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PROJECTIVELY UNSTABLE ELLIPTIC SURFACES

BY

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1. Introduction

Let k be an algebraically closed field. Following Artin [1], we call a flat proper map of k-schemes

$$X \xrightarrow{p} \mathbf{P}^1$$

a *rational Weierstrass fibration* if X is a reduced and irreducible rational surface over k, every geometric fibre of p is an irreducible curve of arithmetic genus 1, and a section $s : \mathbf{P}^1 \to X$ of p is given, not passing through the nodes or cusps of the fibres. Moreover, we will assume that X is normal, and that the generic fibre of p is smooth; in this case, we may resolve the singularities of X and obtain a rational elliptic surface

$$\tilde{X} \xrightarrow{\tilde{p}} \mathbf{P}^1$$

(with section) which we call the *induced elliptic surface*. In this situation one may represent X in Weierstrass form by the equation

$$y^2 = x^3 + A(t)x + B(t)$$

where A is a quartic and B a sextic polynomial in the parameter t of $\mathbf{P}^1$. The polynomials A and B are unique only up to an action of $SL(2) \times k^*$ on the space V of pairs of such polynomials. In [2] we studied the stability (in the sense of geometric invariant theory) of this action and proved the following.

THEOREM 1.1. (a) *The pair (A, B) is properly stable if and only if the induced elliptic surface $\tilde{X}$ has only reduced fibres.*

(b) *The pair (A, B) is strictly semi-stable if and only if $\tilde{X}$ has a fibre of type I_N^*.*

(c) *The pair (A, B) is unstable if and only if $\tilde{X}$ has a fibre of type* II*, III* *or* IV*.

In this paper we wish to study a different parameter space for rational Weierstrass fibrations, by embedding X into $\mathbf{P}^n$ as a cycle of degree d. The

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cycle X will then be represented by a point of the corresponding Chow variety on which $SL(n + 1)$ will act via change of coordinates in $\mathbf{P}^n$, and we conjecture that the stability of the Chow point of X is given by the same criteria as in Theorem 1.1. The main result of this work is the verification of this for a particular class of embeddings.

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2. Chow Forms and Chow Stability

Let X be a cycle of pure dimension r with degree d in $\mathbf{P}^n$. Let $[x_0, \ldots, x_n]$ be coordinates in $\mathbf{P}^n$. For every point $c = [c_0, \ldots, c_n]$ of $(\mathbf{P}^n)^*$, write H_c for the hyperplane defined by $\Sigma_{i=0}^n c_i x_i = 0$.

THEOREM 2.1. *In the above situation, there is an irreducible polynomial Φ_X, multihomogeneous of degree d in each of the $r + 1$ sets of variables $c^j = [c_0^j, \ldots, c_n^j]$, $j = 0, \ldots, r$, such that*

$$\Phi_X(c_i^j) = 0 \text{ iff } X \cap H_c^0 \cap H_c^1 \cap \cdots \cap H_c^r \neq \emptyset.$$

Moreover, Φ_X is unique up to scalar and determines the variety X. Φ_X is called the Chow form of X.

Proof. See [5].

The Chow form of X can be considered as a point of

$$\mathbf{P}\left(S^d\left(\bigwedge^{r+1}(k^{n+1})\right)^*\right),$$

identifying k^{n+1} with $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ by choosing the coordinates $[x_0, \ldots x_n]$ of $\mathbf{P}^n$. $SL(n + 1)$ acts naturally on this space, the action is irreducible, and the orbits are Chow forms Φ_X of projectively equivalent cycles X.

DEFINITION 2.2. In the above situation, a variety $X \subset \mathbf{P}^n$ is called *Chow-stable* if its Chow form is stable for the natural action of $SL(n + 1)$. *Chow-semi-stability* and *Chow-instability* are defined similarly.

Let $X \to \mathbf{P}^1$ be a rational Weierstrass fibration, inducing the elliptic surface $\tilde{X} \to \mathbf{P}^1$. Let D be an ample divisor on X. Embed X in projective space via the linear system $|MD|$ for some suitably large M. The following conjecture is motivated by Theorem 1.1:

CONJECTURE 2.3. (a) If $\tilde{X}$ has only reduced fibres, then X is Chow-stable for the given embedding.

(b) If $\tilde{X}$ has a fibre of type I_N^*, then X is Chow-semi-stable.

(c) If $\tilde{X}$ has a fibre of type II*, III*, or IV*, then X is Chow-unstable.

The rest of this paper will be devoted to proving a special case of (c):

THEOREM 2.4. *Let X be a rational Weierstrass fibration inducing the rational elliptic surface $\tilde{X}$ with section S. Let D_0 be the divisor $3S + 6F$ on $\tilde{X}$, where F denotes the fibre. Then for M sufficiently large, the image of $\tilde{X}$ mapped to $\mathbf{P}^n$ via the linear system $|MD_0|$ is X, and $X \subset \mathbf{P}^n$ is Chow-unstable if $\tilde{X}$ has a fibre of type* IV*.

That D is ample on X is clear by Nakai's criterion. Since D does not meet any component of any fibre which does not meet S, the image of $\tilde{X}$ is indeed the Weierstrass fibration X. The important statement of the theorem is the instability claim. We will prove that $X \subset \mathbf{P}^n$ is Chow-unstable by using Mumford's criterion (see [4, Theorem 2.9], or [3, Chapter I]), which we briefly recall in the next section.

Remarks. (1) The use of $D_0 = 3S + 6F$ instead of $S + 2F$ is merely a convenience, enabling one to perform the calculations of Section 5 more easily.

