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THE MINIMAL PRIME SPECTRUM OF A REDUCED RING

BY

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Introduction

Throughout this discussion R will be a commutative ring with 1. We say R is a *reduced ring* if it has no nilpotent elements other than 0. Of course, this is equivalent to saying that the intersection of the minimal prime ideals of R is 0. The purpose of this paper is to study min R, the minimal prime spectrum of R, in order to obtain information about $Q(R)$, the classical ring of quotients of R; and $E(R)$, the injective envelope of R; as well as other properties of R.

Some of this information is already known. Thus in order to present more detailed results, a good deal of background information has to be used, imposing a severe strain on the general reader unfamiliar with the subject. Further compounding the problem is that much of the information is scattered wholesale about the literature. An even deeper difficulty is that this information, while relatively elementary in character, is usually thrown off as pieces of debris from general construction in the theory of non-commutative rings, or category and sheaf theory, so that no easy route to the subject is available.

In order to overcome these problems we shall present statements and proofs of most relevant facts about a reduced ring and its minimal prime spectrum including folklore and elementary exercises, as well as the work of other authors, giving attributions only for the deeper results.

In §1 we give some of the necessary background material. Because we are interested only in commutative rings, and specifically reduced rings, much of this material has been greatly simplified. We conclude §1 with an interesting contrast between reduced rings and non-reduced Noetherian rings.

The notion of a Von-Neumann regular ring, VNR, plays a key role in the subject. The definition that we use (among the many possible equivalent definitions) is that every principal ideal is a direct summand of the ring. This definition (in contrast to the definition of a semi-simple ring as a ring in which every ideal is a direct summand of the ring) shows that it is the set of principal ideals that matters. This definition gives rise immediately

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to two successively weaker ones: a commutative ring is a PIP if every principal ideal is projective; and it is a PIF if every principal ideal is flat. All of these rings are reduced rings.

In §2 we review the list of known equivalent conditions for a PIF to be a PIP, and add a few of our own. This problem dates back to the problem of finding necessary and sufficient conditions for a ring of weak global dimension 1 to be a semi-hereditary ring; and has been worked on by Hattori, Endo, Vasconcelos, and Quentel. One of the surprising results we prove is that a PIF ring R is a PIP if and only if C_M is a divisible R_M-module for all injective R-modules C and maximal ideals M of R. This provides new examples of rings whose injective modules do not localize.

Looking at the problem one ideal at a time, we find necessary and sufficient conditions for a principal ideal of a reduced ring (or a PIF) to be a projective ideal. Since a principal ideal is projective if and only if its annihilator is a direct summand of the ring, we analyze the conditions for an ideal to be a direct summand of R in terms of properties of subsets of min R.

If R is a reduced ring, one of the known theorems about $E(R)$ is that it is a self-injective VNR. If $P \in \min R$, then R_P is a field; and hence $E(R)$ is a direct summand of ΠR_P ($P \in \min R$). We prove in §3 that $E(R)$ is a subdirect product of the R_P's. There are examples where $E(R) \simeq \Pi R_P$ ($P \in \min R$); and others where $E(R) \simeq \Pi P_\beta$ ($P_\beta \in \Gamma \subset_{\neq} \min R$). Thus it is clear that the structure of $E(R)$ can be quite complex.

If R is a reduced ring and $\min R = \{P_1, \ldots, P_n\}$ is finite, then

$$Q(R) \simeq R_{P_1} \oplus \cdots \oplus R_{P_n} \simeq E(R).$$

There are two interesting generalizations of this theorem. On the one hand Quentel and others have shown (see Proposition 1.16) that min R is compact if and only if $E(R)$ is flat. On the other hand we show that min R is totally disconnected if and only if $E(R) \simeq \Pi R_P$ ($P \in \min R$) (see Proposition 3.5).

In §3 we prove that if $\{P_\beta\}$, $\beta \in \mathscr{B}$, is a subset of min R, then $E(R) \simeq \Pi R_{P_\beta}$ ($\beta \in \mathscr{B}$) if and only if the P_β's are distinct, every P_β is a non-essential ideal of R, and $\cap_\beta P_\beta = 0$. In this case $\{P_\beta\}$, $\beta \in \mathscr{B}$, is the set of all non-essential minimal prime ideals of R. Hence the decomposition (if it exists) is unique.

In the general case, if $\{P_\beta\}$, $\beta \in \mathscr{B}$, is the set of all distinct non-essential minimal prime ideals of R, and we let $I = \cap P_\beta$, then $E(R/I) \simeq \Pi R_{P_\beta}$; $E(R/I)$ is a ring direct summand of $E(R)$; the complementary summand is $E(R/K)$, where $K = \operatorname{ann}_R I$; and R/K is a reduced ring with no non-essential minimal prime ideals. Furthermore, there is a 1-1 correspondence between the non-essential minimal prime ideals of $E(R)$ and those of R given by contraction; and the corresponding localizations are isomorphic.

In §4 we give a number of examples to show that the general theorems of this paper provide efficient methods of deciding whether or not a ring is a PIF, or a PIP, and also of computing $E(R)$. In particular, we produce an example of a ring $R = K[[x]]$ (where K is a hereditary VNR) that is not a PIF; a fortiori, not a ring of w.gl.dim 1. Nevertheless, $Q(R)$ is a VNR, $E(R)$ is a direct product of copies of $k((x))$ where k is a field, and R is an essential extension of a semi-hereditary ring.

1. Preliminaries

DEFINITION. Let R be a commutative ring and A an R-module. We shall let $E(A)$ denote the injective envelope of A.

PROPOSITION 1.1. *Let R be a reduced ring, and let $\{P_\alpha\}$, $\alpha \subset \mathcal{A}$, be the set of minimal prime ideals of R.*

(1) *R_{P_α} is the quotient field of R/P_α, and hence is an injective R-module.*

(2) *$E(R)$ is a direct summand of $\Pi_\alpha R_{P_\alpha}$.*

(3) *$\cup_\alpha P_\alpha$ is the set of all zero divisors of R.*

Proof. (1) Let $O_\alpha = \{r \in R | ur = 0$ for some $u \in R - P_\alpha\}$. Then O_α is an ideal of R and $O_\alpha \subset P_\alpha$. Since $P_\alpha R_{P_\alpha}$ is the only prime ideal of R_{P_α}, every element of $P_\alpha R_{P_\alpha}$ is nilpotent. Thus if $p \in P$, there exists $u \in R - P_\alpha$ and $n > 0$ such that $up^n = 0$. Hence $(up)^n = 0$, and since R is reduced, $up = 0$. Thus $O_\alpha = P_\alpha$, and hence $P_\alpha R_{P_\alpha} = 0$. Therefore, R_{P_α} is the quotient field of R/P_α; and since R_{P_α} is a flat R-module, R_{P_α} is an injective R-module.

(2) It follows from (1) that $\Pi_\alpha R_{P_\alpha}$ is an injective R-module; and that the canonical map $R \to \Pi_\alpha R_{P_\alpha}$ has kernel equal to $\cap P_\alpha = 0$, and hence is a monomorphism. Thus the canonical map extends to a monomorphism: $E(R) \to \Pi_\alpha R_{P_\alpha}$.

(3) It follows from (1) that every element of $\cup_\alpha P_\alpha$ is a zero divisor in R. Conversely, let $x \in R$, $x \neq 0$ be a zero divisor in R. Then there exists $y \in R$, $y \neq 0$ such that $xy = 0$. Since $\cap P_\alpha = 0$, there exists P_β such that $y \notin P_\beta$; and hence $x \in P_\beta$.

DEFINITION. Let A be a subset of an R-module B. Then define $\mathrm{Ann}_R A = \{r \in R | rA = 0\}$.

PROPOSITION 1.2. *Let R be a reduced ring.*

(1) *A prime ideal P of R is a minimal prime ideal of R iff for all $x \in P$, $\mathrm{Ann}_R x \not\subset P$.*

(2) *Let J be a finitely generated ideal of R. Then J is contained in a minimal prime ideal of R if and only if $\mathrm{Ann}_R J \neq 0$.*

(3) *If $x \in R$ and $y \in \mathrm{Ann}_R x$, then $\mathrm{Ann}_R(Rx + Ry) = 0$ iff $x - y$ is not a zero divisor in R.*

Proof. (1) If P is a minimal prime ideal of R and $x \in P$, then by Proposition 1.1(1), there exists $u \in R - P_\alpha$ such that $ux = 0$. Conversely, suppose that for all $x \in P$, $\mathrm{Ann}_R x \not\subset P$. Suppose P_1 is a prime ideal of R and $P_1 \subset_{\neq} P$. Then there exists $x \in P - P_1$, and hence $\mathrm{Ann}_R x \subset P_1 \subset P$. This contradiction shows that P is a minimal prime ideal of R.

(2) Let $J = Ra_1 + \cdots + Ra_n$, and let $I = \mathrm{Ann}_R J$. Suppose that $J \subset P$, a minimal prime ideal of R. Then by Proposition 1.1(1), there exist elements $u_i \in R - P$ such that $ua_i = 0$ for all $i = 1, \ldots, n$. Let $u = u_1 u_2 \ldots u_n$; then $u \notin P$ and $u \in I$. Conversely, suppose that $I \neq 0$. Then there is a minimal prime ideal P of R such that $I \not\subset P$, and hence $J \subset P$.

(3) Assume that $\mathrm{Ann}_R(Rx + Ry) = 0$, and suppose that $t \in R$ and $t(x - y) = 0$. Then $tx = ty$ and hence $(tx)^2 = 0$. Therefore, $tx = 0 = ty$ and hence $t \in \mathrm{Ann}_R(Rx + Ry) = 0$. Thus $x - y$ is not a zero divisor in R. The converse assertion is trivial.

DEFINITION. A commutative ring R is said to be a *Von-Neumann regular ring* (*VNR*) if every principal ideal of R is a direct summand of R; i.e., is generated by an idempotent of R.

PROPOSITION 1.3 [8, Theorem 1.16]. *Let R be a commutative ring. Then R is a VNR iff R is reduced and every prime ideal of R is minimal. In this case every ideal of R is an intersection of prime ideals of R.*

Proof. Assume that R is a VNR. Let I be an ideal of R and x an element of R such that $x^n \in I$ for some $n > 0$. Since $Rx = Re$, where $e^2 = e$, we have $x \in I$, showing that I is an intersection of prime ideals of R. In particular, taking $I = 0$, we see that R is reduced. Now let $I = P$ be a prime ideal of R. Then $1 - e \in \mathrm{Ann}_R x$ and $1 - e \notin P$. Hence by Proposition 1.2(1), P is a minimal prime ideal of R.

Conversely, suppose that R is reduced and that every prime ideal of R is minimal. Let $0 \neq x \in R$ and $I = \mathrm{Ann}_R x$. Since R is reduced, $Rx \cap I = 0$; and by Proposition 1.2(1), $Rx + I$ is not contained in any minimal prime ideal of R. Therefore, $Rx + I = R$, Rx is a direct summand of R; and hence R is a VNR.

DEFINITION. Let R be a commutative ring and let S be the set of non-zero divisors of R. Then R_S is the *classical ring of quotients of* R, and we shall denote it by $Q(R)$.

PROPOSITION 1.4 [12, Proposition 9]. *Let R be a reduced ring. Then the following statements are equivalent:*

(1) *$Q(R)$ is a VNR.*

(2) *If I is an ideal of R contained in the union of the minimal prime ideals of R, then I is contained in one of them.*

(3) *If J is a finitely generated ideal of R, then there exist $b \in J$ and $a \in \mathrm{Ann}_R J$ such that $a + b$ is not a zero divisor in R.*

(4) *If* $b \in R$ *then there exists* $a \in \mathrm{Ann}_R b$ *such that* $\mathrm{Ann}_R(Ra + Rb) = 0$.

Proof. (1) ⇒ (2) Suppose that I is contained in the union of the minimal prime ideals of R. Then by Proposition 1.1(3), every element of I is a zero divisor in R. Thus $QI \neq Q$, and so $QI \subset \mathcal{P}$, a maximal ideal of Q; and $I \subset \mathcal{P} \cap R$. By Proposition 1.3, $\mathcal{P}$ is a minimal prime ideal of Q; and since Q is a localization of R, $\mathcal{P} \cap R$ is a minimal prime ideal of R.

(2) ⇒ (3) Let $J = Rb_1 + \cdots + Rb_n$ be a finitely generated ideal of R, and $I = \mathrm{Ann}_R J$. Suppose that there do not exist elements $b \in J$ and $a \in I$ such that $a + b$ is a non-zero divisor in R. Then $I + J$ is contained in the union of the minimal prime ideals of R, and hence by hypothesis, there exists a minimal prime P of R such that $I + J \subset P$. By Proposition 1.2(1), there exists $c_i \in \mathrm{Ann}_R\, b_i$ with $c_i \notin P$. Let $c = c_1c_2 \ldots c_n$; then $c \in I$ and $c \notin P$. This contradiction proves that there exists $b \in J$ and $a \in I$ where $a + b$ is a non-zero divisor in R.

(3) ⇒ (4) This follows from Proposition 1.2(3).

(4) ⇒ (1) Let $q \in Q = Q(R)$; then there exists $b \in R$ with $Qq = Qb$. By hypothesis, there is an $a \in \mathrm{Ann}_R b$ such that $\mathrm{Ann}_R(Ra + Rb) = 0$. By Proposition 1.2(3), $b - a$ is not a zero divisor in R. Thus, $Qb + Qa = Q$; and since Q is reduced, $Qb \cap Qa = 0$. Therefore, Qb is a direct summand of Q, and hence Q is a VNR.

PROPOSITION 1.5. *Let R be a commutative ring and* $\{P_1, \ldots, P_n\}$ *a finite set of distinct minimal prime ideals of R. Let* $S = R - \cup_{i=1}^n P_i$; *then* $R_S \simeq R_{P_1} \oplus \cdots \oplus R_{P_n}$.

Proof. $\{(P_1)_S, \ldots, (P_n)_S\}$ is the set of all prime ideals of R_S, and each of them is both maximal and minimal in R_S. Moreover, $(R_S)_{(P_i)_S} \simeq R_{P_i}$ for $i = 1, \ldots, n$. Thus without loss of generality we can assume that $\{P_1, \ldots, P_n\}$ is the set of all prime ideals of R, and that each of them is both maximal and minimal in R.

Let $O_i = \{r \in R \mid$ there exists $u \in R - P_i$ with $ur = 0\}$. Since $P_iR_{P_i}$ is the only minimal prime ideal of R_{P_i}, every element of P_i is nilpotent modulo O_i. Thus P_i is the only prime ideal of R containing O_i. Therefore $R/O_i = R_{P_i}$, $i = 1, \ldots, n$; and $O_i + O_j = R$, $i \neq j$. The annihilator of an element of $\cap_{i=1}^n O_i$ is not contained in any maximal ideal of R and thus $\cap_{i=1}^n O_i = 0$. Hence by the Chinese Remainder Theorem,

$$R \simeq R/O_1 \oplus \cdots \oplus R/O_n = R_{P_1} \oplus \cdots \oplus R_{P_n}.$$

PROPOSITION 1.6. *Let R be a reduced ring with only a finite number of minimal prime ideals* $\{P_1, \ldots, P_n\}$. *Then* $Q(R) \simeq R_{P_1} \oplus \cdots \oplus R_{P_n} \simeq E(R)$. *Hence Q(R) is a self-injective VNR (in fact a semi-simple ring), and E(R) is a flat R-module.*

Proof. By Proposition 1.5, $Q(R) \simeq R_{P_1} \oplus \cdots \oplus R_{P_n}$. Thus by Proposition 1.1 R_{P_i} is a field, and $Q(R)$ is an injective R-module. Since $Q(R)$ is an essential extension of R, we have $Q(R) \simeq E(R)$.

PROPOSITION 1.7. *If R is a reduced, self-injective ring, then R is a VNR. In this case if J is any ideal of R and $I = \mathrm{Ann}_R J$, then $R = E(J) \oplus I$.*

Proof. Since R is reduced, $I \cap J = 0$; in addition, $J \oplus I \subset R$ is an essential extension. Thus $R = E(R) = E(J \oplus I) = E(J) \oplus E(I)$. Now $E(I)J \subset E(I) \cap E(J) = 0$, and thus $E(I) \subset I$. Hence $I = E(I)$, and $R = E(J) \oplus I$.

Now suppose that $J = Ra$, $a \in R$. Then by the preceding paragraph we have $I = Re$, where $e^2 = e$; and hence $Ra \subset E(Ra) = R(1 - e)$. Since $\mathrm{Ann}_R(1 - e) = Re = \mathrm{Ann}_R a$, there is an R-homomorphism $f : Ra \to R(1 - e)$ with $f(a) = 1 - e$. Since R is self-injective, f extends to an R-homomorphism from R into R. Thus there exists $t \in R$ such that $1 - e = f(a) = ta$. Therefore, $R(1 - e) \subset Ra$, and so $R(1 - e) = Ra$. Thus Ra is a direct summand of R and hence R is a VNR.

PROPOSITION 1.8. *Let R be a VNR and let $\{x_\gamma\}$, $\gamma \in \Gamma$, be a set of generators for an ideal I of R. If $f : I \to R$ is an R-homomorphism, then there exists a set of elements $\{a_\gamma\}$, $\gamma \in \Gamma$, in R such that $f(x_\gamma) = a_\gamma x_\gamma$ for $\gamma \in \Gamma$; and the system of congruences*

$$y \equiv a_\gamma \bmod(\mathrm{Ann}_R\, x_\gamma), \quad \gamma \in \Gamma,$$

is finitely solvable. Conversely, if the system is finitely solvable, then there is an R-homomorphism f: $I \to R$ so that $f(x_\gamma) = a_\gamma x_\gamma$, $\gamma \in \Gamma$.

