$ is a closed ideal of S it must equal S. Therefore $xY = Yx = Y$, where $x = \beta(s)$. It follows that $yY = Yy = Y$ for all y in the closed subsemigroup of Y generated by x. In particular, there exists an idempotent $e \varepsilon Y$ such that $eY = Ye = eYe = Y$. Since S is topologically simple and multiplication in Y is jointly continuous, Y is simple. Therefore $Y = K(Y) = eYe$, so Y is a group (see section 2). Taking (Y, β) to be the AP compactification of S gives $AP(S) = SAP(S)$ (see section III.10 of [3]).

We conclude this section with some applications of the above results to the theory of semigroup representations. A *representation* of the semitopological semigroup S on the Banach space E is a homomorphism U from S into the semigroup $L(E)$ of all bounded linear operators on E, where multiplication in $L(E)$ is operator composition. The representation U is said to be *strongly* (respectively, *weakly*) *continuous* if for each $x \varepsilon E$ the mapping $U(\cdot)x$ is continuous in the norm (respectively, weak) topology of

E. A strongly (respectively, weakly) continuous representation $U : S \to L(E)$ is *almost periodic* (respectively, *weakly almost periodic*), abbreviated AP (respectively, WAP) if for each $x \in E$ the set $U(S)x$ is relatively norm compact (respectively, relatively weakly compact). A strongly continuous representation U is said to be *strongly almost periodic,* abbreviated SAP, if E is the closed linear span of all $U(S)$-invariant finite dimensional subspaces D of E with the property that $U(S)|_D$ is contained in a bounded group of operators in $L(D)$ (with identity the identity operator). The right translation representation $s \to R(s)$ is WAP on $WAP(S)$, AP on $AP(S)$ and SAP on $SAP(S)$ [5].

Part (c) of the following proposition was proved in [9]. The other parts are proved in a similar manner. (See also [6, p. 159] in connection with the converse of part (b).)

4.10. PROPOSITION. (a) *If $AP(S) = AP(T)|_S$, then every AP representation of S extends to an AP representation of T. The converse holds if S has a topological left identity.*

(b) *If $WAP(S) = WAP(T)|_S$, then every WAP representation of S extends to a WAP representation of T. The converse holds if S has a topological left identity.*

(c) *$SAP(S) = SAP(T)|_S$ if and only if every SAP representation of S extends to a SAP representation of T.*

4.11. COROLLARY. *If T is topologically left and right simple, then every WAP representation of S extends to a WAP representation of T.*

4.12. COROLLARY. *If T is topologically simple, then every SAP representation of S extends to a SAP representation of T.*

5. Application to Flows

In this section we consider the problem of extending the phase semigroup of a flow.

If Z is a compact topological space, then Z^Z will denote the set of all self maps of Z and $C(Z, Z)$ the subset of continuous maps. We shall always assume that Z^Z and $C(Z, Z)$ carry the semigroup operation of function composition and the topology of pointwise convergence. With respect to these, Z^Z is obviously a compact right topological semigroup and $C(Z, Z)$ a semitopological subsemigroup. A *flow* is a triple (S, Z, π), where S is a semitopological semigroup, Z a compact topological space, and $\pi : S \to C(Z, Z)$ a continuous homomorphism. As is customary, we shall write sz for $\pi(s)(z)$ and Sz for the set $\{sz : s \in S\}$. The symbol π is often suppressed from the notation (S, Z, π) when no confusion can result. The closure of $\pi(S)$ in Z^Z is called the *Ellis semigroup* of the flow and is denoted by $E(S, Z)$. It is easily verified that $E(S, Z)$ is a subsemigroup of Z^Z and hence $(E(S, Z), \pi)$ is a right topological compactification of S (see Section 2).

A flow (S, Z) is *distal* if $x = y$ whenever x and y are members of Z such that $\lim_k s_k x = \lim_k s_k y$ for some net (s_k) in S. From the relation $\mathrm{Cl}(Sz) = E(S, Z)(z)$ it is easy to see that (S, Z) is distal if and only if each member of $E(S, Z)$ is injective. The structure theory of compact right topological semigroups (see Section 2) may be used to give a quick proof of the following well-known result of R. Ellis [7, Proposition 5.3] : (S, Z) is distal if and only if $E(S, Z)$ is a group with identity the identity operator.

A flow (S, Z, π) is *point transitive* if $\mathrm{Cl}(Sz) = Z$ for some $z \in Z$; *equicontinuous* if $\pi(S)$ is an equicontinuous family; and *quasiequicontinuous* if $E(S, Z) \subset C(Z, Z)$. If (S, Z) is equicontinuous (respectively, quasiequicontinuous), then $E(S, Z)$ is a topological (respectively, semitopological) semigroup. The converse holds if (S, Z) is point transitive. Clearly, every equicontinuous flow is *jointly continuous,* i.e., the mapping $(s, z) \to sz$ is continuous.

Let S be a dense subsemigroup of a semitopological semigroup T. We say that the flow (S, Z, π) extends to the flow $(T, Z, \overline{\pi})$ if $\overline{\pi}|_S = \pi$. (This is not to be confused with the usual notion of extension of a flow as a homomorphic preimage.) Note that if (S, Z) extends to (T, Z) then $E(S, Z) = E(T, Z)$. Consequently, (T, Z) is equicontinuous, or distal, or quasiequicontinuous if and only if (S, Z) has the same property.

The following theorem provides the connection between the problem of extending flows and the problem of extending continuous, real-valued functions.