(2) Of the three singular fibres II*, III*, and IV* of $\tilde{X}$, the fibre IV* induces in some sense the "mildest" singularity on the Weierstrass fibration X (an $E6$ rational double point as opposed to $E7$ and $E8$). Hence Theorem 2.4 could be viewed as the strongest of three possible results in this direction. The other two results are true:

THEOREM 2.5. *With the notation of Theorem* 2.4, *for M sufficiently large the image of $\tilde{X}$ via the linear system $|MD_0|$ is Chow-unstable if $\tilde{X}$ has a fibre of type* II* *or* III*.

Here it is convenient to use $D_0 = 6S + 12F$ for the II* case and $4S + 8F$ for the III* case.

The calculations necessary in these two cases are completely analogous to the case proved here for a fibre of type IV* and we chose to present the calculation in only this one case. It would be preferable to give a uniform proof for all three cases, but as yet the estimates required in each case are sufficiently different to make this infeasible.

3. Multiplicities and Mumford's Criterion for Chow Stability

In this section we will define the various quantities used in the stability criterion and state the relevant facts about them. For proofs and for more detail, see [3] and [4].

PROPOSITION 3.1. *Suppose X is a k-variety of dimension r, L is an invertible sheaf on X and $I \subset \mathcal{O}_X$ is an ideal sheaf such that $Z = \operatorname{Supp}(\mathcal{O}_X/I)$ is* *proper over k. Then there is a polynomial $P(n, m)$ of total degree less than or equal to r, such that*

$$\chi(L^n/I^m L^n) = P(n, m)$$

for large m. Moreover, the polynomial $P(n, n)$ can be written in the form

$$P(n, n) = en^r/r! + (\text{lower order terms in } n)$$

for some **integer** *e.*

For any polynomial $f(n)$ of degree at most r, define the *normalized leading coefficient* of f to be the number e, such that

$$f(n) = \frac{en^r}{r!} + (\text{lower order terms});$$

denote it by $\mathrm{n\ell c}(f)$. With this terminology, we have the

DEFINITION 3.2. In the situation of Proposition 3.1, we denote by $e_L(I)$ (the multiplicity of I measured via L) the integer

$$\mathrm{n\ell c}\ \chi(L^n/I^n L^n).$$

The criterion for instability is expressed in terms of such a multiplicity. To calculate this number, one uses the following:

PROPOSITION 3.3. *If (in the situation of Proposition 3.1), L and $I \cdot L$ are generated by their sections, then*

$$\left| h^o(L^n/I^n L^n) - e_L(I)\frac{n^r}{r!} \right| = O(n^{r-1}).$$

This fact enables one to compute $e_L(I)$ in terms of the given h^o. This, however, is often not powerful enough; to compute the h^o, we use the following result:

PROPOSITION 3.4. *Assume X, I, L as above. Suppose in addition we are given a diagram*

$$\begin{array}{ccl} X & \underset{\neq}{\supset} & X_o = f^{-1}(0) \\ f\downarrow & & \downarrow \\ \operatorname{Spec} A & \ni & 0 \end{array}$$

where f is proper, and a finite-dimensional vector space $W \subset \Gamma(X, L)$ which

(i) *generates IL, and*
(ii) *defines a closed immersion $X - X_o \hookrightarrow \mathbf{P}(W^*)$.*

Then the dimensions of the kernel and cokernel of the map

$$\left[\frac{\Gamma(X, L^n)}{\begin{array}{l}\text{A-submodule generated}\\ \text{by the image of } W^{\otimes n}\end{array}}\right] \to \Gamma(L^n/I^nL^n)$$

are both $O(n^{r-1})$, enabling one to calculate $e_L(I)$ by computing the dimension of $\{\Gamma(X, L^n)/A\text{-submodule generated by the image of } W^{\otimes n}\}$ for large n.

To see how these multiplicities are used in computing stability, consider the following situation:

Let $X \subset \mathbf{P}^n$ be a projective variety of dimension r. Fix coordinates $[x_0, \ldots, x_n]$ on $\mathbf{P}^n$. Denote by Φ_X the Chow form of X. Let $\lambda : G_m \to SL(n + 1)$ be the 1-parameter subgroup of $SL(n + 1)$ defined by

$$\lambda(t) = \begin{bmatrix} t^{r_o} & & & \\ & t^{r_1} & & \\ & & \ddots & \\ & & & t^{r_n} \end{bmatrix} \cdot t^{-\Sigma r_i/n+1}$$

where $r_0 \geqslant r_1 \geqslant \cdots \geqslant r_n \geqslant 0$; we say that r_i is the *weight* of the coordinate x_i.

Define an ideal sheaf $I \subset \mathcal{O}_{X \times A^1}$ by letting $I\,[\mathcal{O}_X(1) \otimes \mathcal{O}_{A^1}]$ be the subsheaf generated by $\{t^{r_i}x_i\}$ for $i = 0, 1, \ldots, n$.