Proof. Let $f : I \to R$ be an R-homomorphism; and let $x \in I$. Since R is a VNR, we have $Rx = Rx^2$, and thus there is an $a \in I$ with $f(x) = ax$. In particular there is a set of elements $\{a_\gamma\}$, $\gamma \in \Gamma$, in I with $f(x_\gamma) = a_\gamma x_\gamma$, $\gamma \in \Gamma$. Let $\{x_{\gamma_1}, \ldots, x_{\gamma_n}\}$ be any finite subset of the generators $\{x_\gamma\}$. Then there is an $x \in I$ with $Rx_{\gamma_1} + \cdots + Rx_{\gamma_n} = Rx$; and hence there are elements $s_i \in R$ with $x_{\gamma_i} = s_i x$, $i = 1, \ldots, n$. Since $f(x) = ax$, we have

$$a_{\gamma_i} x_{\gamma_i} = f(x_{\gamma_i}) = s_i f(x) = a s_i x = a x_{\gamma_i}.$$

Therefore $a \equiv a_{\gamma_i} \bmod(\mathrm{Ann}_R\, x_{\gamma_i})$, and the system of congruences $y \equiv a_\gamma \bmod(\mathrm{Ann}_R\, x_\gamma)$ is finitely solvable.

Conversely, suppose that the system is finitely solvable, and define $f : I \to R$ by $f(x_\gamma) = a_\gamma x_j$, and extend f linearly to all of I. With the notations of the preceding paragraph, suppose that $\Sigma_{i=0}^n r_i x_{\gamma_i} = 0$, where $r_i \in R$. Since $a_{\gamma_i} x_{\gamma_i} = a x_{\gamma_i}$, we have $\Sigma\, r_i a_{\gamma_i} x_{\gamma_i} = a \Sigma\, r_i x_{\gamma_i} = 0$. Thus, f is a well defined R-homomorphism.

DEFINITION. Let R be a commutative ring with 1, and

$$y \equiv a_\gamma \,(\text{mod}\, I_\gamma), \quad \gamma \in \Gamma,$$

a system of congruences where the a_γ's are elements of R and the I_γ's are principal ideals of R. If every such system that is finitely solvable has a simultaneous solution, we shall say that R is *linearly compact on principal ideals*.

PROPOSITION 1.9. *Let R be a reduced ring. Then R is self-injective iff R is a VNR and is linearly compact on principal ideals.*

Proof. By Proposition 1.7 we can assume that R is a VNR. Then the principal ideals of R are exactly the annihilators of elements of R. Now R is self-injective iff every R-homomorphism from an ideal I of R can be realized by multiplication by an element $a \in R$. By Proposition 1.9 the R-homomorphisms from I into R arise from finitely solvable systems of congruences $y \equiv a_\gamma$ mod($\text{Ann}_R\, x_\gamma$) where the a_γ's are in R and the x_γ's generate I. It is immediate that f is multiplication by an element $a \in R$ iff a is a simultaneous solution of the system of congruences. Therefore, R is self-injective iff R is linearly compact on principal ideals.

DEFINITION. Let R be a commutative ring with 1, $E = E(R)$, and $H = \text{Hom}_R(E, E)$. Let

$$\mathcal{I} = \{f \in H | f(1) = 0\}.$$

Then $\mathcal{I}$ is a left ideal of H. Define $\phi : H \to E$ by $\phi(h) = h(1)$, $h \in H$. Then ϕ is an H-homomorphism of H onto E with Ker $\phi = \mathcal{I}$. Thus $E \simeq H/\mathcal{I}$ is a cyclic H-module with generator $1 \in R$. We have $\phi(I) = 1$, where I is the identity map on E. Since E is a faithful R-module we have a canonical injection $R \subset H$ sending 1 to I. If $Q = Q(R)$, then it is readily seen that E is the Q-injective envelope of Q; and that the injection $R \subset E$ extends to an injection $Q \subset E$.

PROPOSITION 1.10 [10, Proposition 3, p. 95]. *Let R be a commutative ring. Then the following statements are equivalent:*

(1) *$\mathcal{I}$ is a two-sided ideal of H.*
(2) *$\mathcal{I} = 0$.*
(3) *$H \simeq E$ as H-modules.*
(4) *E is a projective H-module.*
(5) *H is a commutative ring.*

Proof. The implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (4), and (5) $\Rightarrow$ (1) are trivial. For (2) $\Rightarrow$ (3) we observe that ϕ is then an isomorphism. And for (1) $\Rightarrow$

(2), let $f \in \mathcal{I}$ and $x \in E$. Since there is an $h \in H$ with $h(1) = x$, and $fh \in \mathcal{I}$, we have $f(x) = 0$. Thus $f = 0$ and hence $\mathcal{I} = 0$.

(4) $\Rightarrow$ (2) Since ϕ is onto and E is projective, there is an H-homomorphism $\lambda : E \to H$ with $\phi\lambda = I$. Let $g = \lambda(1)$; then $1 = I(1) = \phi\lambda(1) = \phi(g) = g(1)$. Thus Ker $g \cap R = 0$, and since E is an essential extension of R, we have Ker $g = 0$. But then Im g is an injective R-module containing R, and hence Im $g = E$. Thus g is an isomorphism. Hence $I = g^{-1}g = g^{-1}(\lambda(1)) = \lambda(g^{-1}(1))$ and thus $I \in$ Im λ. Since Im λ is a left ideal of H, we have Im $\lambda = H$. Thus, if $f \in \mathcal{I} =$ Ker ϕ, then there is an $x \in E$ with $f = \lambda(x)$, and hence $0 = \phi(f) = \phi\lambda(x) = x$. Therefore, $f = \lambda(0) = 0$, and so $\mathcal{I} = 0$.

(2) $\Rightarrow$ (5) Let $h \in H$ and $h(1) = x \in E$. Define $h_x : E \to E$ as follows: if $y \in E$, then there is a unique $g \in H$ with $g(1) = y$ because $\mathcal{I} = 0$. We define $h_x(y) = g(x)$. It is obvious that h_x is an Abelian group homomorphism of E into E. Let $k \in H$; then kg is the unique element of H such that $(kg)(1) = k(y)$. Hence $h_x(ky) = kg(x) = kh_x(y)$. Thus $h_xk = kh_x$, $k \in H$. Therefore $h_x \in H$; and in fact h_x is in the center of H. Now $h_x(1) = I(x) = x = h(1)$. Hence $h_x - h \in \mathcal{I} = 0$. Thus $h_x = h$, and so h is in the center of H. Thus H is a commutative ring.

DEFINITION. Let B be a submodule of an R-module A, and let $x \in A$. Then we define

$$(B : x) = \{r \in R | rx \in B\}.$$

We say that B is an *essential submodule* of A (or A an *essential extension* of B) if every non-zero submodule of A has a non-zero intersection with B.

PROPOSITION 1.11. *Let R be a commutative ring, $E = E(R)$, and $H = \mathrm{Hom}_R(E, E)$. Let $x \in E$; then $(R : x)$ is an essential ideal of R. Moreover, $\exists f \in H \ni f(1) = 0$ and $f(x) \neq 0$ iff $\mathrm{Ann}_R(R : x) \neq 0$.*

Proof. Let $r \in R$; if $rx = 0$, then $r \in (R : x)$; while if $rx \neq 0$, then there is a $t \in R$ with $0 \neq trx \in R$, and hence $0 \neq tr \in (R : x)$. Therefore $(R : x)$ is an essential ideal of R.

If there is an $f \in H$ with $f(1) = 0$ and $f(x) \neq 0$, then there is an $a \in R$ such that $af(x) = s \in R$ and $s \neq 0$. Clearly $s \in \mathrm{Ann}_R(R : x)$. Conversely, if there is $0 \neq s \in \mathrm{Ann}_R(R : x)$, then there is an R-homomorphism $g : R + Rx \to Rs$ with $g(1) = 0$ and $g(x) = s$. Because E is injective, g extends to an element of $f \in H$.

PROPOSITION 1.12 [10, Proposition 1, p. 102]. *Let R be a reduced ring and $E = E(R)$. Then E is a commutative, self-injective VNR, and $\mathrm{Hom}_R(E, E) = \mathrm{Hom}_E(E, E) \simeq E$.*

Proof. Let $H = \mathrm{Hom}_R(E, E)$, and suppose that there is an $f \in H$ and $x \in E$ with $f(1) = 0$ and $f(x) \neq 0$. By Proposition 1.11, $\mathrm{Ann}_R(R : x) \neq 0$ and $(R : x)$ is an essential ideal of R. But then $(R : x) \cap \mathrm{Ann}_R(R : x) \neq 0$, contradicting the fact that R is a reduced ring. Hence the map $\phi : H \to E$ defined by $\phi(h) = h(1)$, $h \in H$, is an H-isomorphism. Therefore, by Proposition 1.10, H is a commutative ring extension of R. Since ϕ is the identity on R, we can use ϕ to give E the structure of a commutative ring extension of R such that $\mathrm{Hom}_R(E, E) = \mathrm{Hom}_E(E, E)$.

Let $x \in E$, $x \neq 0$, and suppose that $x^2 = 0$. Since E is an essential extension of R there exists $r \in R$ with $0 \neq rx \in R$. But $(rx)^2 = r^2x^2 = 0$; and this contradiction shows that E is a reduced ring.

Let $\mathcal{I}$ be an ideal of E and $f : \mathcal{I} \to E$ and E-homomorphism. Because E is R-injective, f extends to an element $g \in H$. But then g is an E-homomorphism, and hence E is a self-injective ring. Thus by Proposition 1.9, E is a VNR.

Remarks. Let R be a reduced ring and $E = E(R)$. Since $\mathrm{Hom}_R(E, E) = \mathrm{Hom}_H(E, E) \simeq E$ is a commutative ring extension of R, it follows readily that if A is another injective envelope of R with a commutative ring structure extending that of R, and if $\theta : E \to A$ is an R-homomorphism that is the identity on R, then θ is a ring isomorphism.

PROPOSITION 1.13. *Let R be a reduced ring and $E = E(R)$. Suppose that A and B are R-submodules of E such that $E = A \oplus B$. Then A and B are ideals of E, and* $\mathrm{Hom}_R(A, B) = 0$.

Proof. Let f be the element of $\mathrm{Hom}_R(E, E)$ that is 0 on A and the identity on B. Then by Proposition 1.12, f is multiplication by $e \in E$ and $e^2 = e$. Thus $B = Ee$ and $A = E(1 - e)$; and hence A and B are ideals of E. If $g \in \mathrm{Hom}_R(A, B)$, define $h \in \mathrm{Hom}_R(E, E)$ to be g on A and 0 on B. Then h is multiplication by $y \in E$, and hence $g(A) = yA \subset A \cap B = 0$. Thus $g = 0$.

DEFINITION. Let R be a reduced ring and let min R be the minimal prime spectrum of R. If $x \in R$, define $D(x) = \{P \in \min R | x \notin P\}$. Then the sets of the form $D(x)$ form a basis for the Zariski topology on min R. When we say that min R is *compact,* we mean that it is compact in this topology.

PROPOSITION 1.14 [12, Lemma 1]. *Let R be a reduced ring, and let A be a commutative ring extension of R.*

(1) *If every prime ideal of A contracts to a minimal prime ideal of R, then* min R *is compact.*

(2) *Assume that A is a VNR. Then A is a flat R-module iff every prime ideal of A contracts to a minimal prime ideal of R. Hence in this case* min R *is compact.*

Proof. (1) Suppose that we have an open cover of min R. Then without loss of generality we can assume it is of the form min $R = \cup_\lambda D(x_\lambda)$, $\lambda \in \Lambda$, $x_\lambda \in R$. Let

$$D_A(x_\lambda) = \{\mathcal{P} \in A | x_\lambda \notin \mathcal{P}\};$$

then as a consequence of our hypothesis we have spec $A = \cup_\lambda D_A(x_\lambda)$. Since the spec of any commutative ring is compact, there exist $x_{\lambda_1}, \ldots, x_{\lambda_n}$ such that

$$\operatorname{spec} A = \bigcup_{i=1}^{n} D_A(x_{\lambda_i}).$$

Let $P \in \min R$; since $R_P \subset A_P$, and R_P is a field, it is easily seen that there is a prime ideal $\mathcal{P}$ of A with $\mathcal{P} \cap R = P$. It follows from this that min $R = \cup_{i=1}^{n} D(x_{\lambda_i})$. Thus min R is compact.

(2) Assume that A is a flat R-module. Let $\mathcal{P}$ be a prime ideal of A, and let $P = \mathcal{P} \cap R$, and suppose that there is a prime ideal P_1 of R, $P_1 \subset_{\neq} P$. Then there is a $p \in P - P_1$, and we have an exact sequence

$$0 \to R/P_1 \xrightarrow{p} R/P_1.$$

Since A is flat over R, we have an exact sequence

$$0 \to A/P_1A \xrightarrow{p} A/P_1A.$$

However, since A is a VNR, there exists $u \in A - \mathcal{P}$ with $pu = 0$. This contradiction shows that P is a minimal prime ideal of R.

Conversely, assume that if $\mathcal{P}$ is a prime ideal of A, then $\mathcal{P} \cap R = P$ is a minimal prime ideal of R. Then since R is reduced, R_P is a field. Now $A_{\mathcal{P}}$ is an R_P-module, and hence $A_{\mathcal{P}}$ is flat over R_P, and thus over R. Thus $\Sigma \oplus A_{\mathcal{P}}$, $\mathcal{P}$ maximal in A, is a flat R-module. Since $\Sigma \oplus A_{\mathcal{P}}$ is a faithfully flat A-module, it follows that A is flat over R. (See [1, Proposition 7, Chapter I, §4].)

PROPOSITION 1.15 [12, Proposition 9]. *Let R be a reduced ring. Then the following statements are equivalent:*

(1) *$Q = Q(R)$ is a VNR.*

(2) min R *is compact; and if a finitely generated ideal is contained in the union of the minimal prime ideals of R, then it is contained in one of them.*

Proof. (1) $\Rightarrow$ (2) Since Q is a localization of R, it is a flat R-module. Thus min R is compact by Proposition 1.14(2). The latter part of (2) follows immediately from Proposition 1.4(2).

(2) $\Rightarrow$ (1) Suppose that Q is not a VNR. Then by Proposition 1.3, Q has a maximal ideal $\mathcal{P}$ that is not a minimal prime ideal of Q. Then $\mathcal{P} \cap$

R is not a minimal prime ideal of R, because Q is a localization of R. Hence $\mathcal{P} \cap R \supsetneq P$, where $P \in \min R$. Choose $b \in (\mathcal{P} \cap R) - P$; and then

$$\min R = D(b) \cup \{\cup D(a) | a \in P\}.$$

Since $\min R$ is compact, there exist $a_1, \ldots, a_n \in P$ with $\min R = D(b) \cup D(a_1) \cup \cdots \cup D(a_n)$. Let $J = Rb + Ra_1 + \cdots + Ra_n$; then J is not contained in any minimal prime ideal of R. Hence by hypothesis J is not contained in the union of the minimal prime ideals of R; and thus J contains an element that is not a zero. Hence $QJ = Q$. But $QJ \subset \mathcal{P}$; and this contradiction shows that Q is a VNR.

Remarks. Quentel has produced an example of a reduced ring R where $\min R$ is compact, but $Q(R)$ is not a VNR. Thus the latter part of statement (2) in Proposition 1.15 is not redundant.

The following proposition is also in part due to Quentel although with a proof that depends on considerable machinery.

PROPOSITION 1.16 [12, Proposition 3]. *Let R be a reduced ring. Then the following statements are equivalent:*

(1) $\min R$ *is compact.*
(2) *If $b \in R$, then there is a finitely generated ideal $J \subset \mathrm{Ann}_R b$ with* $\mathrm{Ann}_R(Rb + J) = 0$.
(3) ΠR_P, $P \in \min R$, *is a flat R-module.*
(4) $E(R)$ *is a flat R-module.*
(5) *If $\mathcal{P}$ is a prime ideal of $E(R)$, then* $\mathcal{P} \cap R \in \min R$.

Proof. (1) $\Rightarrow$ (2) Let $b \in R$; then by Proposition 1.2(1), $Rb + \mathrm{Ann}_R b$ is not contained in any minimal prime ideal of R. Thus

$$\min R = D(b) \cup \{\cup D(a) | a \in \mathrm{Ann}_R b\}.$$

Since $\min R$ is compact, there exist $a_1, \ldots, a_n \in \mathrm{Ann}_R b$ such that

$$\min R = D(b) \cup D(a_1) \cup \cdots \cup D(a_n).$$

Thus if $J = Ra_1 + \cdots + Ra_n$, then $Rb + J$ is not contained in any minimal prime ideal of R. Hence by Proposition 1.2(2), $\mathrm{Ann}_R(Rb + J) = 0$.

(2) $\Rightarrow$ (3) Let $\Pi = \Pi R_{P_\alpha}$, $P_\alpha \in \min R$; and let $I \neq 0$ be an ideal of R. In order to prove that Π is flat it is sufficient to prove that $\mathrm{Tor}_1^R(R/I, \Pi) = 0$. But $\mathrm{Tor}_1^R(R/I, \Pi)$ is isomorphic to the kernel of the canonical map $\theta : I \otimes_R \Pi \to I\Pi$. Thus it is sufficient to prove that $\mathrm{Ker}\, \theta = 0$.