5.1. THEOREM. *Let S be a dense subsemigroup of a semitopological semigroup T.*

(a) *If $LMC(S) \subset LMC(T)|_S$, then every point transitive flow (S, Z) extends to a flow (T, Z).*

(b) *If $LUC(S) \subset LUC(T)|_S$, then every jointly continuous, point transitive flow (S, Z) extends to a flow (T, Z) with the same properties.*

(c) *If $WAP(S) \subset WAP(T)|_S$, then every quasiequicontinuous, point transitive flow (S, Z) extends to a flow (T, Z) with the same properties.*

(d) *If $AP(S) \subset AP(T)|_S$, then every equicontinuous, point transitive flow (S, Z) extends to a flow (T, Z) with the same properties.*

(e) *$SAP(S) \subset SAP(T)|_S$ if and only if every point transitive, equicontinuous, distal flow (S, Z) extends to a flow (T, Z) with the same properties.*

If S has a right identity, then the converses of (a)–(d) *hold.*

Proof. We prove only (c) and its converse; the proofs of the other parts are entirely similar.

Assume that $WAP(S) \subset WAP(T)|_S$, and let (S, Z, π) be a quasiequicontinuous, point transitive flow with $z_0 \in Z$ such that $\mathrm{Cl}(Sz_0) = Z$. Let $R : WAP(T) \to WAP(S)$ denote the restriction mapping,

$$R^* : WAP(S)^* \to WAP(T)^*$$

its dual, and let (X, α) and (Y, β) denote the canonical WAP compactifications of S and T, respectively. Then $\theta = R^*|_X$ is a topological isomorphism from X onto Y such that $\theta \circ \alpha = \beta \circ \iota$, where $\iota : S \to T$ denotes the inclusion mapping (Lemma 2.2). Since $(E(S, Z), \pi)$ is a semitopological compactification of S, there exists a continuous homomorphism $\gamma : X \to E(S, Z)$ such that $\gamma \circ \alpha = \pi$ (see Section 2). Let $\overline{\pi}$ denote the continuous homomorphism

$$\gamma \circ \theta^{-1} \circ \beta : T \to E(S, Z).$$

Since $\overline{\pi} \circ \iota = \gamma \circ \theta^{-1} \circ \beta \circ \iota = \gamma \circ \alpha = \pi$, the proof will be complete once we show that $\overline{\pi}(t)$ is a continuous mapping for $t \varepsilon T$. To this end let (z_n) be a net in Z converging to z, and let $(\overline{\pi}(t)(z_k))$ be a convergent subnet of $(\overline{\pi}(t)(z_n))$. Since $E(S, Z)(z_0) = Z$ and $\gamma \circ \theta^{-1} : Y \to E(S, Z)$ is surjective, there exists for each k a point $y_k \varepsilon Y$ such that $z_k = \gamma \circ \theta^{-1}(y_k)(z_0)$. We may assume that (y_k) converges to some $y \varepsilon Y$. Then

$$z = \lim_k \gamma \circ \theta^{-1}(y_k)(z_0) = \gamma \circ \theta^{-1}(y)(z_0)$$

and hence

$$\begin{aligned} \overline{\pi}(t)(z) &= \gamma \circ \theta^{-1}(\beta(t))(z) \\ &= [\gamma \circ \theta^{-1}(\beta(t))] \circ [\gamma \circ \theta^{-1}(y)](z_0) \\ &= \gamma \circ \theta^{-1}(\beta(t)y)(z_0) \\ &= \lim_k \gamma \circ \theta^{-1}(\beta(t)y_k)(z_0) \\ &= \lim_k \overline{\pi}(t)(z_k). \end{aligned}$$

Therefore $\overline{\pi}(t)(z)$ is the unique limit point of $(\overline{\pi}(t)(z_n))$, so $\overline{\pi}(t((z_n) \to \overline{\pi}(t)(z)$.

For the converse of (c), let e be a right identity for S, and assume that every point transitive, quasiequicontinuous flow (S, Z) extends to a flow (T, Z) (necessarily of the same type). Let (X, α) denote the canonical *WAP* compactification of S and define $\pi : S \to X^X$ by $\pi(s)(x) = \alpha(s)x$ $(s \varepsilon S, x \varepsilon X)$. Then (S, X, π) is obviously a flow, and the existence of a right identity in S implies that (S, X) is point transitive and that the mapping $x \to$ (left multiplication by x) is a topological isomorphism from X onto $E(S, X)$. Therefore (S, X, π) extends to a flow $(T, X, \overline{\pi})$. Given $f \varepsilon WAP(S)$, define $g \varepsilon C(T)$ by $g(t) = [\overline{\pi}(t)(\alpha(e))](f)$. Obviously $g|_S = f$, so by 4.1, $g \varepsilon WAP(T)$.

5.2. COROLLARY. *If T is a semitopological group and S has the f.i.p. for right ideals, then every jointly continuous, point transitive flow (S, Z) extends to a flow (T, Z) of the same type.*

Proof. 5.1(b) and 3.3.

5.3. COROLLARY. *If T is topologically left and right simple, then every quasiequicontinuous, point transitive flow (S, Z) extends to a flow (T, Z) of the same type.*

Proof. 5.1(c) and 4.3.

5.4. COROLLARY. *If T is topologically simple, then every equicontinuous, point transitive, distal flow (S, Z) extends to a flow (T, Z) of the same type. If, in addition, there exists some $s \varepsilon S$ such that $\mathrm{Cl}(sS) = \mathrm{Cl}(Ss)$, then every equicontinuous flow (S, Z) is necessarily distal, and therefore, if it is point transitive, extends to a flow (T, Z) of the same type.*

Proof. The first part follows from 5.1(e) and 4.7. For the second part, apply 4.9 to $(Y, \beta) = (E(S, Z), \pi)$, recalling 2.4(b).

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