Denote by $e(I)$ the multiplicity of I measured via $O_X(1) \otimes \mathcal{O}_{A^1}$:

$$e(I) \equiv e_{\mathcal{O}_X(1)\otimes\mathcal{O}_{A^1}}(I).$$

We can now state Mumford's criterion for stability:

THEOREM 3.5. *Let $X \subset \mathbf{P}^n$ as above. Then X is Chow-stable (respectively Chow-semi-stable) if for every choice of coordinates $[x_o, \ldots, x_n]$ in $\mathbf{P}^n$ and for every 1-parameter subgroup λ of $SL(n + 1)$ given as above,*

$$(3.6) \qquad e(I) < \frac{(r + 1)\deg X}{n + 1} \sum_{i=0}^{n} r_i \quad (\textit{respectively} \leqslant).$$

The Chow-stability of a variety X in $\mathbf{P}^n$ is, even with the criterion above, still no easy matter to verify. A priori one must check all the 1-parameter subgroups of $SL(n + 1)$, and compute the appropriate multiplicity in each case. However, if one is intent on proving that a certain variety X is Chow-*un*stable, then one need only find a single 1-parameter subgroup for which the inequality (3.6) fails. The inexperienced stability reckoner may liken this search to that of the needle in a haystack, but in fact this subgroup has been seen empirically to be associated, when it exists, to a fairly concrete geometric property of the variety; in the case of a rational elliptic surface with a fibre of type IV*, the special singular fibre is the focal point for the "destabilizing" subgroup. Let us now turn our attention to this case, and with the aid of the Propositions 3.3 and 3.4, make the appropriate calculations exhibiting the surface as unstable.

4. The Destabilizing Subgroup

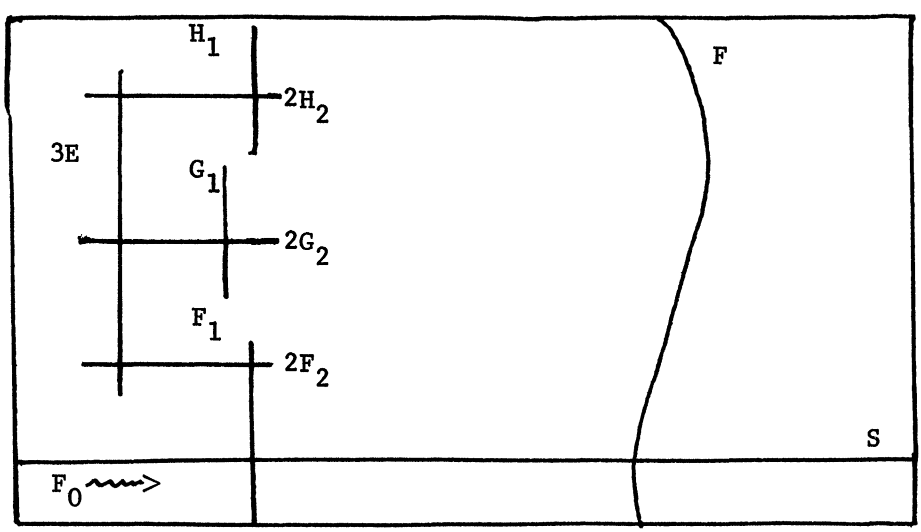
Let us fix the following notation:

$\tilde{X}$ a minimal rational elliptic surface with section and a fibre of type IV*;
X the associated Weierstrass fibration;
S the given section of $\tilde{X}$;
F the class of a fibre of $\tilde{X}$;
$K_{\tilde{X}} = -F$ the canonical divisor on $\tilde{X}$;
F_0 the singular IV* fibre;
F_1, F_2, G_1, G_2, H_1, H_2, and E the seven rational components of F_0, so that

$$F_0 = F_1 + 2F_2 + G_1 + 2G_2 + H_1 + 2H_2 + 3E, \qquad (F_1 \cdot S) = 1,$$

$$(F_1 \cdot F_2) = (F_2 \cdot E) = (G_1 \cdot G_2) = (G_2 \cdot E) = (H_1 \cdot H_2) = (H_2 \cdot E) = 1,$$

and all other pairwise intersections are 0;
$D = 3MS + 6MF$, the embedding divisor for X.
One may wish to picture the elliptic surface $\tilde{X}$ by the following graph:



Note that

$$(S^2) = -1, (F^2) = 0, (S \cdot F) = 1,$$

and

$$(F_1^2) = (F_2^2) = (G_1^2) = (G_2^2) = (H_1^2) = (H_2^2) = (E^2) = -2.$$

Thus the degree of X in $\mathbf{P}^n$ is given by

$$\deg X = (D^2) = 9M^2(S^2) + 36M(S \cdot F) = 27M^2. \tag{4.1}$$

Since D is very ample on X, for large enough M the higher cohomology groups $H^1(\tilde{X}, \mathcal{O}(D))$ and $H^2(\tilde{X}, \mathcal{O}(D))$ vanish; we can therefore calculate the dimension n of the ambient projective space $\mathbf{P}^n$ by computing the euler characteristic of the sheaf $\mathcal{O}(D)$:

$$\begin{aligned} n + 1 &= h^o(\tilde{X}, \mathcal{O}(D)) \\ &= \chi(\mathcal{O}(D)) \\ &= \frac{(D \cdot D - K_{\tilde{X}})}{2} + \chi(\mathcal{O}_{\tilde{X}}) \text{ (by Riemann-Roch)} \\ &= \frac{27M^2 + 3M + 2}{2}. \end{aligned} \tag{4.2}$$

We can now describe the 1-parameter subgroup which will exhibit $X \subset \mathbf{P}^n$ as Chow-unstable. Consider the following filtration of $\Gamma(\tilde{X}, \mathcal{O}(D))$:

$$\begin{aligned} (0) \subset \Gamma(D - (18ME)) \subset \Gamma(D - (18M - 1)E) \\ \subset \cdots \subset \Gamma(D - 2E) \subset \Gamma(D - E) \subset \Gamma(D) \end{aligned} \tag{4.3}$$

where $\Gamma(D - KE)$ is used to abbreviate $\Gamma(\tilde{X}, \mathcal{O}(D - KE))$. Choose a basis $[x_o, \ldots, x_n]$ of $\Gamma(D)$, inducing coordinates in $\mathbf{P}^n$, compatible with this filtration. Let λ be the 1-parameter subgroup of $SL(n + 1)$ defined by

$$\lambda(t) \cdot x_i = t^{r_i} x_i \tag{4.4}$$

where the exponent r_i is computed using the following recipe:

(4.5) If x_i is non-zero in $\Gamma(D - KE)/\Gamma(D - (K + 1)E)$ then $r_i = 18M - K$.