Let $b \in I$ and $\langle x_\alpha \rangle \in \Pi$, where $x_\alpha \in R_{P_\alpha}$. We will show that we can write $b \otimes \langle x_\alpha \rangle$ in the form where $x_\alpha = 0$, for all α such that $b \in P_\alpha$. By hypothesis there is a finitely generated ideal

$$J = Ra_1 + \cdots + Ra_n \quad \text{with } J \subset \mathrm{Ann}_R b \text{ and } \mathrm{Ann}_R(Rb + J) = 0.$$

By Proposition 1.2, $\min R = D(b) \cup D(a_1) \cup \cdots \cup D(a_n)$. Thus we can take subsets $\overline{D}(a_i) \subset D(a_i)$ such that we have the disjoint union

$$\min R = D(b) \cup \overline{D}(a_1) \cup \cdots \cup \overline{D}(a_n).$$

Consider a fixed integer i, $1 \leq i \leq n$. If $P_\alpha \in \overline{D}(a_i)$, then a_i is a unit in R_{P_α}; and we can write $x_\alpha = a_i(y_\alpha(i))$, where $y_\alpha(i) \in R_{P_\alpha}$. If $P_\alpha \notin \overline{D}(a_i)$, we put $y_\alpha(i) = 0$. If $P_\alpha \in D(b)$, we let $y_\alpha(0) = x_\alpha$; and if $P_\alpha \notin D(b)$ we let $y_\alpha(0) = 0$. It is then clear that

$$\langle x_\alpha \rangle = \langle y_\alpha(0) \rangle + a_1 \langle y_\alpha(1) \rangle + \cdots + a_n \langle y_\alpha(n) \rangle.$$

Since $a_i \in \mathrm{Ann}_R b$, we have $b \otimes \langle x_\alpha \rangle = b \otimes \langle y_\alpha(0) \rangle$. Thus without loss of generality we can assume that $x_\alpha = 0$ if $b \in P_\alpha$.

Now if $b \otimes \langle x_\alpha \rangle \in \mathrm{Ker}\, \theta$, then $x_\alpha = 0$ if $b \in P_\alpha$, and $bx_\alpha = 0$ if $b \notin P_\alpha$. But if $b \notin P_\alpha$, then b is a unit in R_{P_α}, and so $x_\alpha = 0$ for all α. Therefore $b \otimes \langle x_\alpha \rangle = 0$. In general, suppose that $x \in \mathrm{Ker}\, \theta$ and

$$x = (b_1 \otimes \langle x_\alpha(1) \rangle) + \cdots + (b_k \otimes \langle x_\alpha(k) \rangle) \quad \text{where } b_i \in I \text{ and } x_\alpha(i) \in R_{P_\alpha}.$$

We shall prove that $x = 0$ by induction on k, the case $k = 1$ having already been proved.

As we have demonstrated, we can assume that $x_\alpha(1) = 0$ if $b_1 \in P_\alpha$. Now

$$0 = \theta(x) = \langle b_1 x_\alpha(1) + \cdots + b_k x_\alpha(k) \rangle,$$

and hence $b_1 x_\alpha(1) + \cdots + b_k x_\alpha(k) = 0$, for all α. For all α such that $b_1 \notin P_\alpha$, b_1 is a unit in R_{P_α}, and hence if $i > 1$ we can write $x_\alpha(i) = -b_1 y_\alpha(i)$, where $y_\alpha(i) \in R_{P_\alpha}$. Thus

$$b_1[x_\alpha(1) - b_2 y_\alpha(2) - \cdots - b_k y_\alpha(k)] = 0.$$

But since b_1 is a unit in R_{P_α}, we have $x_\alpha(1) = b_2 y_\alpha(2) + \cdots + b_k y_\alpha(k)$. It is now clear that by substitution we can write x as

$$x = (b_2 \otimes \langle z_\alpha(2) \rangle) + \cdots + (b_k \otimes \langle z_\alpha(k) \rangle).$$

Hence $x = 0$ by induction on k. Thus $\mathrm{Ker}\, \theta = 0$, and so ΠR_{P_α} is flat.

(3) $\Rightarrow$ (4) Since $E(R)$ is a direct summand of ΠR_{P_α} by Proposition 1.1(2), $E(R)$ is also a flat R-module.

(4) $\Rightarrow$ (5) By Proposition 1.12, $E(R)$ is a commutative VNR containing R. Hence by Proposition 1.14(2), the prime ideals of $E(R)$ contract to minimal prime ideals of R.

(5) $\Rightarrow$ (1) This is an immediate consequence of Proposition 1.14(1).

Remarks. (1) If R is a reduced coherent ring, then $\min R$ is compact. For one of the definitions of a coherent ring is that every direct product of flat R-modules is flat. Thus ΠR_{P_α} is flat, and hence $\min R$ is compact by Proposition 1.16.

(2) If R is a reduced ring, then $\mathrm{Hom}_R(E(R), E(R))$ is always a commutative ring; but $E(R)$ is flat iff $\min R$ is compact. However, if R is a Noetherian

ring (not necessarily reduced) then min R is always finite (hence compact); but as we shall show in Proposition 1.18, $E(R)$ is flat iff $\mathrm{Hom}_R(E(R), E(R))$ is a commutative ring.

DEFINITION. We shall say that an ideal is *irreducible* if it is not the intersection of two properly larger ideals.

Portions of the following two propositions are contained in [3, Theorem 3].

PROPOSITION 1.17. *Let R be a commutative, Noetherian, local ring with maximal ideal M. Then the following statements are equivalent:*

(1) *R is self-injective.*
(2) *M is the only prime ideal of R and 0 is an irreducible ideal of R.*
(3) *$R \simeq E(R/M)$.*
(4) *$E(R/M)$ is a flat R-module.*

Proof. (1) ⇒ (2) Let $0 = Q_1 \cap \cdots \cap Q_n$ be an irredundant decomposition of 0 in R, where Q_i is an irreducible P_i primary ideal. Then by [11, Theorem 2.3],

$$E(R) \simeq E(R/P_1) \oplus \cdots \oplus E(R/P_n).$$

But $R \simeq E(R)$ and R is indecomposable. Hence $n = 1$ and $R \simeq E(R/P_1)$. Thus every element of $R - P_1$ is a unit in R. Therefore, $P_1 = M$ and $0 = Q_1$ is irreducible and M-primary. Hence M is the only prime ideal of R.

(2) ⇒ (3) Since 0 is an irreducible M-primary ideal of R, we have $E(R) = E(R/M)$. Now R has finite length and $L(R) = L(\mathrm{Hom}_R(R, E(R/M)) = L(E(R/M))$. Since $R \subset E(R/M)$, we have $R = E(R/M)$.

(3) ⇒ (4) Trivial.

(4) ⇒ (1) Let $E = E(R/M)$ and let I be an ideal of R. Then be [2, Ch. VI, Proposition 5.3],

$$\mathrm{Hom}_R(\mathrm{Ext}^1_R(R/I, R), E) \simeq \mathrm{Tor}\,^R_1(\mathrm{Hom}_R(R, E), R/I) = 0$$

because E is flat. Therefore, $\mathrm{Ext}^1_R(R/I, R) = 0$ showing that R is self-injective.

PROPOSITION 1.18. *Let R be a commutative, Noetherian ring, and let $\{P_1, \ldots, P_n\}$ be the prime ideals belonging to 0 in R. Let $O_i = \{r \in R \mid ur = 0$ for some $u \in R - P_i\}$. Then the following statements are equivalent:*

(1) *Every P_i is a minimal prime ideal of R, and O_i is an irreducible ideal of R.*
(2) *$E(R) \simeq R_{P_1} \oplus \cdots \oplus R_{P_n}$.*
(3) *$E(R) \simeq Q(R)$.*

(4) $\operatorname{Hom}_R(E(R), E(R))$ *is a commutative ring.*
(5) $E(R)$ *is a flat R-module.*

Proof. (1) $\Rightarrow$ (2) O_i is a P_i-primary ideal because P_i is minimal; and $0 = O_1 \cap \cdots \cap O_n$ is a normal, irreducible decomposition of 0 in R. Thus

$$E(R) \simeq E(R/P_1) \oplus \cdots \oplus E(R/P_n)$$

by [11, Theorem 2.3]. Since 0 is an irreducible ideal of R_{P_i}, and $P_iR_{P_i}$ is the only prime ideal of R_{P_i}, it follows from Proposition 1.17 that

$$E(R/P_i) \simeq E(R_{P_i}/P_iR_{P_i}) \simeq R_{P_i}.$$

Therefore $E(R) \simeq R_{P_1} \oplus \cdots \oplus R_{P_n}$.

(2) $\Rightarrow$ (3) Since every R_{P_i} is a self-injective ring, it follows from Proposition 1.17 that every P_i is a minimal prime ideal of R. Since $S = R - \cup_{i=1}^n P_i$ is the set of non-zero divisors of R, we have by Proposition 1.5 that

$$Q(R) = R_S \simeq R_{P_1} \oplus \cdots \oplus R_{P_n}.$$

Therefore, $E(R) = Q(R)$.

(3) $\Rightarrow$ (4) $\operatorname{Hom}_R(E(R), E(R)) \simeq \operatorname{Hom}_R(R_S, R_S) \simeq R_S$ is a commutative ring.

(4) $\Rightarrow$ (5) By Proposition 1.10, $E(R) \simeq \operatorname{Hom}_R(E(R), E(R))$. Let I be an ideal of R. By [2, Ch. VI, Proposition 5.2] we have

$$\operatorname{Tor}_1^R(\operatorname{Hom}_R(E(R), E(R)), R/I) \simeq \operatorname{Hom}_R(\operatorname{Ext}_R^1(R/I, E(R)), E(R)) = 0.$$

Therefore, $\operatorname{Hom}_R(E(R), E(R))$ is a flat R-module.

(5) $\Rightarrow$ (1) Let $E_i = E(R/P_i)$; then $E(R) = E_1^{k_1} \oplus \cdots \oplus E_n^{k_n}$. Hence E_i is a flat R-module, $i = 1, \ldots, n$. Since E_i is an R_{P_i}-module, E_i is a flat R_{P_i}-module. Hence by Proposition 1.17, $P_iR_{P_i}$ is the only prime ideal of R_{P_i} and 0 is irreducible in R_{P_i}. Therefore, P_i is a minimal prime ideal of R and O_i is an irreducible ideal of R.

2. PIF Rings

DEFINITIONS. Let R be a commutative ring. We shall say that R is a PIF if every principal ideal of R is flat; and we shall say that R is a PIP if every principal ideal of R is projective. If P is a prime ideal of R we shall define

$$O_P = \{r \in R \mid \text{there is } u \in R - P \text{ with } ur = 0\}.$$

In this section we shall give necessary and sufficient conditions for a PIF to be a PIP. Then we shall focus on a single ideal and give necessary and sufficient conditions for it to be a direct summand of R in terms of the properties of subsets of $\min R$. From these considerations we shall be able to give necessary and sufficient conditions for a principal ideal of a PIF to be a projective ideal.

The following proposition characterizes PIF rings as those rings that are locally integral domains.

PROPOSITION 2.1. *Let R be a commutative ring. Then the following statements are equivalent:*

(1) *R is a PIF.*
(2) *R_M is an integral domain for all maximal ideals M of R.*
(3) *R is reduced; and a maximal ideal M of R contains only one minimal prime ideal P of R.*

In this case, $P = O_M$; and $R_P = Q(R_M)$, the quotient field of R_M.

Proof. (1) $\Rightarrow$ (2) Let M be a maximal ideal of R. Then every principal ideal of R_M is flat, hence free over R_M. Thus R_M has no zero divisors.

(2) $\Rightarrow$ (1) Let $a \in R$; then either $R_Ma = 0$ or R_Ma is R_M-free $\forall$ maximal ideals M of R. Hence $\text{w.dim}_R\, Ra = \sup_M(\text{w.dim}_{R_M} R_Ma) = 0$. Thus Ra is a flat ideal of R.

(2) $\Rightarrow$ (3) Let M be a maximal ideal of R. Then 0_M is contained in every prime ideal of R contained in M. On the other hand $R/O_M \subset R_M$, and hence O_M is a prime ideal of R. Thus O_M is the unique minimal prime ideal of R contained in M. The annihilator of an element of $\cap O_M$ (where M ranges over all of the maximal ideals of R) is not contained in any maximal ideal of R. Hence $\cap O_M = 0$ and thus R is reduced. If $O_M = P$, then R_P is a field by Proposition 1.1, and clearly $R_P = Q(R_M)$.

(3) $\Rightarrow$ (2) Let M be a maximal ideal of R. Then R_M is reduced and has only one minimal prime ideal. Therefore, R_M is an integral domain.

PROPOSITION 2.2. *Let R be a commutative ring with only a finite number of minimal prime ideals. Then the following statements are equivalent:*

(1) *R is a PIF.*
(2) *R is a PIP.*
(3) *R is a finite direct sum of integral domains.*

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3) Let $\{P_1, \dots, P_n\}$ be the minimal prime ideals of R. Then R is reduced and so $\cap_{i=1}^n P_i = 0$. By Proposition 2.1, $P_i + P_j = R$, $i \neq j$. Thus by the Chinese Remainder Theorem, $R \simeq R/P_1 \oplus \cdots \oplus R/P_n$.

(3) $\Rightarrow$ (2) If R is a finite direct sum of integral domains, then every principal ideal of R is a direct sum of principal ideals over each of these domains. Hence it is obvious that principal ideals of R are projective.

PROPOSITION 2.3. *Let R be a reduced ring and J a finitely generated flat ideal of R. If $\exists$ elements $a \in J$ and $b \in \text{Ann}_R J$ such that $a + b$ is not a zero divisor in R, then J is a projective ideal of R.*

Proof. Let $I = Rb + J$; then $I = Rb \oplus J$ because R is reduced and $b \in \mathrm{Ann}_R J$. Let P be a prime ideal of R. If $J_P \neq 0$, then J_P is a free R_P-*ideal of rank 1 and* $bJ_P = 0$. Therefore, $R_P b = 0$, and so $I_P = J_P$ is free of rank 1 over R_P. On the other hand suppose that $J_P = 0$. Then $I_P = R_P b = R_P(a + b)$ is free of rank 1 over R_P because $a + b$ is not a zero divisor in R_P. Thus I_P is free of rank 1 if P is a prime ideal of R. Hence by [1, Ch. II, §5, Theorem 2] I is a projective ideal of R. Since J is a direct summand of I, J is also a projective ideal of R.

PROPOSITION 2.4. *Let R be a reduced ring such that $Q(R)$ is a VNR. Then every finitely generated flat ideal of R is projective.*

Proof. This is an immediate consequence of Propositions 1.4(3) and 2.3.

PROPOSITION 2.5. *Let R be a commutative ring and I an ideal of R. Then the following statements are equivalent:*

(1) *R/I is a flat R-module.*
(2) *$I \cap K = IK$ for any ideal K of R.*
(3) *If $a \in I$, then there exists $c \in I$ with $(1 - c)a = 0$.*
(4) *If J is a finitely generated ideal of R, $J \subset I$, then there is a $c \in I$ with $(1 - c)J = 0$.*
(5) *$I_M = 0$ or R_M for any maximal ideal M of R.*

If I and K are ideals of R such that R/I and R/K are flat, then $R/(I + K)$ is also flat.

Proof. (1) ⇒ (2) If A is an R-module, then $A \otimes_R R/I \simeq A/IA$. Let K be an ideal of R; then since R/I is flat we have a commutative diagram with exact rows and vertical isomorphisms:

$$\begin{array}{ccccccc} 0 \to & K/IK & \to & R/I & \to & R/(I+K) & \to 0 \\ & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\ 0 \to & (I+K)/I & \to & R/I & \to & R/(I+K) & \to 0. \end{array}$$

Since the kernel of the canonical map $K/IK \to (I + K)/I$ is $(I \cap K)/IK$, we have $I \cap K = IK$.

(2) ⇒ (3) Let $a \in I$; then $Ra = Ia$. Hence there is a $c \in I$ with $(1 - c)a = 0$.

(3) ⇒ (4) Let $J = Ra_1 + \cdots + Ra_n$, where $a_i \in I$. By hypothesis, there are $c_i \in I$ with $(1 - c_i)a_i = 0$, $i = 1, \ldots, n$. Then $(1 - c_1)(1 - c_2) \cdots (1 - c_n) = 1 - c$, where $c \in I$ and $(1 - c)J = 0$.

(4) ⇒ (5) Let M be a maximal ideal of R. If $I \not\subset M$, then $I_M = R_M$. Hence assume that $I \subset M$. If $a \in I$, then there is a $c \in I$ with $(1 - c)a = 0$; and since $1 - c \notin M$, it follows that the image of a in R_M is 0. Therefore, $I_M = 0$.

(5) $\Rightarrow$ (1) Since $(R/I)_M = R_M$ or 0, it follows that $(R/I)_M$ is a flat R_M-module $\forall$ maximal ideals M of R. Therefore because flatness is determined locally, R/I is a flat R-module.

Assume that I and K are ideals of R such that R/I and R/K are flat R-modules. Let $x = a + b$, where $a \in I$ and $b \in K$. Then there are $c \in I$ and $d \in K$ such that $(1 - c)a = 0$ and $(1 - d)b = 0$. Thus $(1 - c)(1 - d)x = 0$, and $(1 - c)(1 - d) = 1 - y$, where $y \in I + K$. Thus $R/(I + K)$ is a flat R-module by (3).

The following proposition is due to Vasconcelos [14, Proposition 3.4].

PROPOSITION 2.6. *Let R be a reduced ring such that* min R *is compact. Then every principal flat ideal of R is projective.*

Proof. Let Rb be a flat ideal of R and let $I = \mathrm{Ann}_R b$. By Proposition 1.16, there is a finitely generated ideal $J \subset I$ such that $\mathrm{Ann}_R(J + Rb) = 0$. Since $R/I \simeq Rb$ is flat, there exists $a \in I$ with $(1 - a)J = 0$ by Proposition 2.5. Thus $J \subset Ra$ and hence $\mathrm{Ann}_R(Ra + Rb) = 0$. By Proposition 1.2(3), $a + b$ is not a zero divisor in R. Thus by Proposition 2.3, Rb is a projective ideal of R.

DEFINITION. A commutative ring R is said to be *semi-hereditary* if every finitely generated ideal of R is projective.

Portions of the next proposition are due to Hattori [9], Endo [6], Vasconcelos [13], [14], and Quentel [12].

PROPOSITION 2.7. *Let R be a PIF; then the following statements are equivalent:*

(1) *R is a PIP.*
(2) *Every finitely generated flat ideal of R is projective.*
(3) *$Q(R)$ is a semi-hereditary ring.*
(4) *$Q(R)$ is a VNR.*
(5) min *R is compact.*
(6) *$E(R)$ is a flat R-module.*

Proof. (4) $\Rightarrow$ (2) is Proposition 2.4; and (2) $\Rightarrow$ (1) is trivial. (4) $\Rightarrow$ (5) follows from Proposition 1.15; and (5) $\Rightarrow$ (1) is Proposition 2.6. (5) $\Leftrightarrow$ (6) is Proposition 1.16.

(1) $\Rightarrow$ (4) Let $a \in R$ and $I = \mathrm{Ann}_R a$; then I is a direct summand of R, and hence $I = Re$, where $e^2 = e$. Now $\mathrm{Ann}_R(Re + Ra) = 0$ by Proposition 1.2, and so $Q(R)$ is a VNR by Proposition 1.4.

(4) $\Rightarrow$ (3) is elementary.

(3) $\Rightarrow$ (4) Since principal ideals of $Q(R)$ are projective $Q(R)$-modules, and $Q(R)$ is its own ring of quotients, $Q(R)$ is a VNR by (1) $\Rightarrow$ (4).

Remarks. The assumption in Proposition 2.7 that R is a PIF, is necessary. There exist many examples where R is a reduced ring and $Q(R) = E(R)$ is a VNR, but R is not a PIF. The easiest example is the following. Let R be a quasi-local reduced ring with only a finite number of minimal prime ideals but such that R is not a domain. (Take any Noetherian local ring that is not a domain and factor out the intersection of the minimal prime ideals.) By Proposition 1.6, $Q(R) = E(R)$ is a semi-simple ring. If R were a PIF, then by Proposition 2.2, R would be a finite direct sum of integral domains. But since R is quasi-local, it is indecomposable. This contradiction shows that R is not a PIF.

DEFINITIONS. Let R be a commutative ring. The *weak global* dimension of R (w.gl.dim R) is defined to be the smallest non-negative integer n (if any such exist) such that $\operatorname{Tor}^R_{n+1}$ is the *0*-functor. Otherwise, w.gl.dim $R = \infty$.

Thus w.gl.dim $R = 0$ iff every R-module is flat. It is not hard to see that w.gl.dim $R = 0$ iff R is a VNR. It also follows from the definitions that w.gl.dim $R \leq 1$ iff every submodule of a flat R-module is flat iff every ideal of R is flat.

Remarks. (1) Let R be a commutative ring such that w.gl.dim $R \leq 1$. Then each of the 6 conditions of Proposition 2.7 is equivalent to R being a semi-hereditary ring. Thus Proposition 2.7 is a generalization of the results of Hattori [9], Endo [6], Vasconcelos [13] and Quentel [12].

(2) It is well known (see [7, Corollary 11.30]) that a finitely generated flat ideal is projective iff it is finitely presented. Since a *coherent ring* is defined to be a ring such that finitely generated ideals are finitely presented, it follows that if w.gl.dim $R \leq 1$, then R is semi-hereditary iff R is a coherent ring. One would therefore expect that the conditions for a PIF to be a PIP would involve some weakened form of coherence. Since a ring R is coherent iff a direct product of flat R-modules is flat [4, Theorem 2.1], we have a better understanding of why a PIF is a PIP iff min R is compact (Proposition 2.7). For as we have seen in Proposition 1.16, if R is a reduced ring, then min R is compact iff ΠR_P ($P \in \min R$) is a flat R-module.

In a recent paper (*Commutative coherent rings,* Canad. J. Math., vol. 34 (1982), pp. 1240–1244) we have shown that a commutative ring R is a coherent ring iff $\operatorname{Hom}_R(B, C)$ is flat for injective R-modules B and C. Thus condition (4) of the next proposition is seen to be related to coherence, since flat modules over integral domains are torsion-free. Furthermore, it follows from Proposition 2.7 that if w.gl.dim $R \leq 1$, then each of the four conditions of the next proposition is equivalent to R being a semi-hereditary ring.

PROPOSITION 2.8. *Let R be a PIF. Then the following statements are equivalent*:

(1) *R is a PIP.*

(2) *$Q(R)_M$ is the quotient-field of R_M for any maximal ideal M of R.*

(3) *If C is an injective R-module, then C_M is a divisible R_M-module for any maximal ideal M of R.*

(4) *If C is an injective R-module, then* $\mathrm{Hom}_R(C, E(R/M))$ *is a torsion-free R_M-module for any maximal ideal M of R.*

Proof. (1) ⇒ (2) Let $Q = Q(R)$ and S the set of non-zero divisors in R so that $Q = R_S$. Let M be a maximal ideal of R, P the unique minimal prime ideal of R contained in M, and $\bar{S}$ the image of S in R/P. Then $Q/PQ \simeq (R/P)_{\bar{S}} \subset R_P$ because R_P is the quotient field of R/P. But Q is a VNR by Proposition 2.7, and so Q/PQ is a field. Thus $Q/PQ \simeq R_P$. But by Proposition 2.1, R_P is the quotient field of R_M and $P_M = 0$. Therefore, $(PQ)_M = 0$, and we have $R_P \simeq (R_P)_M \simeq Q_M/(PQ)_M = Q_M$.

(2) ⇒ (3) Let C be an injective R-module. For each $x \in C$, there is an R-homomorphism $f : Q \to C$ with $f(1) = x$. Hence there is an R-module F that is a direct sum of copies of Q and an R-surjection $g : F \to C \to 0$. Let M be a maximal ideal of R. Then we have an R_M-surjection $g_M : F_M \to C_M \to 0$. But by hypothesis F_M is a direct sum of copies of the quotient field of R_M. Therefore, C_M is a divisible R_M-module.

(3) ⇒ (4) Let C be an injective R-module and M a maximal ideal of R. Then $E(R/M)$ (the injective envelope of R/M) is an R_M-module; and hence

$$\mathrm{Hom}_R(C, E(R/M)) \simeq \mathrm{Hom}_R(C, \mathrm{Hom}_{R_M}(R_M, E(R/M))$$
$$\simeq \mathrm{Hom}_{R_M}(C_M, E(R/M)).$$

Now $\mathrm{Hom}_{R_M}(C_M, E(R/M))$ is R_M-torsion-free because C_M is a divisible R_M-module.

(4) ⇒ (1) Let $a \in R$, and suppose that Ra is not a projective R-module. Then $hd_R(R/Ra) > 1$; and thus there is a B (a homomorphic image of an injective R-module) such that

$$\mathrm{Ext}\,^1_R(R/Ra, B) \neq 0.$$

Hence there is a maximal ideal M of R such that $\mathrm{Hom}_R(\mathrm{Ext}^1_R(R/Ra, B), D) = 0$ where $D = E(R/M)$. Because R/Ra is finitely presented, there is a canonical surjection

$$\mathrm{Tor}_1^R(Hom_R(B, D), R/Ra) \to \mathrm{Hom}_R(\mathrm{Ext}^1_R(R/Ra, B), D).$$

Thus $\mathrm{Tor}_1^R(\mathrm{Hom}_R(B, D), R/Ra) \neq 0$. Because D is an R_M-module, we have

$$Tor_1^R(Hom_R(B, D), R/Ra) \simeq \mathrm{Tor}_1^{R_M}(\mathrm{Hom}_{R_M}(B_M, D), R_M/R_m a).$$

But $R_M a$ is a flat R_M-module; and $\mathrm{Hom}_{R_M}(B_M, D)$ is R_M-torsion-free by hypothesis. Thus $\mathrm{Tor}\,_1^{R_M}\mathrm{Hom}_{R_M}(B_M, D), R_M/R_M a) = 0$. This contradiction proves that Ra is a projective ideal of R.

Remarks. Let R be a PIF that is not a PIP (there is an example due to Vasconcelos of such a ring that we shall reproduce in §4). Then injective R-modules do not localize. In fact, according to Proposition 2.8, there is an injective R-module C and a maximal ideal M of R such that C_M is not even a divisible R_M-module. Again this is related to a lack of coherence. For further results on this subject and its relation to coherence see [5].

We now turn our attention to the problem of finding out when a single principal ideal is projective, or when an ideal is a direct summand of the ring.

DEFINITION. Let R be a reduced ring and $\mathscr{C}$ a subset of min $R = \{P_\alpha\}$. Let $\mathscr{C}'$ denote the complement of $\mathscr{C}$ in min R, and let $J_\mathscr{C} = \cap P_\sigma$, $P_\sigma \in \mathscr{C}$. It is immediate that $\mathscr{C}$ is a closed subset of min R iff $P_\alpha \supset J_\mathscr{C}$ implies that $P_\alpha \in \mathscr{C}$. If $\mathscr{C}$ is closed, it is also easy to verify that $\mathrm{Ann}_R J_\mathscr{C} = J_{\mathscr{C}'}$. We shall say that $\mathscr{C}$ is a *good* subset of min R if $J_{\mathscr{C}'} \not\subset \cup P_\gamma$, $P_\gamma \in \mathscr{C}$.

PROPOSITION 2.9. *Let R be a reduced ring.*

(1) *$\mathscr{C}$ is a good subset of* min R *iff there exists $a \in R$ with $\mathscr{C} = D(a)$. In this case $\mathscr{C}$ is both open and closed in* min R; $\mathrm{Ann}_R a = J_\mathscr{C}$; *and* $\mathrm{Ann}_R(\mathrm{Ann}_R a) = J_{\mathscr{C}'}$.

(2) *$\mathscr{C}$ good implies that $\mathscr{C}'$ is good for any subset $\mathscr{C}$ of* min R *iff $Q(R)$ is a VNR.*

Proof. (1) $\mathscr{C}$ is a good subset of min R iff $J_{\mathscr{C}'} \not\subset \cup P_\sigma$, $P_\sigma \in \mathscr{C}$ iff there exists a $\in J_{\mathscr{C}'}$, $a \notin \cup P_\gamma$, $P_\gamma \in \mathscr{C}$ iff there exists $a \in R$ with $D(a) = \mathscr{C}$. Suppose that $\mathscr{C} = D(a)$, then $\mathscr{C}$ is open in min P. By Proposition 1.1, $\mathscr{C}$ is the set of minimal prime ideals of R that contain $\mathrm{Ann}_R a$. Thus $\mathscr{C}$ is closed in min R and $J_\mathscr{C} \supset \mathrm{Ann}_R a$. On the other hand, $J_\mathscr{C} a \subset (J_\mathscr{C} \cap J_{\mathscr{C}'}) = 0$, and so $\mathrm{Ann}_R a = J_\mathscr{C}$. Since $\mathscr{C}$ is closed, $\mathrm{Ann}_R J_\mathscr{C} = J_{\mathscr{C}'}$.

(2) Let $\mathscr{C}$ be a good subset of min R. Then there exists $a \in J_{\mathscr{C}'}$, $a \notin \cup P_\gamma$, $P_\gamma \in \mathscr{C}$, and hence $\mathrm{Ann}_R a = J_\mathscr{C}$. Now $\mathscr{C}'$ is a good subset of min R iff there exists $b \in J_\mathscr{C}$ such that $b \notin \cup P_\delta$, $P_\delta \in \mathscr{C}'$, iff there exists b $\in \mathrm{Ann}_R a$ such that $a + b$ is not in any minimal prime ideal of R. Hence by Proposition 1.4(3), $\mathscr{C}$ good implies $\mathscr{C}'$ good for any subset $\mathscr{C}$ of min R iff $Q(R)$ is a VNR.

PROPOSITION 2.10. *Let R be a reduced ring and J an ideal of R such that R/J is a flat R-module. Then R/J is a reduced ring and $J = J_\mathscr{C}$, where $\mathscr{C}$ is a closed subset of* min R. *If R is a PIF, or a PIP, then so is R/J.*

Proof. Let $x \in J$; then by Proposition 2.5 there is a $b \in J$ such that $(1 - b)x = 0$. Thus if P is any prime ideal of R containing J, then x is contained in every minimal prime ideal of R contained in P. Thus if $\mathscr{C}$ is the set of minimal prime ideals of R containing J, then $\mathscr{C}$ is a non-empty closed subset of min R and $J \subset J_{\mathscr{C}}$. Now suppose that $y \in R$ and $y^n = x \in J$; then $(1 - b)y^n = 0$, and so $((1 - b)y)^n = 0$. Therefore $(1 - b)y = 0$, and hence $y \in J$. Therefore, $J = J_{\mathscr{C}}$, and R/J is a reduced ring.

Assume that R is a PIF; let M be a maximal ideal of R containing J; and let P be the unique minimal prime ideal of R contained in M. As we have seen, $J \subset P$, and so P/J is the unique minimal prime ideal of R/J contained in M/J. Hence R/J is a PIF by Proposition 2.1.

Let $r \in R$, and $c \in (J : r)$, and $x = rc \in J$. Then there is a $b \in J$ with $(1 - b)cr = 0$ and so

$$(1 - b)c = a \in \operatorname{Ann}_R r.$$

Thus $(J : r) = J + \operatorname{Ann}_R r$. Let $I = \operatorname{Ann}_R r$ and assume that Rr is a projective R-module. Then $I = Re$, where $e^2 = e$; and hence $(J : r)/J = (Re + J)/J$ is generated by an idempotent element of R/J. Since $(J : r)/J$ is the annihilator in R/J of $r + J$, we see that $r + J$ generates a projective ideal of R/J. Hence if R is a PIP, then so is R/J.

PROPOSITION 2.11. *Let R be a* PIF *and $\mathscr{C}$ a finite subset of* min R. *Then $R/J_{\mathscr{C}}$ is a flat R-module.*

Proof. Let M be a maximal ideal of R and let P be the unique maximal prime ideal of R contained in M. If $P \in \mathscr{C}$, then $J_{\mathscr{C}} \subset P$, and so $(J_{\mathscr{C}})_M \subset P_M = 0$. If $P \notin \mathscr{C}$, and $J_{\mathscr{C}} \subset M$, then there is a $P' \in \mathscr{C}$ with $P' \subset M$, and hence $P' = P$. This contradiction shows that $J_{\mathscr{C}} \not\subset M$ and hence $(J_{\mathscr{C}})_M = R_M$. Thus $(J_{\mathscr{C}})_M = 0$ or R_M, and hence $R/J_{\mathscr{C}}$ is a flat R-module by Proposition 2.5.

Remarks. If a principal ideal of a commutative ring R is a projective R-module, then its annihilator is a direct summand of R. In the next proposition we characterize the direct summands of a reduced ring R in terms of the subsets of min R. We note that if $\mathscr{C}$ is a subset of min R and $\overline{\mathscr{C}} = \{P \in \min R \mid P \supset J_{\mathscr{C}}\}$, then $\overline{\mathscr{C}}$ is closed and $J_{\mathscr{C}} = J_{\overline{\mathscr{C}}}$. Thus the restriction in (1) of the next proposition that $\mathscr{C}$ be closed is no restriction at all on the ideal $J_{\mathscr{C}}$. Moreover, by Proposition 2.10, every direct summand of R is of the form $J_{\mathscr{C}}$, where $\mathscr{C}$ is a closed subset of min R.

PROPOSITION 2.12. *Let R be a reduced ring and $\mathscr{C}$ a subset of* min R. *Then the following statements are equivalent*:

(1) *$\mathscr{C}$ is closed and $J_{\mathscr{C}}$ is a direct summand of R.*
(2) *$\mathscr{C}$ is a good subset of* min R *and $R/J_{\mathscr{C}'}$, is flat.*
(3) *$\mathscr{C}$ is both open and closed in* min R *and $R/J_{\mathscr{C}}$ and $R/J_{\mathscr{C}'}$, are flat.*

In this case both $\mathscr{C}$ and $\mathscr{C}'$ are good subsets of min R. *If R is a PIF, then this condition is equivalent to the other three.*

Proof. (1) $\Rightarrow$ (2) $J_{\mathscr{C}} = Re$, where $e^2 = e$. Since $\mathscr{C}$ is closed, if $P \in \mathscr{C}'$, then $J_{\mathscr{C}} \not\subset P$, and so $e \notin P$. Thus $J_{\mathscr{C}} \not\subset \cup P$, $P \in \mathscr{C}'$, and so $\mathscr{C}'$ is a good subset of R. Since $\mathscr{C}$ is closed, $J_{\mathscr{C}'} = \mathrm{Ann}_R J_{\mathscr{C}} = R(1 - e)$. Moreover, since $\mathscr{C}'$ is closed by Proposition 2.9, the same argument we have just used shows that $\mathscr{C}'' = \mathscr{C}$ is a good subset of min R. Finally, $R/J_{\mathscr{C}'} \simeq Re$ is R-projective.