Thus as a diagonal matrix $\lambda(t)$ has the form

$$\lambda(t) = \begin{bmatrix} t^{18M} &&&&&&&&&&&& \\ & \ddots &&&&&&&&&&& \\ && t^{18M} &&&&&&&&&& \\ &&& t^{18M-1} &&&&&&&&& \\ &&&& \ddots &&&&&&&& \\ &&&&& t^{18M-1} &&&&&&& \\ &&&&&& t^{18M-2} &&&&&& \\ &&&&&&& \ddots &&&&& \\ &&&&&&&& t^{18M-K} &&&& \\ &&&&&&&&& \ddots &&& \\ &&&&&&&&&& t^{18M-K} && \\ &&&&&&&&&&& t^{18M-K-1} & \\ &&&&&&&&&&&& \ddots \\ &&&&&&&&&&&&& t^1 \\ &&&&&&&&&&&&&& t^0 \\ &&&&&&&&&&&&&&& \ddots \\ &&&&&&&&&&&&&&&& t^0 \end{bmatrix} \cdot t^{-\Sigma r_i/n+1}. \tag{4.6}$$

We say that r_i is the *weight* of the basis element x_i.

I claim that this choice of coordinates and this 1-parameter subgroup exhibits $X \subset \mathbf{P}^n$ as Chow-unstable. Using (3.6), we must compute the multiplicity $e(I)$ and the sum of the weights r_i, and show that

$$e(I) > \frac{(3)(27M^2)}{(27M^2 + 3M + 2)/2} \sum_{i=0}^{n} r_i. \tag{4.7}$$

For both of these calculations we must know the dimensions of the spaces $\Gamma(D - KE)/\Gamma(D - (K + 1)E)$; denote this dimension by d_K:

$$d_K = \dim \Gamma(D - KE)/\Gamma(D - (K + 1)E). \tag{4.8}$$

Using this notation, we have

$$\sum_{i=0}^{n} r_i = \sum_{K=0}^{18M} (18M - K) d_K \tag{4.9}$$

from the definition (4.5) of the r_i's. As we will see in Section 6, knowledge of the d_K's are also crucial in computing the multiplicity $e(I)$.

Remarks. (1) The above filtration of $\Gamma(D)$ is clearly given by the order of zero along the curve E which is of multiplicity three in the fibre of type IV*. It is for this reason that we use $D = 3MS + 6MF$ instead of $MS + 2MF$.

(2) The filtrations used in the proofs for the cases of the fibres of types II* and III* are also given by the order of zero along the unique curve of maximum multiplicity in the singular fibre in question. This curve is the multiplicity four curve in the III* fibre and the multiplicity six curve in the II* fibre.

5. The Computation of the $\mathbf{d}_K$

We will use the following technique for computing the dimensions d_K. The exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(D - (K + 1)E) \to \mathcal{O}_{\tilde{X}}(D - KE) \to \mathcal{O}_E((D - KE) \cdot E) \to 0$$

gives rise to the exact sequence of sections

$$0 \to \Gamma(D - (K + 1)\, E) \to \Gamma(D - KE) \to \Gamma(E, \mathcal{O}_E(2K)),$$

identifying $\mathcal{O}_E((D - KE) \cdot E)$ with $\mathcal{O}_E(2K)$. Since $\dim \Gamma(E, \mathcal{O}_E(2K)) = 2K + 1$, ($E$ is a smooth rational curve), the dimension d_K is bounded above by $2K + 1$:

$$d_K \leqslant 2K + 1.$$

This bound is *not* sharp; the main part of the ensuing argument is devoted to its sharpening. We need:

LEMMA 5.1. *Let A be a divisor on the surface $\tilde{X}$, and let C be a smooth irreducible curve on $\tilde{X}$. Assume $(A \cdot C) < 0$. Then C is a base curve of the linear system $|A|$, or equivalently*

$$\Gamma(X, \mathcal{O}_{\tilde{X}}(A)) \xleftarrow{\sim} \Gamma(X, \mathcal{O}_{\tilde{X}}(A - C)).$$

Proof. From the short exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(A - C) \to \mathcal{O}_{\tilde{X}}(A) \to \mathcal{O}_C \otimes \mathcal{O}_X(A) \to 0$$

we have the sequence of sections

$$0 \to \Gamma(X, \mathcal{O}_{\tilde{X}}(A - C)) \to \Gamma(X, \mathcal{O}_{\tilde{X}}(A)) \to \Gamma(C, \mathcal{O}_{\tilde{X}}(A)|_C),$$

and, by assumption, the degree of $\mathcal{O}_X(A)|_C$ on C is negative; hence $\Gamma(C, \mathcal{O}_X(A)|_C) = (0)$, proving the required isomorphism, *Q.E.D.*

Using the above lemma we can show that the linear system $|D - KE|$ has actual base locus B_K of the form

$$(5.2) \qquad B_K = s(K)S + f_1(K)F_1 + f_2(K)F_2 + g_1(K)G_1 + g_2(K)G_2 + h_1(K)H_1 + h_2(K)H_2 + e(K)E$$

instead of only the a priori base locus KE. Let us illustrate this by computing the "extra" base locus for the system $|D - E|$, as follows.