(2) $\Rightarrow$ (3) By Proposition 2.9, $\mathscr{C}$ is both open and closed in min R; $\mathscr{C} = D(a)$ for some element $a \in R$; $\mathrm{Ann}_R a = J_{\mathscr{C}}$, and $\mathrm{Ann}_R J_{\mathscr{C}} = J_{\mathscr{C}'}$. Therefore $a \in J_{\mathscr{C}'}$; and hence by Proposition 2.5, there exists $b \in J_{\mathscr{C}'}$ with $(1 - b)a = 0$. But then $1 - b \in \mathrm{Ann}_R a = J_{\mathscr{C}}$ and so $J_{\mathscr{C}} + J_{\mathscr{C}'} = R$. Since $J_{\mathscr{C}} \cap J_{\mathscr{C}'} = 0$, $J_{\mathscr{C}}$ and $J_{\mathscr{C}'}$ are direct summands of R; and $R/J_{\mathscr{C}} \simeq J_{\mathscr{C}'}$ and $R/J_{\mathscr{C}'} \simeq J_{\mathscr{C}}$ are R-projective.

(3) $\Rightarrow$ (1) By Proposition 2.5, $R/(J_{\mathscr{C}} + J_{\mathscr{C}'})$ is a flat R-module. Suppose that $J_{\mathscr{C}} + J_{\mathscr{C}'} \neq R$; then by Proposition 2.10, $J_{\mathscr{C}} + J_{\mathscr{C}'} = J_{\mathscr{D}}$ where $\mathscr{D}$ is a non-empty subset of min R. Let $P \in \mathscr{D}$; then $P \supset J_{\mathscr{D}} \supset J_{\mathscr{C}}$; and since $\mathscr{C}$ is closed $P \in \mathscr{C}$. Similarly $P \in \mathscr{C}'$. This contradiction shows that $J_{\mathscr{C}} + J_{\mathscr{C}'} = R$. Since $J_{\mathscr{C}} \cap J_{\mathscr{C}'} = 0$, $J_{\mathscr{C}}$ is a direct summand of R.

In the course of proving (1) $\Rightarrow$ (2) we showed that both $\mathscr{C}$ and $\mathscr{C}'$ are good subsets of min R. Conversely, suppose that R is a PIF and that both $\mathscr{C}$ and $\mathscr{C}'$ are good subsets of min R. By Proposition 2.9, there exists $b \in R$ such that $D(b) = \mathscr{C}'$ and $J_{\mathscr{C}'} = \mathrm{Ann}_R b$. Therefore $R/J_{\mathscr{C}'} \simeq Rb$; and since R is a PIF, $R/J_{\mathscr{C}'}$ is a flat R-module. Thus we have proved (2).

PROPOSITION 2.13. *Let R be a PIF; $a \in R$; $I = \mathrm{Ann}_R a$; and $J = \mathrm{Ann}_R I$. Then the following statements are equivalent*:

(1) *Ra is a projective ideal of R.*
(2) *$\mathrm{Hom}_R(I, R)$ is a flat R-module.*
(3) *R/J is a flat R-module.*
(4) *There exists $b \in R$ such that $J = \mathrm{Ann}_R b$.*
(5) *If $\{P_\gamma\} = \mathscr{D}$ is a subset of* min R *and $I \subset \cup_\gamma P_\gamma$, then there is a $P_{\gamma_0} \in \mathscr{D}$ with $I \subset P_{\gamma_0}$.*

Proof. (1) $\Rightarrow$ (2) I is a direct summand of R, and hence $\mathrm{Hom}_R(I, R)$ is a projective R-module.

(2) $\Rightarrow$ (3) We have an exact sequence

$$0 \to \mathrm{Hom}_R(R/I, R) \to R \to \mathrm{Hom}_R(I, R).$$

Since $\mathrm{Hom}_R(R/I, R) \simeq \mathrm{Ann}_R I = J$, we have an embedding $R/J \subset \mathrm{Hom}_R(I, R)$. Let M be a maximal ideal of R. Then $R_M/J_M \subset \mathrm{Hom}_R(I, R)_M$. Now $\mathrm{Hom}_R(I, R)_M$ is a flat R_M-module; and R_M is an integral domain by Proposition 2.1. Thus $\mathrm{Hom}_R(I, R)_M$ is a torsion-free R_M-module; and hence so is R_M/J_M. Therefore $J_M = 0$ or R_M; and thus by Proposition 2.5, R/J is a flat R-module.

(3) $\Rightarrow$ (4) Let $\mathscr{C} = D(a)$; then by Proposition 2.9, $\mathscr{C}$ is a good subset of min R; $J_{\mathscr{C}} = I$; and $J_{\mathscr{C}'} = J$. By hypothesis, $R/J_{\mathscr{C}'}$ is flat; and thus by

Proposition 2.12, $J_{\mathscr{C}}$ is a direct summand of R. Therefore, $J_{\mathscr{C}} = Re$, where $e^2 = e$; and hence $J_{\mathscr{C}'} = \mathrm{Ann}_R e$.

(4) $\Rightarrow$ (5) Clearly $I = \mathrm{Ann}_R J$, and since $J = \mathrm{Ann}_R b$, we have $b \in I$. Suppose that $\{P_\alpha\} = \mathscr{D}$ is a subset of min R and that $I \subset \cup_\gamma P_\gamma$. Then there exists $P_{\gamma_0} \in \mathscr{D}$ such that $b \in P_{\gamma_0}$. Hence by Proposition 1.2(1), $J \not\subset P_{\gamma_0}$. However, since $IJ = 0$, we have $I \subset P_{\gamma_0}$.

(5) $\Rightarrow$ (1) By Proposition 2.9, $I = J_{\mathscr{C}}$, where $\mathscr{C}$ is a good subset of min R. If $J_{\mathscr{C}} \subset \cup_\alpha P_\alpha$, $P_\alpha \in \mathscr{C}'$; then by hypothesis there is a $P_{\alpha_0} \in \mathscr{C}'$ with $J_{\mathscr{C}} \subset P_{\alpha_0}$. But $\mathscr{C}$ is closed, and hence $P_{\alpha_0} \in \mathscr{C}$. This contradiction shows that $\mathscr{C}'$ is a good subset of min R. Hence by Proposition 2.12, I is a direct summand of R, and hence Ra is a projective ideal of R.

The following proposition is a generalization of Proposition 2.2.

Proposition 2.14. *Let R be a PIF and $a \in R$; and suppose that a is an element of only a finite number of primes $P_1, \ldots, P_n$ in* min R.

(1) *Ra is a projective ideal of R.*
(2) $R \simeq (\cap_{i=1}^n P_i) \oplus (\Sigma_{i=1}^n \oplus R/P_i)$.
(3) *Every P_i is a direct summand of R.*

Proof. (1) Let $\mathscr{C} = D(a)$; then $J_{\mathscr{C}} = \mathrm{Ann}_R a$ by Proposition 2.9(1). Now $\mathscr{C}' = \{P_i, \ldots, P_n\}$ and $J_{\mathscr{C}} \not\subset P_i$ for all i since $\mathscr{C}$ is closed. Therefore $J_{\mathscr{C}} \not\subset \cup_{i=1}^n P_i$, and hence $\mathscr{C}'$ is a good subset of min R. Therefore, by Proposition 2.12, $J_{\mathscr{C}} \oplus J_{\mathscr{C}'} = R$. Thus $Ra \simeq R/J_{\mathscr{C}} \simeq J_{\mathscr{C}'}$ is a projective ideal of R.

(2) By Proposition 2.1, every maximal ideal of R contains a unique minimal prime ideal of R. Thus $P_i + P_j = R$, $i \neq j$. Hence by the Chinese Remainder Theorem,

$$R/J_{\mathscr{C}'} \simeq \sum_{i=1}^{n} \oplus R/P_i.$$

Since $J_{\mathscr{C}'} = \cap_{i=1}^n P_i$ and $R = J_{\mathscr{C}} \oplus J_{\mathscr{C}'}$, we have $R \simeq (\cap_{i=1}^n P_i) \oplus (\Sigma_{i=1}^n \oplus R/P_i)$.

(3) Since R/P_i is isomorphic to a direct summand of R, there is an idempotent $e_i \in R$ so that $P_i = \mathrm{Ann}_R\, e_i = R(1 - e_i)$. Hence P_i is a direct summand of R, for $i = 1, \ldots, n$.

Proposition 2.15. *Let R be a commutative ring such that* w.gl.dim $R \leq 1$. *Then the following statements are equivalent*:

(1) *R is a semi-hereditary ring.*
(2) *$R/J_{\mathscr{C}}$ is flat for all subsets $\mathscr{C}$ of* min R.
(3) *$R/J_{\mathscr{C}}$ is a semi-hereditary ring for all subsets $\mathscr{C}$ of* min R.
(4) *If $\mathscr{C}$ is any subset of* min R *and M is a maximal ideal of R, then $J_{\mathscr{C}} \subset M$ iff $J_{\mathscr{C}} \subset O_M$, the unique minimal prime ideal of R contained in M.*

Proof. (1) $\Rightarrow$ (2) Let $\mathscr{C} = \{P_\alpha\}$ be a subset of min R. Since R_{P_α} is a flat R-module, and R is a coherent ring, ΠR_{P_α} ($P \in \mathscr{C}$) is a flat R-module. Since w.gl.dim $R \leq 1$, and $R/J_{\mathscr{C}} \subset \Pi R_{P_\alpha}$ ($P_\alpha \in \mathscr{C}$), it follows that $R/J_{\mathscr{C}}$ is a flat R-module.

(2) $\Rightarrow$ (4) Let $\mathscr{C}$ be a subset of min R and M a maximal ideal of $R \ni J_{\mathscr{C}} \subset M$. By Proposition 2.5(5), we have $(J_{\mathscr{C}})_M = 0$. Thus $J_{\mathscr{C}} \subset O_M$, the unique minimal prime ideal of R contained in M.

(4) $\Rightarrow$ (2) Let $\mathscr{C}$ be a subset of min R and M a maximal ideal of R. If $J_{\mathscr{C}} \not\subset M$, then $(J_{\mathscr{C}})_M = R_M$; while if $J_{\mathscr{C}} \subset M$, then $J_{\mathscr{C}} \subset O_M$ and hence $(J_{\mathscr{C}})_M = 0$. Thus by Proposition 2.5, $R/J_{\mathscr{C}}$ is a flat R-module.

(2) $\Rightarrow$ (1) Let $\mathscr{C}$ be a good subset of min R. Then by Proposition 2.12, $\mathscr{C}'$ is also a good subset of min R. Thus by Proposition 2.9, $Q(R)$ is a VNR. Therefore by Proposition 2.7, R is a semi-hereditary ring.

(3) $\Rightarrow$ (1) Take $\mathscr{C} =$ min R, so that $J_{\mathscr{C}} = 0$.

(1) $\Rightarrow$ (3) Let $\mathscr{C}$ be a subset of min R. Since (1) $\Rightarrow$ (2), $R/J_{\mathscr{C}}$ is a flat R-module. Therefore w.gl.dim $R/J_{\mathscr{C}} \leq$ w.gl.dim $R \leq 1$. By Proposition 2.10, $R/J_{\mathscr{C}}$ is a PIP. Therefore by Proposition 2.7, $R/J_{\mathscr{C}}$ is a semi-hereditary ring.

Remarks. (1) Let R be a commutative semi-hereditary ring, and let J be an ideal of R. Then it follows from Propositions 2.12 and 2.15 that J is a direct summand of R iff $J = J_{\mathscr{C}}$ where $\mathscr{C}$ is an open and closed subset of min R. An equivalent formulation of this fact is that a subset $\mathscr{C}$ of min R is good $\Leftrightarrow$ $\mathscr{C}$ is open and closed in min R.

(2) In order to illustrate how close a PIF is to being an integral domain, we make the following definitions. Let R be a PIF, min $R = \{P_\alpha\}$, and $C = \Pi R_{P_\alpha}$. If A is an R-module, we say that A is a *torsion R-module* if $\mathrm{Hom}_R(A, C) = 0$; and we say that A is *torsion-free* if A has no non-zero torsion submodules. We define $t(A)$ to be the sum of all of the torsion submodules of A. Then $t(A)$ is the unique largest torsion submodule of A, and $A/t(A)$ is torsion-free.

Since C is an injective R-module, submodules as well as factor modules of torsion modules are again torsion modules. Thus $t(A)$ has the usual properties of a torsion-functor. It is easy to verify that $t(A) = \cap \operatorname{Ker} f$, $f \in \mathrm{Hom}_R(A, C)$; and that A is torsion-free iff A can be embedded in a direct product of copies of C.

If M is a maximal ideal of R, and P is the unique minimal prime ideal of R contained in M, then by Proposition 2.1, R_M is an integral domain and R_P is the quotient field of R_M. It follows readily from this that A is a torsion R-module iff A_M is a torsion R_M-module $\forall$ maximal ideals of R. As a consequence it can easily be shown that if A is a flat R-module, then A is a torsion-free R-module.

3. The Injective Envelope of a Reduced Ring

Let R be a reduced ring and $\{P_\alpha\} =$ min R. By Proposition 1.1, $E(R)$ is a direct summand of ΠR_{P_α}, $P_\alpha \in$ min R; and by Proposition 1.12, $E(R)$ is a self-injective VNR. In this section we shall describe some of the structure of $E(R)$, and also exactly how it sits in ΠR_{P_α}.

PROPOSITION 3.1. *Let R be a reduced ring and $\{P_\alpha\} = \min R$. Then $E(R)$ is a subdirect product of the R_{P_α}'s.*

Proof. Let $E = E(R)$. By Proposition 1.1 we can identify E with its image in ΠR_{P_α}. Let e_α be the identity element of R_{P_α} so that $1 = \langle e_\alpha \rangle$. Let $P_\beta \in \min R$; then the projection of ΠR_{P_α} onto R_{P_β} induces an R-homomorphism $f_\beta\colon E \to R_{P_\beta}$ such that $f(1) = e_\beta$. We wish to show that f_β is onto.

Let $a \in R - P_\beta$; since E is a VNR, Ea is a direct summand of E. Hence there is an ideal F of E with $E = Ea \oplus F$. Thus $F = \operatorname{Ann}_E a$. Now a is a unit in R_{P_β}; and hence if $x \in F$, then $0 = f_\beta(ax) = af_\beta(x)$ shows that $f_\beta(x) = 0$. Thus $F \subset \operatorname{Ker} f_\beta$, and so $f_\beta(E) = f_\beta(Ea) = af_\beta(E)$. Therefore, $f_\beta(E)$ is an R/P_β-divisible submodule of R_{P_β}. Since R_{P_β} is the quotient field of R/P_β it follows that $f_\beta(E) = R_{P_\beta}$.

This shows that E is a subdirect product of the R_{P_α}'s.

DEFINITION. Let R be a commutative ring with 1, $\{P_\alpha\}$, $\alpha \in \mathscr{A}$, a collection of distinct prime ideals of R, and let $A = \Pi_\alpha R_{P_\alpha}$. With componentwise addition and multiplication, A is a commutative ring with identity and we have a canonical ring homomorphism $R \to A$. We identify $\operatorname{Hom}_A(A, A)$ with A in the usual way via left multiplication by elements of A. For each $\alpha \in \mathscr{A}$, let

$$O_\alpha = \{r \in R \mid sr = 0 \text{ for some } s \in R - P_\alpha\},$$

and let $J_\alpha = \cap O_\beta$, $\beta \neq \alpha$. If I and J are ideals of R, we define

$$(I : J) = \{r \in R \mid rJ \subset I\}.$$

PROPOSITION 3.2. *If $(O_\alpha : J_\alpha) = O_\alpha$, $\alpha \in \mathscr{A}$, then* $\operatorname{Hom}_R(A, A) = A$.

Proof. Let $B_\alpha = \Pi R_{P_\beta}$, $\beta \neq \alpha$. We shall first prove that $\operatorname{Hom}_R(B_\alpha, R_{P_\alpha}) = 0$, $\alpha \in \mathscr{A}$. Let $f \in \operatorname{Hom}_R(B_\alpha, R_{P_\alpha})$, $x = \langle x_\beta \rangle \in B_\alpha$, and $a \in J_\alpha$. Since $R_{P_\beta} O_\beta = 0$ for all β, we have $ax = \langle ax_\beta \rangle = 0$. Thus $af(x) = 0$. Now $f(x) = r/s$, where $r \in R$ and $s \in R - P_\alpha$. Since $af(x) = 0$, there exists $u \in R - P_\alpha$ with $uar = 0$. Thus $ar \in O_\alpha$ for $a \in J_\alpha$, and so $r \in (O_\alpha : J_\alpha) = O_\alpha$. Therefore, $r/s = 0$, and hence $f = 0$.

Let $i_\alpha : R_{P_\alpha} \to A$ and $\Pi_\alpha : A \to R_{P_\alpha}$ be the canonical inclusion and projection maps, respectively. Let $f \in \operatorname{Hom}_R(A, A)$ and define $f_\alpha : R_{P_\alpha} \to R_{P_\alpha}$ by $f_\alpha = \Pi_\alpha f i_\alpha$; then f_α is multiplication by $q_\alpha \in R_{P_\alpha}$. We let $q = \langle q_\alpha \rangle \in A$, and we shall show that f is multiplication by q.

Let $x = \langle x_\alpha \rangle \in A$, where $x_\alpha \in R_{P_\alpha}$. For each $\alpha \in \mathscr{A}$, we can write $x = i_\alpha(x_\alpha) + y_\alpha$, where $y_\alpha \in B_\alpha$. Let $h_\alpha = f|B_\alpha$; then $\Pi_\alpha h_\alpha \in \operatorname{Hom}_R(B_\alpha, R_{P_\alpha}) = 0$. Thus

$$\Pi_\alpha f(y_\alpha) = \Pi_\alpha h_\alpha(y_\alpha) = 0.$$

Hence $\Pi_\alpha f(x) = \Pi_\alpha fi_\alpha(x_\alpha) + \Pi_\alpha f(y_\alpha) = f_\alpha(x_\alpha)$, $\alpha \in \mathcal{A}$. Therefore,

$$f(x) = \langle f_\alpha(x_\alpha)\rangle = \langle q_\alpha x_\alpha\rangle = q\langle x_\alpha\rangle = qx \text{ for all } x \in A.$$

Thus $\mathrm{Hom}_R(A, A) = \mathrm{A}$.

PROPOSITION 3.3. *Let R be a commutative ring; with the preceding notation, assume that $\cap_\alpha O_\alpha = 0$, and that R_{P_α} is a self-injective ring for $\alpha \in \mathcal{A}$. Then the following statements are equivalent*:

(1) $J_\alpha \neq 0$ *and* $J_\alpha \cap \mathrm{Ann}_R J_\alpha = 0$, $\alpha \in \mathcal{A}$.
(2) $\mathrm{Ann}_R J_\alpha = O_\alpha$, $\alpha \in \mathcal{A}$.
(3) $\mathrm{Hom}_R(A, A) = A$.
(4) $R \subset A$ *is an essential extension and* $J_\alpha \cap \mathrm{Ann}_R J_\alpha = 0$, $\alpha \in \mathcal{A}$.
(5) $E(R) \simeq A$, $(O_\beta : O_\alpha) = O_\beta$ *for* $\beta \neq \alpha$, *and* $J_\alpha \cap \mathrm{Ann}_R J_\alpha = 0$ *for all* $\alpha \in \mathcal{A}$.

Proof. (1) $\Rightarrow$ (2) Let $I_\alpha = \mathrm{Ann}_R J_\alpha$. Then

$$\mathrm{Ann}_{R_{P_\alpha}}(R_{P_\alpha} J_\alpha) = R_{P_\alpha} I_\alpha.$$

In one direction the inclusion is obvious. On the other hand, let

$$x = r/v \in \mathrm{Ann}_{R_{P_\alpha}}(R_{P_\alpha} J_\alpha)$$

where $r \in R$ and $v \in R - P_\alpha$. If $a \in J_\alpha$, then $(ra)/v = 0$ in R_{P_α}, and so there exists $u \in R - P_\alpha$ such that $ura = 0$ in R. Therefore, $ra \in O_{P_\alpha}$, and hence $rJ_\alpha \subset O_{P_\alpha} \cap J_\alpha = 0$. Therefore, $r \in \mathrm{Ann}_R J_\alpha = I_\alpha$; and thus $x \in R_{P_\alpha} I_\alpha$.

Now $R_{P_\alpha} I_\alpha \cap R_{P_\alpha} J_\alpha = R_{P_\alpha}(I_\alpha \cap J_\alpha) = R_{P_\alpha} 0 = 0$ and

$$R_{P_\alpha} I_\alpha \oplus R_{P_\alpha} J_\alpha = \mathrm{Ann}_{R_{P_\alpha}}(R_{P_\alpha} J_\alpha) \oplus R_{P_\alpha} J_\alpha$$

is an essential R_{P_α}-submodule of R_{P_α}. Thus, since R_{P_α} is self-injective, we have

$$R_{P_\alpha} = E(R_{P_\alpha} I_\alpha) \oplus E(R_{P_\alpha} J_\alpha).$$

But R_{P_α} is a quasi-local ring, and hence decomposable. If $R_{P_\alpha} J_\alpha = 0$, then $J_\alpha \subset O_{P_\alpha}$, and so $J_\alpha = J_\alpha \cap O_{P_\alpha} = 0$, contrary to hypothesis. Therefore $R_{P_\alpha} I_\alpha = 0$, and so $I_\alpha \subset O_\alpha$. Since $O_\alpha J_\alpha = 0$, we have $O_\alpha \subset I_\alpha$. Thus $O_\alpha = I_\alpha = \mathrm{Ann}_R J_\alpha$, $\alpha \in \mathcal{A}$.

(2) $\Rightarrow$ (3) Since $\cap_\alpha O_\alpha = 0$, we have $(O_\alpha : J_\alpha) = \mathrm{Ann}_R J_\alpha$, $\alpha \in \mathcal{A}$. Hence $\mathrm{Hom}_R(A, A) = A$ by Proposition 3.2.

(3) $\Rightarrow$ (4) The kernel of the canonical map $R \to A$ is $\cap_\alpha O_\alpha = 0$, and hence $R \subset A$. Since A is an injective R-module we have $E(R) \subset A$. Thus $A = E(R) \oplus X$, where X is an R-submodule of A and $1 \in E(R)$. Let $f : A \to A$ be the R-homomorphism that is the identity on $E(R)$ and O on X. By hypothesis, f is multiplication by $q \in A$. Hence $q = q \cdot 1 = f(1)$

$= 1$. Thus if $x \in X$, we have $0 = f(x) = qx = 1 \cdot x$. Therefore, $X = O$, and $A = E(R)$.

Let β be a fixed index and let $t \in J_\beta \cap \operatorname{Ann}_R J_\beta$. Let $y_\alpha = 0$, $\alpha \neq \beta$, let y_β be the image of t in R_{P_β}, and let $y = \langle y_\alpha \rangle$. Let x_α be the identity of R_{P_α}, $\alpha \neq \beta$, let $x_\beta = 0$, and let $x = \langle x_\alpha \rangle$. Then $\operatorname{Ann}_R x = J_\beta$, and $J_\beta \subset \operatorname{Ann}_R y$. Thus there is an R-homomorphism $f : Rx \to Ry$ such that $f(x) = y$. Since A is an injective R-module, f extends to an R-homomorphism from A to A. By hypothesis, this homomorphism is multiplication by an element $q = \langle q_\alpha \rangle \in A$. Therefore, $qx = y$ and so $0 = q_\beta x_\beta = y_\beta$. Thus $t \in O_\beta$, and since $t \in J_\beta$, we have $t \in O_\beta \cap J_\beta = 0$. Thus $J_\beta \cap \operatorname{Ann}_R J_\beta = 0$.

(4) $\Rightarrow$ (5) Since A is injective, we have $E(R) = A$. Let α,β be a fixed pair of indices, $\alpha \neq \beta$, let $s \in (O_\beta : O_\alpha)$ and suppose $s \notin O_\beta$. Let $y_\gamma = 0$, $\gamma \neq \beta$, let y_β be the image of s in R_{P_β}, and let $y = \langle y_\gamma \rangle$. Since A is an essential extension of R, there is an $r \in R$ with $ry = a \neq 0 \in R$. We have identified a with the element $\langle a_\gamma \rangle$, where a_γ is the image of a in R_{P_γ} for all γ. Therefore $ry_\gamma = a_\gamma$, $\gamma \in \mathscr{A}$. Therefore, $a_\gamma = 0$, $\gamma \neq \beta$, and thus $a \in J_\beta$. Since $s \in (O_\beta : O_\alpha)$ we have $O_\alpha s \subset O_\beta$. Therefore, $O_\alpha y_\beta = 0$, and hence $O_\alpha y = 0$. Thus $O_\alpha a = rO_\alpha y = r \cdot 0 = 0$. Since $J_\beta \subset O_\alpha$, we have $J_\beta a = 0$. Therefore, $a \in J_\beta \cap \operatorname{Ann}_R J_\beta = 0$. This contradiction shows that $(O_\beta : O_\alpha) = O_\beta$.

(5) $\Rightarrow$ (1) Suppose $J_\alpha = 0$. Let $B_\alpha = \Pi R_{P_\beta}$, $\beta \neq \alpha$. Since J_α is the kernel of the canonical map $R \to B_\alpha$, we have $R \subset B_\alpha$. Since B is injective, we have $A \simeq E(R) \subset B_\alpha$. Now $R_{P_\alpha} \subset A$, and hence there is an $x = \langle x_\beta \rangle \in B_\alpha$ with $\operatorname{Ann}_R x = O_\alpha$. Now there exists $\beta \neq \alpha$ so that $x_\beta \neq 0$, and we have $O_\alpha \subset \operatorname{Ann}_R x_\beta$. Since $x_\beta = t/u$ where $t \in R$ and $u \in R - P_\beta$, we have $O_\alpha t \subset O_\beta$. Thus by hypothesis, $t \in (O_\beta : O_\alpha) = O_\beta$. But then $x_\beta = 0$. This contradiction shows that $J_\alpha \neq 0$.

Note. It is easy to see directly that $\operatorname{Ann}_R J_\beta = O_\beta$ implies $(O_\beta : O_\alpha) = O_\beta$. For suppose that $t \in (O_\beta : O_\alpha)$. Since $J_\beta \subset O_\alpha$, we have $J_\beta t \subset O_\beta$. Thus $J_\beta t \subset O_\beta \cap J_\beta = 0$. Therefore, $t \in \operatorname{Ann}_R J_\beta = O_\beta$ by assumption.

DEFINITION. We shall let n-min R denote the set of those minimal prime ideals of R that are not essential ideals of R.

PROPOSITION 3.4. *Let R be a reduced ring.*

(1) *Let P be a prime ideal of R. Then $P \in$ n-*min *R iff* $\operatorname{Ann}_R P \neq 0$.

(2) *If P is a prime ideal of R and $0 \neq a \in \operatorname{Ann}_R P$, then $P = \operatorname{Ann}_R a$; and a is an element of every prime ideal in* min *R that is not equal to P.*

Proof. (1) Suppose that $P \in$ n-min R. Then there exists $a \in R$, $a \neq 0$, $Ra \cap P = 0$. Thus $Pa = 0$, and so $\operatorname{Ann}_R P \neq 0$. Conversely, suppose that $\operatorname{Ann}_R P \neq 0$, and let $0 \neq a \in \operatorname{Ann}_R P$. Because R is reduced $Ra \cap P = 0$; and hence P is a non-essential ideal of R. Suppose that P_1 is a prime ideal of R and that $P_1 \subset P$. Since $a \notin P$, and $Pa = 0 \subset P_1$, we have $P \subset P_1$. Thus $P \in$ n-min R.

(2) Since $Pa = 0$, we have $P \subset \text{Ann}_R a$. But $a \notin P$; and $a \cdot (\text{Ann}_R a) = 0 \subset P$ implies $\text{Ann}_R a \subset P$. Let $P' \in$ min R and $P' \neq P$. Then $aP = 0 \subset P'$ implies $a \in P'$.

PROPOSITION 3.5. *Let R be a reduced ring; let $\{P_\alpha\}$, $\alpha \in \mathcal{A}$, be a subset of* min R, *and let $A = \Pi_\alpha R_{P_\alpha}$. Assume that $\cap P_\alpha = 0$. Then the following statements are equivalent:*

(1) $\{\cap_\beta P_\beta \mid \beta \neq \alpha\} \neq 0$, $\alpha \in \mathcal{A}$.
(2) $\text{Ann}_R P_\alpha \neq 0$, $\alpha \in \mathcal{A}$ (i.e., $P_\alpha \in$ n-min R, $\alpha \in \mathcal{A}$).
(3) $\text{Hom}_R(A, A) = A$.
(4) *$R \subset A$ is an essential extension.*
(5) $E(R) \simeq A$.

Thus $E(R) \simeq \Pi R_{P_\alpha}$ where P_α ranges over all elements of min R *if and only if* min R *is totally disconnected.*

Proof. Let $O_\alpha = \{r \in R \mid ur = 0$ for some $u \in R - P_\alpha\}$. By Proposition 1.1, $O_\alpha = P_\alpha$ and R_{P_α} is a self-injective ring for all α. Thus with the notation of Proposition 3.3, $J_\alpha = \cap_\beta P_\beta$, $\beta \neq \alpha$. Since R is reduced, we have $J_\alpha \cap \text{Ann}_R J_\alpha = 0$ for all α. It is also obvious that $(P_\beta : P_\alpha) = P_\beta$, $\beta \neq \alpha$. Thus the equivalence of (1)–(5) is a consequence of Proposition 3.3. The final statement of Proposition 3.5 follows from (1) and the fact that an element P in min R is an open set in min R if and only if there exists an element x of R such that P is the only prime ideal in min R that does not contain x.

PROPOSITION 3.6. *Let R be a reduced ring and $\{P_\beta\}$, $\beta \in \mathcal{B}$, be a subset of* min R *so that $E(R) \simeq \Pi R_{P_\beta}$ $(\beta \in \mathcal{B})$.*

(1) *$\cap P_\beta = 0$; and the P_β's are all distinct.*
(2) $\{P_\beta\} =$ n-min R.

Thus the representation $E(R) \simeq \Pi R_{P_\beta}$ (if it exists) is unique.

Proof. (1) Since $\cap P_\beta$ annihilates $E(R)$, we have $\cap P_\beta = 0$. Suppose that $P_{\beta_1} = P_{\beta_2} = P$ for $\beta_1 \neq \beta_2$. Then there are elements x and y of $E(R)$ such that $\text{Ann}_R x = P = \text{Ann}_R y$ and $Rx \cap Ry = 0$. Now there are $r, t \in R$ with $0 \neq rx = a \in R$ and $0 \neq ty = b \in R$. We have $Pa = 0 = Pb$, and thus $a \notin P$ and $b \notin P$. However, $ab \in Rx \cap Ry = 0 \in P$. This contradiction shows that the P_β's are distinct.

(2) If $\beta \in \mathcal{B}$, then there exists nonzero $a_\beta \in R$ with $P_\beta a_\beta = 0$, and hence $P_\beta \in$ n-min R by Proposition 3.4. Conversely, let $P \in$ n-min R and $0 \neq a \in \text{Ann}_R P$. Since $a \in \Pi R_{P_\beta}$, there exists $\beta \in \mathcal{B}$ such that $\text{Ann}_R a \subset P_\beta$. Thus $P = P_\beta$, and so $\{P_\beta\} =$ n-min R.

PROPOSITION 3.7. *Let R be a reduced ring such that* min R *is compact. Then the following statements are equivalent*:

(1) $E(R) \simeq \Pi_\alpha R_{P_\alpha}$, *where P_α ranges over all elements of* min R.
(2) *Every minimal prime ideal of R is non-essential.*
(3) min R *is finite.*

Proof. (1) ⇒ (2) follows from Proposition 3.6; and (3) ⇒ (1) is Proposition 1.6.

(2) ⇒ (3) For each $\alpha \in \mathscr{A}$, there exists nonzero $a_\alpha \in \mathrm{Ann}_R P_\alpha$. By Proposition 3.4, $a_\alpha \in \cap P_\beta$, $\beta \neq \alpha$. Thus $P_\alpha = D(a_\alpha)$ is an open subset of min R. Therefore, min R is finite.

PROPOSITION 3.8. *Let R be a commutative ring and let $\{P_\alpha\}$ =* min R. *Then* min R *is finite iff $P_\alpha \not\subset \{\cup P_\beta \mid \beta \neq \alpha\}$ for all α.*

Proof. If min R is finite, then the assertion is an elementary and well-known fact. On the other hand, assume that $P_\alpha \not\subset \{\cup P_\beta \mid \beta \neq \alpha\}$ for all α. By factoring out $\{\cap P_\alpha \mid \alpha \in \mathscr{A}\}$ we can assume without loss of generality that R is reduced. Let $A = \Pi R_{P_\alpha}$, then A is a commutative ring and $R \subset A$.

Suppose that min R is not finite. Then $\Sigma_\alpha \oplus R_{P_\alpha}$ is a proper ideal of A, and hence is contained in a maximal ideal $\mathscr{M}$ of A. Then $\mathscr{M} \cap R$ contains a minimal prime ideal P_γ of R. By hypothesis there is an $a \in P_\gamma$ such that $a \notin \{\cup P_\beta \mid \beta \neq \gamma\}$. Let a_α be the image of a in R_{P_α} for all α; then by our identification $R \subset A$ we have $a = \langle a_\alpha \rangle$. For $\beta \neq \gamma$, a_β is a unit in R_{P_β} because R_{P_β} is a field and $a_\beta \neq 0$. Let $u_\beta = a_\beta^{-1}$ for $\beta \neq \gamma$, let $u_\gamma = 0$, and let $u = \langle u_\alpha \rangle \in A$. Since $a \in P_\gamma \subset \mathscr{M}$, we have $ua \in \mathscr{M}$. But ua is the element of A that is the identity at every component $\beta \neq \gamma$ and is 0 at the γ-component. Since $\Sigma_\alpha \oplus R_{P_\alpha}$ is also contained in $\mathscr{M}$ we see that $1 \in \mathscr{M}$. This contradiction shows that min R is finite.