(i) $D - E \cdot H_2 = -1$, so, by the lemma, $|D - E| = |D - E - H_2|$.

(ii) $D - E - H_2 \cdot G_2 = -1$, so $|D - E| = |D - E - H_2 - G_2|$.

(iii) $D - E - H_2 - G_2 \cdot F_2 = -1$, so $|D - E| = |D - E - H_2 - G_2 - F_2|$.

(iv) $D - E - H_2 - G_2 - F_2 \cdot E = -1$, so $|D - E| = |D - 2E - H_2 - G_2 - F_2|$.

(v) $D - 2E - H_2 - G_2 - F_2 \cdot G_1 = -1$, so

$$|D - E| = |D - 2E - H_2 - G_2 - F_2 - G_1|.$$

(vi) $D - 2E - H_2 - G_2 - F_2 - G_1 \cdot H_1 = -1$, so

$$|D - E| = |D - 2E - H_2 - G_2 - F_2 - G_1 - H_1|.$$

(vii) $D - 2E - H_2 - G_2 - F_2 - G_1 - H_1 \cdot H_2 = -1$, so

$$|D - E| = |D - 2E - 2H_2 - G_2 - F_2 - G_1 - H_1|.$$

(viii) $D - 2E - 2H_2 - G_2 - F_2 - G_1 - H_1 \cdot G_2 = -1$, so

$$|D - E| = |D - 2E - 2H_2 - 2G_2 - F_2 - G_1 - H_1|.$$

(ix) $D - 2E - 2H_2 - 2G_2 - F_2 - G_1 - H_1 \cdot E = -1$, so

$$|D - E| = |D - 3E - 2H_2 - 2G_2 - F_2 - G_1 - H_1|.$$

(x) $D - 3E - 2H_2 - 2G_2 - F_2 - G_1 - H_1 \cdot F_2 = -1$, so

$$|D - E| = |D - 3E - 2H_2 - 2G_2 - 2F_2 - G_1 - H_1|.$$

The reader may check that the divisor

$$D - 3E - 2H_2 - 2G_2 - 2F_2 - G_1 - H_1$$

meets all of the curves S, F_1, F_2, G_1, G_2, H_1, H_2 and E non-negatively, and therefore no further inferences regarding the system $|D - E|$ may be made, using the lemma. Thus we set $B_1 = 3E + 2H_2 + 2G_2 + 2F_2 + G_1 + H_1$, and we then have

$$|D - E| = |D - B_1|. \tag{5.3}$$

Note that then

$$|D - B_1| = |D - 3E| = |D - 2E| = |D - E| \tag{5.4}$$

since $3E < B_1$. To sharpen the bound on d_0, d_1, etc., note that $d_0 \leqslant 1$ in any case, and by (5.4), we have $d_1 = d_2 = 0$. We set $B_2 = B_3 = B_1$ in keeping with our notation. To use B_3 to sharpen d_3, consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\tilde{X}}(D - 4E) \arrow[r] & \mathcal{O}_{\tilde{X}}(D - 3E) \arrow[r] & \mathcal{O}_E(D - 3E \cdot E) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\tilde{X}}(D - B_3 - E) \arrow[r] \arrow[u] & \mathcal{O}_{\tilde{X}}(D - B_3) \arrow[r] \arrow[u] & \mathcal{O}_E(D - B_3 \cdot E) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

This induces a diagram of global sections

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(D - 4E) \arrow[r] & \Gamma(D - 3E) \arrow[r] & \Gamma(E, D - 3E \cdot E) \\
0 \arrow[r] & \Gamma(D - B_3 - E) \arrow[r] \arrow[u] & \Gamma(D - B_3) \arrow[r] \arrow[u, "\wr"] & \Gamma(E, D - B_3 \cdot E),
\end{tikzcd}$$

where the middle vertical map is an isomorphism by the construction of B_3. Hence the image of $\Gamma(D - 3E)$ in $\Gamma(E, D - 3E \cdot E)$ is contained in the image of $\Gamma(E, D - B_3 \cdot E)$ in $\Gamma(E, D - 3E \cdot E)$; thus its dimension is bounded by the dimension of $\Gamma(E, D - B_3 \cdot E)$. But the dimension of the image of $\Gamma(D - 3E)$ is equal to d_3 by definition of d_3; we then have

$$d_3 \leqslant \dim \Gamma(E, D - B_3 \cdot E) = \dim \Gamma(E, \mathcal{O}_E) = 1 \tag{5.5}$$

since $D - B_3 \cdot E = 0$ and E is a smooth rational curve. To bound d_4 we compute B_4 with the same procedure used to compute B_1; we find that

$$B_4 = 4E + 2F_2 + 3G_2 + 3H_2 + 2G_1 + 2H_1.$$

A similar argument as that used above, applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(D - 5E) \arrow[r] & \Gamma(D - 4E) \arrow[r] & \Gamma(E, D - 4E \cdot E) \\
0 \arrow[r] & \Gamma(D - B_4 - E) \arrow[r] \arrow[u] & \Gamma(D - B_4) \arrow[r] \arrow[u, "\wr"] & \Gamma(E, D - B_4 \cdot E), \arrow[u]
\end{tikzcd}$$

shows that

$$d_4 \leqslant \dim \Gamma(E, D - B_4 \cdot E) = \dim \Gamma(E, \mathcal{O}_E) = 1 \tag{5.6}$$

since $D - B_4 \cdot E = 0$ again. One may compute

$$B_5 = 6E + 3F_2 + 4G_2 + 4H_2 + 2G_1 + 2H_1;$$

hence $B_5 = B_6$ and $d_5 = 0$; moreover,

$$d_6 \leqslant \dim \Gamma(E, D - B_6 \cdot E) = \dim \Gamma(E, \mathcal{O}_E(1)) = 2$$

by the same argument as above. In the general case we have the following result:

PROPOSITION 5.7. *Let B_K be defined as in* (5.2).

(i) *If $e(K) = K$, then $d_K \leqslant (D - B_K \cdot E) + 1$*
(ii) *If $e(K) > K$, then $d_K = 0$.*

Proof. As above consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(D - (K+1)E) \arrow[r] & \Gamma(D - KE) \arrow[r] & \Gamma(E, \mathcal{O}_E(D - KE\,E)) \\
0 \arrow[r] & \Gamma(D - B_K - E) \arrow[r] \arrow[u] & \Gamma(D - B_K) \arrow[r] \arrow[u, "\wr"] & \Gamma(E, \mathcal{O}_E(D - B_K E))
\end{tikzcd}$$

where the middle vertical map is an isomorphism by the construction of B_K. In case (i), we have

$$d_K \leqslant \dim \Gamma(E, \mathcal{O}_E(D - B_K \cdot E)) = (D - B_K \cdot E) + 1,$$

as above. In case (ii), note that the middle vertical isomorphism factors through $\Gamma(D - (K + 1)E)$; hence

$$\Gamma(D - (K + 1)E) \xrightarrow{\sim} (D - KE) \quad \text{and} \quad d_K = 0, \quad Q.E.D.$$

I will omit the tedious but elementary calculation of the B_K, and just present the results in Table 5.8.

Let u_K be the upper bound for d_K given by Proposition 5.7. From Table 5.8, in particular from the last column, we may read off these u_K's when $e(K) = K$.

The crucial observation to be made is given by the following result.

PROPOSITION 5.9. *For every K between* 0 *and* $18M$, $d_K = u_K$.

Proof. Using Table 5.8 and the standard formulas for sums of consecutive integers, one can readily calculate $\Sigma_{K=0}^{18M} u_K$; it turns out to be

$$\frac{1}{2}(27M^2 + 3M + 2).$$

Note here that this is the total dimension $n + 1$ of $\Gamma(D)$ by (4.2); since we know that $\Sigma_{K=0}^{18M} d_K = n + 1$ by definition, and $d_K \leqslant u_K$ for all K, we must in fact have equality for each K, *Q.E.D.*

We can therefore consider the last column of Table 5.8 (except for $K = 1, 2, 5$, and $18M - 1$, when $d_K = 0$) as a table of values for the dimensions d_K. Using these values and the description (4.9) for the sum of the weights,

Table 5.8
Base locus of $D - KE$: $D - B_K$

	$B_K = s(K)S + f_1(K)F_1 + f_2(K)F_2 + g_1(K)G_1 + g_2(K)G_2 + h_1(K)H_1 + h_2(K)H_2 + e(K)E$							
	K	$e(K)$	$f_1(K)$	$f_2(K)$	$g_1(K) = h_1(K)$	$g_2(K) = h_2(K)$	$s(K)$	$(D - B_K \cdot E) + 1$
	1	3	0	2	1	2	0	1
	2	3	0	2	1	2	0	1
	3	3	0	2	1	2	0	1
	4	4	0	2	2	3	0	1
	5	6	0	3	2	4	0	2
	6	6	0	3	2	4	0	2
For $7 \leqslant K \leqslant 6M$	$K \equiv 1(6)$	K	0	$\frac{K+1}{2}$	$\frac{K+2}{3}$	$\frac{2K+1}{3}$	0	$\frac{K-1}{6}$
	$K \equiv 2(6)$	K	0	$\frac{K}{2}$	$\frac{K+1}{3}$	$\frac{2K+2}{3}$	0	$\frac{K-2}{6}$
	$K \equiv 3(6)$	K	0	$\frac{K+1}{2}$	$\frac{K}{3}$	$\frac{2K}{3}$	0	$\frac{K+3}{6}$
	$K \equiv 4(6)$	K	0	$\frac{K}{2}$	$\frac{K+2}{3}$	$\frac{2K+1}{3}$	0	$\frac{K+2}{6}$
	$K \equiv 5(6)$	K	0	$\frac{K+1}{2}$	$\frac{K+1}{3}$	$\frac{2K+2}{3}$	0	$\frac{K-5}{6}$
	$K \equiv 0(6)$	K	0	$\frac{K}{2}$	$\frac{K}{3}$	$\frac{2K}{3}$	0	$\frac{K+6}{6}$
For $6M < K \leqslant 15M$	$K \equiv 1(3)$	K	$\frac{K+2}{3} - 2M$	$\frac{2K+1}{3} - M$	$\frac{K+2}{3}$	$\frac{2K+1}{3}$	0	M
	$K \equiv 2(3)$	K	$\frac{K+1}{3} - 2M$	$\frac{2K+2}{3} - M$	$\frac{K+1}{3}$	$\frac{2K+2}{3}$	0	$M - 1$
	$K \equiv 0(3)$	K	$\frac{K}{3} - 2M$	$\frac{2K}{3} - M$	$\frac{K}{3}$	$\frac{2K}{3}$	0	$M + 1$
For $15M < K \leqslant 18M - 2$	$K \equiv 1(3)$	K	$K - 12\,M$	$K - 6M$	$\frac{K+2}{3}$	$\frac{2K+1}{3}$	$K - 15M$	$6M - \frac{K-1}{3}$
	$K \equiv 2(3)$	K	$K - 12M$	$K - 6M$	$\frac{K+1}{3}$	$\frac{2K+2}{3}$	$K - 15M$	$6M - \frac{K+1}{3}$
	$K \equiv 0(3)$	K	$K - 12M$	$K - 6M$	$\frac{K}{3}$	$\frac{2K}{3}$	$K - 15M$	$6M - \frac{K-3}{3}$
	$18M - 1$	$18M$	$6M$	$12M$	$6M$	$12M$	$3M$	1
	$18M$	$18M$	$6M$	$12M$	$6M$	$12M$	$3M$	1