PROPOSITION 3.9. *Let R be a reduced ring and $E = E(R)$. There is a* 1-1 *correspondence of* n-min E *onto* n-min R *such that if $M \in$* n-min E, *then $M \cap R = P \in$* n-min R. *In this case $E/M \simeq E_M \simeq R_P$ and M is the only prime ideal of E contracting to P.*

Proof. Let $M \in$ n-min E and $P = M \cap R$. Since $(\mathrm{Ann}_E M) \cap R \neq 0$, there is a nonzero $t \in R$ with $Mt = 0$. But then $Pt = 0$; and hence by Proposition 3.4, $P \in$ n-min R. If N is any prime ideal of E satisfying $N \cap R = P$, then $t \notin N$. But $Mt = 0$; and so $M \subset N$. Thus $M = N$, since the prime ideals of E are all minimal.

On the other hand, let $P \in$ n-min R. Since $R_P \subset E_P$, and R_P is a field, there is a prime ideal M of E with $M \cap R = P$. Now there exists $t \in R$ such that $Pt = 0$. Suppose that $Mt \neq 0$. Then there exist $m \in M$ and

$r \in R$ such that $0 \neq s = rmt \in R$. But then $Ps = 0$ and $s \in P$. This contradiction proves that $Mt = 0$, and hence $M \in$ n-min E.

Now we have a canonical injection $R/P \subset E/M$; and this is an essential extension. For let $0 \neq x \in E/M$. Since $t \notin M$, and M is a maximal ideal of E, there is a $y \in Et$ such that $x = y + M$. Also there exists $r \in R$ with $0 \neq ry \in R$. Since $Pry = 0$, $ry \notin P$. Thus $rx = r(y + M)$ is a non-zero element of R/P. Therefore, $R/P \subset E/M$ is an essential extension. But E/M is a field, and thus E/M is the quotient field of R/P. Therefore, since R_P is the quotient field of R/P, and $E_M \simeq E/M$, we have $R_P \simeq E_M$.

PROPOSITION 3.10. *Let R be a reduced ring and J an ideal of R. Let $I = \mathrm{Ann}_R J$, $K = \mathrm{Ann}_R I$, and $E = E(R)$.*

(1) *$E = E(J) \oplus E(I)$; and $E(J)$ and $E(I)$ are ideals of E.*

(2) *$J \subset K$ is an essential extension, and so $E(J) = E(K)$.*

(3) *$E(I) \cap R = I$ and $E(J) \cap R = K$. Thus R/I and R/K are reduced rings.*

(4) *$E(J) = \mathrm{Ann}_E I$; and thus $E(J)$ is a unique submodule of E.*

(5) *$E(I) \simeq E(R/K)$ and $E(K) \simeq E(R/I)$.*

(6) *$E(R/I)$ is an R/I-module, and as such it is the injective envelope of the ring R/I. A similar statement holds for $E(R/K)$.*

(7) *$E \simeq E(R/I) \oplus E(R/K)$ is a ring direct sum decomposition.*

Proof. (1) Since R is reduced, $J \cap I = 0$. It is easily seen that $I \oplus J$ is an essential ideal of R. Thus $E = E(I) \oplus E(J)$. By Proposition 1.13, $E(J)$ and $E(I)$ are ideals of E.

(2) Since $IJ = 0$, we have $J \subset K$. Let t be a non-zero element of K. Since $I \oplus J$ is essential in R, there exists $r \in R$ such that $0 \neq rt = a + b$ where $a \in I$ and $b \in J$. Then $a = rt - b \in I \cap K = 0$; hence $rt = b \in J$. Thus $J \subset K$ is an essential extension. Therefore, $E(J) = E(K)$.

(3) Of course, $I \subset (E(I) \cap R)$. On the other hand, $J \cdot E(I) \subset E(J) \cap E(I) = 0$, and so $(E(I) \cap R) \subset I$. A similar argument shows that $E(K) \cap R = K$. By (2), $E(J) = E(K)$ and so $E(J) \cap R = K$.

By Proposition 1.3, $E(I)$ is an intersection of some prime ideals of E. Thus $I = E(I) \cap R$ is an intersection of some prime ideals of R. Therefore, R/I is a reduced ring. Similarly, R/K is a reduced ring.

(4) We have $I \cdot E(J) \subset E(I) \cap E(J) = 0$, and so $E(J) \subset \mathrm{Ann}_E I$. Because $E(I)$ is an essential extension of I, there is no non-zero element of $E(I)$ that is annihilated by I. Hence $E(J) = \mathrm{Ann}_E I$.

(5) Since $I \oplus K$ is essential in R, it is easy to see that I is isomorphic to an essential R-submodule of R/K. Thus $E(I) \simeq E(R/K)$. Similarly $E(K) \simeq E(R/I)$.

(6) Since $E(K) = E(J) = \mathrm{Ann}_E I$; and $E(K) \simeq E(R/I)$, we see that $E(R/I)$ is annihilated by I. Thus $E(R/I)$ is an R/I-module. Clearly as such it is injective and essential over R/I. By symmetry we have a similar statement for $E(R/K)$.

(7) There is a canonical monomorphism of rings: $R \to R/I \oplus R/K$; and as R-modules it is not difficult to verify that this is an essential extension. Hence we have an induced R-module isomorphism $\theta : E \simeq E(R/I) \oplus E(R/K)$. By (6), $E(R/I) \oplus E(R/K)$ has a ring structure that is compatible with that of R. By the remarks following Proposition 1.12, θ is a ring isomorphism.

PROPOSITION 3.11. *Let R be a reduced ring and let $\{P_\gamma \mid \gamma \in \Gamma\}$ be a non-empty subset of* n-min R. *Let $I = \cap_\gamma P_\gamma$; and let J be the intersection of the minimal primes of R that are not in $\{P_\gamma \mid \gamma \in \Gamma\}$. Let $\bar{R} = R/I$, $\bar{P}_\gamma = P_\gamma/I$, and $\bar{E}$ be the injective envelope of $\bar{R}$ over $\bar{R}$.*

(1) *$\bar{R}$ is a reduced ring; $\{\bar{P}_\gamma \mid \gamma \in \Gamma\}$ =* n-min $\bar{R}$; *and $\bar{E} = \Pi_\gamma \bar{R}_{\bar{P}_\gamma}$.*

(2) *$I = \mathrm{Ann}_R J$; and $\bar{R}_{\bar{P}_\gamma} \simeq R_{P_\gamma}$; moreover, $\bar{E} \simeq E(R/I) \simeq \Pi R_{P_\gamma}$ is a direct summand of $E(R)$.*

Proof. (1) We have $\cap_\gamma \bar{P}_\gamma = 0$, and so $\bar{R}$ is a reduced ring. Now there exists $a_\gamma \notin P_\gamma$ such that $P_\gamma a_\gamma = 0$. But then $\bar{a}_\gamma \neq 0$ and $\bar{P}_\gamma \bar{a}_\gamma = 0$ shows that $\bar{P}_\gamma \in$ n-min $\bar{R}$. By Proposition 3.5 we have $\bar{E} \simeq \Pi_\gamma \bar{R}_{\bar{P}_\gamma}$. Hence by Proposition 3.6, $\{P_\gamma \mid \gamma \in \Gamma\}$ = n-min $\bar{R}$.

(2) $I \cap J$ is the intersection of all of the minimal prime ideals of R, and thus $I \cap J = 0$. Therefore, $I \subset \mathrm{Ann}_R J$. By Proposition 3.4, $P_\gamma = \mathrm{Ann}_R\, a_\gamma$ and $a_\gamma \in J$. Now if $r \in \mathrm{Ann}_R J$, then $ra_\gamma = 0$, and hence $r \in P_\gamma$, $\gamma \in \Gamma$. Thus $r \in I$. Hence $I = \mathrm{Ann}_R J$. Thus by Proposition 3.10, $\bar{E} \simeq E(R/I)$ and $E(R/I)$ is a direct summand of $E(R)$. The only thing remaining to be proved is that $\bar{R}_{\bar{P}_\gamma} \simeq R_{P_\gamma}$. But R_{P_γ} is the quotient field of R/P_γ and $\bar{R}_{\bar{P}_\gamma}$ is the quotient field of $\bar{R}/\bar{P}_\gamma \simeq R/P_\gamma$. Hence $\bar{R}_{\bar{P}_\gamma}$ and R_{P_γ} are isomorphic R-modules.

DEFINITION. Let R be a reduced ring; and $\{P_\beta \mid \beta \in \mathscr{B}\}$ = n-min R; and let $\{P_\delta \mid \delta \in \Delta\}$ be the set of all essential minimal prime ideals of R. Let $\mathscr{I}(R) = \cap_\beta P_\beta$ and $\mathscr{J}(R) = \cap_\delta P_\delta$. (By convention we put the intersection of an empty set of ideals equal to R.)

PROPOSITION 3.12. *Let R be a reduced ring; $E = E(R)$; $I = \mathscr{I}(R)$; $J = \mathscr{J}(R)$; and $K = \mathrm{Ann}_R I$.*

(1) *$I = \mathrm{Ann}_R J$. Thus all of the statements of Proposition 3.10 are true in this case.*

(2) *$E(R/I) \simeq \Pi R_{P_\beta}$ ($\beta \in \mathscr{B}$) is a direct product of fields.*

(3) *$E \simeq \Pi R_{P_\beta} \oplus E(R/K)$.*

(4) If $I \neq 0$, *then R/K is a reduced ring with no non-essential minimal prime ideals and $E(R/K)$ is the R/K-injective envelope of R/K and hence is a self-injective VNR with the same property.*

Proof. (1) and (2) follow from Proposition 3.11, and (3) follows from Proposition 3.10.

(4) Assume $I \neq 0$. Then by Proposition 3.10, R/K is a reduced ring and $E(R/K)$ is the R/K-injective envelope of R/K. Thus $E(R/K)$ is a self-injective VNR.

Suppose that P is a prime ideal of R such that $P \supset K$ and $P/K \in$ n-min(R/K). Then there exists $a \in R - K$ with $Pa \subset K$. Because $K = \text{Ann}_R I$, we have $PaI = 0$. But since $a \notin K$, we have $aI \neq 0$. Then $P \in$ n-min R. By Proposition 3.4, we have $aI \subset J$; and hence $aI \subset I \cap J = 0$. This contradiction shows that R/K has no non-essential minimal prime ideals. Hence by Proposition 3.9, $E(R/K)$ has no non-essential minimal prime ideals.

Remark. Let R be a reduced ring. It is clear from Proposition 3.12 that $E(R)$ is a direct product of fields iff $\mathcal{I}(R) = 0$ iff $\mathcal{J}(R)$ is an essential ideal of R. Thus if R has only a finite number (or no) essential minimal prime ideals, then R is a direct product of fields. On the other hand $E(R)$ has no direct summand that is a field iff every minimal prime ideal of R is essential iff $\mathcal{J}(R) = 0$. In general, $E(R)$ is a direct sum of two rings: one of which is a direct product of fields, and the other having no direct summand that is a field. We shall see by the examples in §4 that both kinds of summands can exist.

PROPOSITION 3.13. *Let R be a commutative, self-injective VNR. Let $\{A_\beta\}$, $\beta \in \mathcal{B}$, be the set of distinct simple submodules of R, and let $P_\beta = \text{Ann}_R A_\beta$. Let $I = \mathcal{I}(R)$, $J = \mathcal{J}(R)$, and $K = \text{Ann}_R I$. Then*

(1) *$\{P_\beta \mid \beta \in \mathcal{B}\}$ is the set of non-essential prime ideals of R; and $R_{P_\beta} \simeq A_\beta$.*

(2) *The sum of the A_β's is direct and $\Sigma \oplus A_\beta = J$.*

(3) *$K = E(J) = \Pi A_\beta$; and K is the intersection of the essential prime ideals of R that do not contain I. Thus $R = I \oplus K = (\cap P_\beta) \oplus \Pi A_\beta$; and if $I \neq 0$, then $I \simeq R/K$ is a self-injective* VNR *with no non-essential prime ideals.*

Proof. We recall that by Proposition 1.3, every prime ideal of R is a minimal prime ideal of R.

(1) Since A_β is simple, P_β is a maximal ideal; and by Proposition 3.4, P_β is a non-essential prime ideal. We have $A_\beta \simeq R/P_\beta \simeq R_{P_\beta}$. On the other hand let P be a non-essential prime ideal of R. By Proposition 3.4, there exists $a \in R$ with $P = \text{Ann}_R a$. Since P is a maximal ideal of R, $Ra \simeq R/P$ is a simple R-module. Hence P is one of the P_β's by definition.

(2) Let A_{β_1} and A_{β_2} be two different simple submodules of R. Then $A_{\beta_i} = Re_i$, $e_i^2 = e_i$ for $i = 1,2$ since R is a VNR. Now $e_1e_2 \in A_{\beta_1} \cap A_{\beta_2} = 0$. And hence e_1, e_2 are orthogonal. It follows from this that the sum of the A_β's is direct. By Proposition 3.4, $\Sigma \oplus A_\beta \subset J$. Since R is a VNR, $\Sigma \oplus A_\beta$ is the intersection of the prime ideals of R that contain it. Since

no P_β can contain $\Sigma \oplus A_\beta$, and J is the intersection of the essential prime ideals of R, we see that $\Sigma \oplus A_\beta = J$.

(3) By Proposition 3.12, $I = \mathrm{Ann}_R J$; and hence by Proposition 1.7, $R = I \oplus E(J)$. Therefore, $E(J) = \mathrm{Ann}_R I = K$. By Propositions 3.10 and 3.12,

$$E(J) \simeq E(R/I) \simeq \Pi R_{P_\beta} \simeq \Pi A_\beta;$$

and since $I = \cap P_\beta$, we have $R = (\cap P_\beta) \oplus \Pi A_\beta$. If $I \neq 0$, then $I \simeq R/K$ is a self-injective VNR with no non-essential minimal prime ideals by Proposition 3.12. Finally, since K is the intersection of the prime ideals of R that contain it, and $R = I \oplus K$, we see that K is the intersection of the prime ideals of R that do not contain I, and these are necessarily essential.

PROPOSITION 3.14. *Let R be a reduced ring.*

(1) *There are* 1-1 *correspondences between the sets of simple submodules* $\{A_\beta\}$ *of* $E(R)$, n-min $E(R) = \{M_\beta\}$ *and* n-min $R = \{P_\beta\}$, *given by* $P_\beta = \mathrm{Ann}_R A_\beta = M_\beta \cap R$.

(2) $\Sigma \oplus A_\beta = \mathscr{I}(E(R))$; *and* $\Pi A_\beta = E(\mathscr{I}(R)) = E(K)$, *where* $K = \mathrm{Ann}_R I$.

(3) $E(\mathscr{I}(R)) = \mathscr{I}(E(R))$.

(4) $E(J(R)) \cap R = K$ *and* $E(\mathscr{I}(R)) \cap R = \mathscr{I}(R)$.

Proof. (1) follows from Propositions 3.9 and 3.13.

(2) By Proposition 3.13 we have $\Sigma \oplus A_\beta = \mathscr{I}(E(R))$; and by Proposition 3.12 we have $E(\mathscr{I}(R)) = E(K) = E(R/I) \simeq \Pi R_{P_\beta} \simeq \Pi A_\beta$.

(3) Now $\mathscr{I}(E(R)) = \cap M_\beta$; and by Proposition 3.12,

$$\mathscr{I}(R) = \cap P_\beta = \cap M_\beta \cap R = \mathscr{I}(E(R)) \cap R.$$

Thus $\mathscr{I}(E(R))$ is an essential extension of $\mathscr{I}(R)$. By Proposition 3.13, $\mathscr{I}(E(R))$ is a direct summand of $E(R)$ and hence R-injective. Thus we have $E(\mathscr{I}(R)) = \mathscr{I}(E(R))$.

(4) follows from Proposition 3.12.

DEFINITION. Let R be a reduced ring and let $P \in \min R$. We shall say that P is *irrelevant* if P is an essential ideal of R and $P \supset \mathscr{I}(R)$. Otherwise an essential minimal prime will be called *relevant*.

PROPOSITION 3.15. *Let R be a reduced ring such that* min R *is compact. Then R has an irrelevant minimal prime ideal iff* n-min R *is infinite.*

Proof. If n-min R is finite, then $\mathscr{I}(R)$ is the intersection of finitely many non-essential minimal primes, and hence these are the only minimal primes that can contain $\mathscr{I}(R)$. Conversely, suppose that $\{P_\beta\}$ = n-min R is infinite. Let A_β be the simple submodule of $E(R)$ corresponding to P_β. Now ΠA_β is the intersection of the relevant essential prime ideals of $E(R)$; and $\Sigma \oplus$

A_β is the intersection of all of the essential prime ideals of $E(R)$ by Proposition 3.13. Since $\Sigma \oplus A_\beta \neq \Pi A_\beta$, $E(R)$ has an irrelevant prime ideal N. Thus

$$N \cap R \supset \mathscr{I}(E(R)) \cap R = E(\mathscr{I}(R)) \cap R = \mathscr{I}(R)$$

by Proposition 3.12. Since min R is compact, $N \cap R$ is a minimal prime ideal of R by Proposition 1.6. By Proposition 3.9, $N \cap R$ is an essential prime ideal of R. Thus $N \cap R$ is an irrelevant prime ideal of R.

4. Examples

In this section we present some examples to illustrate the ideas of this paper.