one can compute this sum by appealing repeatedly to the standard formulas for sums of consecutive integers and their squares. We find that

$$\sum_{i=0}^{n} r_i = \frac{225}{2}M^3 + 9M^2 + \frac{23}{2}M. \tag{5.10}$$

To complete the calculation, let us now compute the multiplicity $e(I)$.

6. The Calculation of the Multiplicity

In this, the least transparent of the series of calculations, we will use the Propositions 3.3 and 3.4. We consider the diagram

$$\begin{array}{ccc} X \times \mathbf{A}^1 & \hookleftarrow & X \\ {\scriptstyle F}\downarrow & & \downarrow \\ \operatorname{Spec} k[t] = \mathbf{A}^1 & \hookleftarrow & 0 \end{array}$$

where f is simply the projection onto the second factor. Let

$$L = \mathcal{O}_X(D) \otimes \mathcal{O}_{\mathbf{A}^1}$$

and let W be the subspace of $\Gamma(X \times \mathbf{A}^1, L)$ which is generated by the elements $\{t^{r_i}x_i\}$, where the x_i's and r_i's are chosen as in Section 4. Let I be the ideal in $\mathcal{O}_{X\times\mathbf{A}^1}$ defined by letting $I \cdot L$ be the subsheaf generated by W. By Theorem 3.5, the multiplicity we are interested in is $e_L(I)$.

Note that Proposition 3.4 applies in this situation: f is proper (X is proper over k), and by definition W generates $I \cdot L$. We need only check that W defines a closed immersion

$$X \times \mathbf{A}^1 - X_0 \hookrightarrow \mathbf{P}(W^*);$$

this follows from the definition of W and the fact that $\mathcal{O}_X(D)$ is very ample for X. We can therefore compute $e(I)$ by computing the dimension of the $\Gamma(X \times \mathbf{A}^1, L^n)/k[t]$-submodule generated by the image of $W^{\otimes n}$ (as in Proposition 3.4), for large n.

Since the ring $k[t]$ is graded by powers of t, we may view the module $\Gamma(X \times \mathbf{A}^1, L^n)$ as graded in this manner; by the Kunneth formula,

$$\begin{aligned} \Gamma(X \times \mathbf{A}^1, L^n) &\cong \Gamma(X, \mathcal{O}_X(nD)) \otimes_k k[t] \\ &\cong \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})(nD)) \otimes_k k[t] \\ &\cong \bigoplus_{L=0}^{\infty} [\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD)) \otimes_k (k \cdot t^L)]. \end{aligned}$$

Since W is generated by homogeneous elements of $\Gamma(X \times \mathbf{A}^1, L)$ (with respect to this grading), so is $W^{\otimes n}$; let $\bigoplus_{L=0}^{\infty} W_L^n \otimes (k \cdot t^L)$ denote the $k[t]$-submodule generated by $W^{\otimes n}$ where W_L^n is the *L-th* graded piece of this submodule. Using this notation, we have

$$(6.1) \qquad (X \times \mathbf{A}^1, L^n)/k[t]\text{-submodule generated by } W^{\otimes n}$$
$$\cong \bigoplus_{L=0}^{\infty} \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD)) \otimes (k \cdot t^L)/W_L^n \otimes (k \cdot t^L).$$

The problem is now reduced to the calculation of the dimension of

$$\Gamma(X, \mathcal{O}_{\tilde{X}}(nD)) \otimes (k \cdot t^L)/W^n \otimes k \cdot t^L$$

for each L, or equivalently

$$(6.2) \qquad \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD))/W_L^n.$$

PROPOSITION 6.3. *In the above situation,*

$$W_L^n \subset \text{image of } \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD - (18Mn - L)E))$$

under the natural inclusion

$$\Gamma(X, \mathcal{O}_X(nD - (18Mn - L)E) \subset \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD)),$$

for $L \leqslant 18Mn$.