Example 1. Let $\mathscr{A}$ be an infinite index set; for each $\alpha \in \mathscr{A}$ let K_α be a field, and let $K = \Pi K_\alpha$ $(\alpha \in \mathscr{A})$. Then K is a self-injective VNR. Let e_α be the element of K that is the identity of K_α at the α-coordinate and 0 elsewhere; and let 1 be the identity of K. For each α let $P_\alpha = K(1 - e_\alpha)$; then P_α is a non-essential prime ideal of K and $\cap P_\alpha = 0$. Thus the P_α's are all of the non-essential prime ideals of K and $K \simeq \Pi K_{P_\alpha}$ by Propositions 3.5 and 3.6. Since $K_{P_\alpha} \simeq K_\alpha$, this is not surprising.

Let $J = \Sigma \oplus Ke_\alpha \simeq \Sigma \oplus K_\alpha$; then J is the sum of all of the simple submodules of K, and by Proposition 3.13, J is the intersection of all of the essential prime ideals of K. It is clear that there are elements a and b in $K - J$ such that $ab = 0$, and thus J is not a prime ideal of K. We put $R = K/J$; and then R is a VNR with an infinite number of prime ideals, and they are all essential in R. For let P be a prime ideal of K containing J; then P is an essential prime ideal of K. The problem is to show that P/J is essential in R.

Suppose that there is an $e \in K - J$ such that $Pe \subset J$. Without loss of generality we can assume that $e^2 = e$. Since R/J is reduced, $e \notin P$. Thus $P = K(1 - e) + J$. If $x \in K$, we define Supp x to be the set of coordinates α in $\mathscr{A}$, where x is not 0. Thus J is the set of elements $x \in K$ such that Supp x is finite; and P is the set of elements $x \in K$ such that Supp $x \subset$ Supp$(1 - e)$ except for a finite number of coordinates.

Now Supp e and Supp$(1 - e)$ are complementary subsets of $\mathscr{A}$. Supp e is not finite because $e \notin J$, and Supp$(1 - e)$ is not finite because $P \neq J$. Thus we can write each of Supp e and Supp$(1 - e)$ as disjoint unions of two infinite sets:

$$\text{Supp}\, e = A_1 \cup A_2 \quad \text{and} \quad \text{Supp}(1 - e) = B_1 \cup B_2.$$

We let c be the element of K such that the α-coordinate of c is the identity of K_α for $\alpha \in A_1 \cup B_1$ and 0 for $\alpha \in A_2 \cup B_2$; and we let d be the element of K such that the α-coordinate of d is the identity of K_α for $\alpha \in A_2 \cup B_2$ and 0 for $\alpha \in A_1 \cup B_1$. Then c and d are not in P, but $cd = 0$. This contradiction shows that every prime ideal of R is essential in R, and hence they are infinite in number.

Remarks. (1) It is an open question whether or not the ring R of example (1) is self-injective.

(2) Let R be any self-injective VNR that is not a finite direct sum of fields; and let J be the sum of all of the simple submodules of R. (J could be 0.) Then R/J is a VNR with an infinite number of prime ideals and they are all essential in R. For by Proposition 3.13 the proof can easily be reduced to the case of Example 1. The question of whether or not R/J is self-injective is a generalization of the open question posed by Example 1.

Example 2. Let D be an integral domain and N the natural numbers. Let $D_n = D$, $n \in N$; and let 1 be the identity of ΠD_n. We put $R = \Sigma \oplus D_n + D \cdot 1$; i.e., R is the set of sequences in ΠD_n that are ultimately constant. In the future we shall denote this ring by $D(\infty)$.

We let e_n be the element of R that is the identity of D at the *n-th* coordinate, and 0 elsewhere, and we put $P_n = R(1 - e_n)$. Then P_n is a non-essential prime ideal of R and $\cap P_n = 0$. Hence by Propositions 3.5 and 3.6, the P_n's are all of the non-essential prime ideals of R, and $E(R) \simeq \Pi R_{P_n}$. Since $R/P_n \simeq D_n = D$, it follows that $R_{P_n} \simeq Q_n$, the quotient field of D, and we have $E(R) \simeq \Pi Q_n$.

It is clear that the annihilator of an element of R is generated by an idempotent element of R, and thus R is a PIP.

Let $J = \Sigma \oplus Re_n \simeq \Sigma \oplus D_n$. Since $R/J \simeq D$, J is a prime ideal of R and is the only essential minimal prime ideal of R. Since $R/P \simeq D$ for every minimal prime ideal P of R, and since $P = O_M$ for every maximal ideal M of R that contains P, we see that w.gl.dim R = w.gl.dim D. Since R is a PIP, it follows from Proposition 2.7 that R is semi-hereditary iff w.gl.dim $D \leq 1$ (i.e., D is a Prüfer domain).

Let Q be the quotient field of D; then it is easily seen that $Q(R) = Q(\infty)$. It is of course easy to verify directly that $Q(\infty)$ is a VNR (so that w.gl.dim $Q(\infty) = 0$). Since $Q(\infty)$ has only a countable number of idempotents, it follows from [8, Corollary 2.15] that every ideal of $Q(\infty)$ is a projective $Q(\infty)$-module. Thus $Q(\infty)$ is a non-Noetherian hereditary ring (i.e., gl.dim $Q(\infty) = 1$). $Q(R)$ is not a self-injective ring, since $E(Q(\infty)) = E(R) \simeq \Pi Q_n$.

Example 3. The following example of a ring R was constructed by Vasconcelos [13, Example 3.2] as an example of a commutative ring of w.gl.dim 1 that is not semi-hereditary. Our chief interest lies in computing $E(R)$ and showing that $E(R) \simeq \Pi R_P$, where P ranges over all of min R, even though min R is infinite. The example is a slight modification of Example 2, but the modification produces some interesting consequences.

Let N be the natural numbers, Z the integers, and $A_n = Z/2Z$, $n \in N$.

Let $A = \Sigma \oplus A_n$ and define addition and multiplication componentwise in A. We let R be the ring obtained by adjoining the identity 1 of Z to A. That is, $R = Z \times A$, where addition is defined componentwise and multiplication is given by the formula

$$(m,a)(m',a') = (mm',ma' + m'a + aa').$$

It is clear that R is a reduced ring. Let $P = (O,A)$. Then $R/P \simeq Z$, and hence P is a prime ideal of R. Since $\text{Ann}_R(P) = (2,0)$, P is a non-essential minimal prime ideal by Proposition 3.4. We have $R_P \simeq Q$, the field of rational numbers; and the only prime ideals properly containing P are of the form $M = (mz,A)$ where $0 \neq m$ is a prime integer. M is a maximal ideal of R, $O_M = P$, and $R_M \simeq Z_{mz}$ is a discrete valuation ring.

Let e_n be the identity of A_n, and let $P_n = R(1,e_n)$. Then $R/P_n \simeq Z/2Z$ is a field, and hence P_n is a maximal ideal of R. But $\text{Ann}_R\, P_n = R(0,e_n)$, and thus P is a non-essential minimal prime ideal of R. Since it is easily seen that a prime ideal of R either contains P, or is equal to P_n for some $n \in N$, we see that $\{P,P_n\}$ is the full set of minimal prime ideals of R, and that they are all non-essential. Thus we have $E(R) \simeq R_P \times \Pi R_{P_n} \simeq Q \times \Pi A_n$ (where $A_n = Z/2Z$). Strangely enough, the multiplication in this direct product is not twisted, but is componentwise multiplication. It follows from Proposition 3.7 that min R is not compact. Since the localizations of R at the prime ideals of R are fields or discrete valuation rings, w.gl.dim $R = 1$; and, a fortiori, R is a PIF. But since min R is not compact, R is not a PIP by Proposition 2.7. We have $Q(R) = Z_{2z} \times A$ (with twisted multiplication) and $Q(R)$ is not a VNR.

Example 4. Let K be a VNR and $R = K[X]$. Then R is a PIP. For let $f(X) \in R$,

$$f(X) = a_0 + a_1X + \cdots + a_nX^n \quad \text{where } a_i \in K;$$

and let $I = \text{Ann}_K(a_0, \ldots, a_n)$. Since K is a VNR, $(a_0, \ldots, a_n) = Ke$, where $e^2 = e \in K$. Thus $I = K \cdot (1 - e)$. Since K is a reduced ring, it follows that if $b^2a_i = 0$, where $b \in K$, then $ba_i = 0$. Using this fact, and an easy calculation, we obtain $\text{Ann}_R(f(X)) = I[X] = R \cdot (1 - e)$. Thus $\text{Ann}_R(f(X))$ is a direct summand of R, and hence $R \cdot f(X)$ is a projective ideal of R.

Let P be a prime ideal of R and $p = P \cap K$; then $p[X]$ is a prime ideal of R contained in P, and hence all of the minimal prime ideals of R are of the form $p[X]$. We have $R/p[X] \simeq (K/p)[X]$; and since K/p is a field, $R/p[X]$ is a principal ideal domain. Thus if P is not a minimal ideal of R, it is a maximal ideal of R; and $O_P = p[X]$ by Proposition 2.1. Thus we see that R_P is a discrete valuation ring, or a field $\forall$ prime ideals P of R. Hence w.gl.dim $R = 1$. Therefore, by Proposition 2.7, R is a semi-hereditary ring.

Let $\{p_\alpha\}$ be the set of all non-essential prime ideals of K, and suppose that $\cap_\alpha p_\alpha = 0$; then $\{p_\alpha[X]\}$ is a set of non-essential minimal prime ideals of R and $\cap_\alpha p_\alpha[X] = 0$. Thus by Propositions 3.5 and 3.6, $\{p_\alpha[X]\}$ is the set of all non-essential minimal prime ideals of R; and in this case, $E(R) \simeq \Pi_\alpha (K/p_\alpha)(X)$.

Example 5. *Let k be a field, and with the notation of Example* 2, *let $K = k(\infty)$, so that K is a hereditary VNR. Let $R = K[[X]]$.*

(1) *R is reduced but is not a PIF.*

(2) *$Q(R)$ is a VNR, and $E(R) \simeq \Pi_n k_n((X))$, where $k_n = k$, $n \in N$.*

(3) *R is a flat essential ring extension of $k[[X]](\infty)$; and the latter ring is semi-hereditary.*

Proof. Let e_n be the element of K that is the identity of k at the *n-th* coordinate, and 0 elsewhere; then $e_n^2 = e_n$, and $p_n = K(1 - e_n)$ is a non-essential prime ideal of K by Example (2). Let $P_n = p_n[[X]] = R(1 - e_n)$; then it is readily verified that P_n is a non-essential prime ideal of R. Since $\cap p_n = 0$, we have $\cap P_n = 0$; and thus R is a reduced ring. Hence by Propositions 3.5 and 3.6, the P_n's are all of the non-essential minimal prime ideals of R and $E(R) \simeq \Pi_n R_{P_n}$. Now $R/P_n \simeq Re_n$, and $Re_n \simeq k_n[[X]]$, where $k_n = k$. Therefore $R_{P_n} \simeq k_n((X))$, and $E(R) \simeq \Pi_n k_n((X))$.

Let $\mathcal{M} = \Sigma_n \oplus Ke_n$; then $\mathcal{M}$ is a maximal ideal of K by example (2). It is readily verified that $M = \mathcal{M} + RX$ is a maximal ideal of R. Let

$$O_M = \{r \in R \mid ur = O \text{ for some } u \in R - M\}.$$

We shall prove that $O_M = R\mathcal{M}$. For let $y \in R\mathcal{M}$; then $y = r_1a_1 + \cdots + r_na_n$, where $r_i \in R$ and $a_i \in \mathcal{M}$. Since K is a VNR, $Ka_1 + \cdots + Ka_n = Ke$, where $e^2 = e$. Then $(1 - e) \in R - M$ and $(1 - e)y = 0$. Therefore, $y \in O_M$. Conversely, let $y \in O_M \subset M$. Then $y = \Sigma_{i=0}^{\infty} a_iX^i$, where $a_0 \in \mathcal{M}$ and $a_i \in K$ for all i; and there exists $u = \Sigma_{i=0}^{\infty} b_iX^i$ such that $b_0 \in K - \mathcal{M}$, $b_i \in K$ and $uy = 0$. Therefore, there exist $c \in K - \mathcal{M}$ and $d \in \mathcal{M}$ such that $1 = cb_0 + d$. Replacing u by $cu \in R - M$, we see that without loss of generality we can assume that $b_0 = 1 - d$, $d \in \mathcal{M}$. Now $a_0b_0 = 0$, and so $a_0 = da_0$. Assume that we have proved that $a_j \in Kd$, $j < i$. Since

$$a_0b_i + a_1b_{i-1} + \cdots + a_{i-1}b_1 + a_ib_0 = 0,$$

we see that $a_i \in Kd$. Thus $y \in Rd \subset R\mathcal{M}$, and so $O_M = R\mathcal{M}$.

To prove that R is not a PIF, it is sufficient by Proposition 2.1 to prove that $O_M = R\mathcal{M}$ is not a prime ideal of R. Let $a_i = e_{2i+1}$ and $b_i = e_{2(i+1)}$, $i \geqslant 0$; let $y = \Sigma_{i=0}^{\infty} a_iX^i$ and $z = \Sigma_{i=0}^{\infty} b_iX^i$; then $yz = 0$. If $y \in R\mathcal{M}$, then there is an $n_0 \in N$ such that $ye_n = 0$, $n \geqslant n_0$. But this is not the case and so $y \notin R\mathcal{M}$. Similarly $z \notin R\mathcal{M}$. Therefore, $R\mathcal{M}$ is not a prime ideal of R.

In order to prove that $Q(R)$ is a VNR, we need to be able to identify the nonzero divisors in R. For this purpose we make the following definitions. If $a \in K$, we define

$$\text{Supp } a = \{n \in N \mid \text{the } n\text{-}th \text{ coordinate of } a \text{ is not } 0\}.$$

And if $y = \Sigma_{i=0}^{\infty} a_i X^i \in R$, we define Supp $y = \cup_{i=0}^{\infty}$ Supp a_i. Let $z = \Sigma_{i=0}^{\infty} b_i X^i \in R$. Then we shall prove that $yz = 0$ iff Supp $y \cap$ Supp $z = \emptyset$, the empty set.

If Supp $y \cap$ Supp $z = \emptyset$, then Supp $a_i \cap$ Supp $b_j = \emptyset$ for all i,j. Therefore $a_i b_j = 0$ for all i,j and so $yz = 0$. Conversely, suppose that $yz = 0$. Let $I = \text{Ann}_K \cup_{i=0}^{\infty} a_i$. Since K is reduced, if $a^2 b = 0$ in K, then $ab = 0$. An easy calculation using this fact shows that $\text{Ann}_R y = I[[X]]$. Thus $b_j \in I$ for all j, and so

$$\text{Supp } a_i \cap \text{Supp } b_j = \emptyset \quad \text{for all } i,j.$$

Hence Supp $y \cap$ Supp $z = \emptyset$. It follows from this fact that y is not a zero-divisor in R iff Supp $y = N$. By Proposition 1.4, to prove that $Q(R)$ is a VNR it is sufficient to prove that if $y \in R$, then there exists $z \in \text{Ann}_R y$ such that $y + z$ is not a zero divisor in R. But if $y = \Sigma_{i=0}^{\infty} a_i X^i$, Supp $y = A \neq N$; and A' is the complement of A in N, it is not difficult to find elements $b_i \in K$ with $\cup_{i=0}^{\infty}$ Supp $b_i = A'$. If we let $z = \Sigma_{i=0}^{\infty} b_i X^i$, then $yz = 0$ and $\text{Supp}(y + z) = N$. Hence $y + z$ is not a zero divisor in R, and therefore, $Q(R)$ is a VNR.

If $a \in K$, define $n(a)$ to be the smallest element of N such that the coordinates of a are constant from $n(a)$ to ∞. If $y = \Sigma_{i=0}^{\infty} a_i X^i \in R$, define $n(y) = \sup_i n(a_i)$. Let $B = \{y \in R \mid n(y) < \infty\}$. Then B is a subring of R containing 1, and $R\mathcal{M} \subset B$. Thus R, as a B-module, is an essential extension of B. We shall prove that $B \simeq k[[X]](\infty)$, and hence by Example 2, B is a semi-hereditary ring.

Let $f_n : K \to k$ be the *n-th* coordinate function; and define $\theta : B \to k[[X]](\infty)$ as follows: if $y = \Sigma_{i=0}^{\infty} a_i X^i \in B$, then $\theta(y) = \langle \Sigma_{i=0}^{\infty} f_n(a_i) X^i \rangle$, an element of $\Pi_n k_n[[X]]$, where $k_n = k$, $n \in N$. If $n_0 = n(y)$, then for $n \geqslant n_0$ we have $f_n(a_i) = f_{n_0}(a_i)$, $i = 0, 1, \ldots, \infty$. Thus, in fact, $\theta(y) \in k[[X]](\infty)$. It is readily verified that θ is a ring isomorphism.

By Example (2), $E(B) = \Pi k_n((X))$; and we have already proved that $E(R) = \Pi k_n((X))$. Hence $E(B) = E(R)$. Since $E(B)$ is a flat B-module by Proposition 2.7, and since w.gl.dim $B \leqslant 1$ and R is a B-submodule of $E(B)$, we see that R is a flat ring extension of B.

We note that if we extend θ with the same definition to a ring homomorphism from R into $\Pi k_n[[X]]$, then θ remains a monomorphism. It is not onto because the latter ring is a PIP. Thus we have

$$k[[X]](\infty) \subset_{\neq} R = k(\infty)[[X]] \subset_{\neq} \Pi_n k_n[[X]]$$

as essential ring extensions.

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