Proof. The subspace W_L^n is generated by monomials $x_{i_1} \cdots\cdot x_{i_n}$ of degree n in the x_i's, such that $r_{i_1} + r_{i_2} + \cdots + r_{i_n} = L$. Recall that by Definition 4.5 of the r_i's, each x_i vanishes to order $18M - r_i$ on E; hence the above monomial vanishes to order

$$\sum_{i=1}^{n} (18M - r_i) = 18Mn - L$$

on E, *Q.E.D.*

Using Proposition 6.3, we have

$$\dim \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD))/W_L^n \geqslant \dim \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD))/\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD - (18Mn - L)E)$$

for $L \leqslant 18Mn$. Hence

$$(6.4) \qquad \begin{aligned} e(I) &= n\ell c\left[\dim \bigoplus_{L=0}^{\infty} (\tilde{X}, \mathcal{O}_{\tilde{X}}(nD)/W_L\right] \\ &\geqslant n\ell c\left[\dim \bigoplus_{L=0}^{18Mn} \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD))/\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD - (18Mn - L)E))\right] \\ &= n\ell c\left[\sum_{L=0}^{18Mn} [h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD)) - h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD - (18Mn - L)E))]\right] \\ &= n\ell c\left[\sum_{L=0}^{18Mn} [h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD)) - h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD - LE))]\right]. \end{aligned}$$

Let $d_K^n = \dim \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD - KE))/\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD - (K+1)E))$. Note that $d_K^1 = d_K$. The values of d_K^n may be read from the last column of Table 5.8 by replacing M by Mn wherever M occurs. Let us re-copy this table below.

Table 6.5

$d_K^n = \dim \Gamma(X, \mathcal{O}_X(nD - KE))/\Gamma(X, \mathcal{O}_X(nD - (K+1)E))$			
K	d_K^n	K	d_K^n
0	1	$6Mn + 1 \leqslant K \leqslant 15Mn$	
1	0	$K \equiv 1\ (3)$	Mn
2	0	$K \equiv 2\ (3)$	$Mn - 1$
3	1	$K \equiv 0\ (3)$	$Mn + 1$
4	1		
5	0		
6	2		
$7 \leqslant K \leqslant 6Mn$		$15Mn + 1 \leqslant K \leqslant 18Mn - 3$	
$K \equiv 1\ (6)$	$K - 1/6$	$K \equiv 1\ (3)$	$6Mn - (K-1)/3$
$K \equiv 2\ (6)$	$K - 2/6$	$K \equiv 2\ (3)$	$6Mn - (K+1)/3$
$K \equiv 3\ (6)$	$K + 3/6$	$K \equiv 0\ (3)$	$6Mn - (K-3)/3$
$K \equiv 4\ (6)$	$K + 2/6$		
$K \equiv 5\ (6)$	$K - 5/6$	$18Mn - 2$	1
$K \equiv 0\ (6)$	$K + 6/6$	$18Mn - 1$	0
		$18Mn$	1

Note that

$$h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD)) - h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(nD - LE) = \sum_{K=0}^{L-1} d_K^n.$$

Hence the estimate (6.4) for $e(I)$ can be written as

$$e(I) \geqslant \mathrm{n}\ell\mathrm{c}\left(\sum_{L=1}^{18Mn}\sum_{K=0}^{L-1} d_K^n\right)$$

$$= \mathrm{n}\ell\mathrm{c}\left(\sum_{K=0}^{18Mn-1}\sum_{L=K+1}^{18Mn} d_K^n\right) \tag{6.6}$$

$$= \mathrm{n}\ell\mathrm{c}\left(\sum_{K=0}^{18Mn-1}[18Mn - K - 1]d_K^n\right).$$

This sum is a cubic polynomial in n; to find its normalized leading coefficient, we take the coefficient of n^3 and multiply by 6. Hence in computing the

$n\ell c$, we may ignore any terms which are $O(n^2)$. Hence

$$e(I) \geqslant n\ell c \sum_{K=0}^{18Mn} [18Mn - K] d_K^n$$

$$= n\ell c \sum_{K=0}^{6Mn} [18Mn - K][K]$$

$$+ \sum_{K=6Mn}^{15Mn} [18Mn - K][Mn] + \sum_{K=15Mn}^{18Mn} [18Mn - K]\left[6Mn - \frac{K}{3}\right],$$

Using Table 6.5. Again invoking the standard formulas we find

$$e(I) \geqslant 675M^3. \tag{6.7}$$

This is the estimate needed to prove Theorem 2.4.

7. The Proof of Instability

By Theorem 3.5, it will suffice to show that, in our above notation,

$$e(I) > \frac{(r + 1)\deg X}{n + 1} \sum_{i=0}^{n} r_i. \tag{7.1}$$

Using the estimate (6.7) for $e(I)$, and the values (4.1), (4.2), and (5.10) for the terms $\deg X$, $n + 1$, and Σr_i respectively, it suffices to show that

$$675M^3 > \frac{(3)(27M^2)}{(27M^2 + 3M + 2)/2}\left(\frac{225}{2} M^3 + 9M^2 + \frac{23}{2} M\right),$$

or

$$675M^3 > \frac{(3)(27M^2)(225M^3 + 18M^2 + 23M)}{27M^2 + 3M + 2}.$$

Clearing denominators, it suffices to show that

$$675M^3(27M^2 + 3M + 2) > (3)(27M^2)(225M^3 + 18M^2 + 23M),$$

or

$$(27)(675)M^5 + 3(675)M^4 + 2(675)M^3$$
$$> (27)(675)M^5 + (3)(27)(18)M^4 + (3)(27)(23)M^3.$$

The coefficients of M^5 in these two polynomials are equal; hence we need only show that

$$3(675)M^4 + 2(675)M^3 > 3(27)(18)M^4 + 3(27)(23)M^3$$

or, dividing through by $27M^3$, that

$$75M + 50 > 54M + 69.$$

This is true for all $M \geqslant 1$, *Q.E.D.*

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