ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1984

CENTRAL LIMIT THEOREMS IN A FINITELY ADDITIVE SETTING[1]

BY

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1. Introduction

Let I be an arbitrary nonempty set. Let I^* denote the set of all finite sequences of elements of I including the empty sequence and $P(I)$ denote the set of all finitely additive probabilities defined on all subsets of I. A *strategy* σ on I is a mapping on I^* into $P(I)$. The strategy σ is called an *independent strategy* if there exists a sequence $\{\gamma_n\}_{n\geq 1}$ of elements of $\mathrm{P}(I)$ such that $\sigma(p) = \gamma_{n+1}$ whenever p is an element of I^* of length n, $n \geq 0$. (The empty sequence has length zero.) In this case we shall denote σ by

$$\gamma_1 \times \gamma_2 \times \ldots \times \gamma_n \times \ldots .$$

If $\gamma_n = \gamma_m$ for all n, $m \geq 1$, then σ will be called an *i.i.d strategy.* If

$$\sigma(i_1, \ldots, i_n) = \sigma(i_n)$$

for all $n \geq 1$ and all $i_1, \ldots, i_n \in I$, then σ will be called a *Markov strategy with stationary transitions.* Let N stand for the set of positive integers and equip $H = I^N$ with the product of discrete topologies. Let $\mathscr{B}$ be the σ-field of subsets of H generated by open sets. Following Dubins and Savage ([4] and [3]), and Purves and Sudderth [9], it can be shown that every strategy σ induces a finitely additive probability on $\mathscr{B}$, unique subject to certain regularity conditions. We shall conveniently denote this probability on $\mathscr{B}$ by σ again. To state the central limit theorems in a finitely additive setting, we shall call a sequence $\{Y_n\}_{n\geq 1}$ of real valued functions on H *a sequence of coordinate mappings* if $Y_n(h)$ depends only on the n-th coordinate of h for all $h \in H$.

THEOREM 1A (Lindeberg theorem). *Let σ be an independent strategy on I. Let $\{Y_n\}_{n\geq 1}$ be a sequence of coordinate mappings on H such that*

$$\int Y_n(h)d\sigma(h) = 0 \quad \textit{and} \quad 0 < \int Y_n^2(h)d\sigma(h) < \infty$$

Received November 30, 1981.

[1]This work was done when the author was at the Indian Statistical Institute, Calcutta.

[2]After the author proved these theorems, R.L. Karandikar obtained a more general result in [7]. We still indicate proofs of these theorems, firstly because several of the techniques and lemmas used in the proof are needed in Section 3. Moreover our proofs are more direct and elementary.

for all n. Let

$$v_n^2 = \int Y_n^2(h)d\sigma(h), \quad S_n(h) = Y_1(h) + \ldots + Y_n(h), \quad s_n^2 = v_1^2 + \ldots + v_n^2$$

for $n \in N$ *and* $h \in H$. *If*

(L) $\frac{1}{s_n^2}\Sigma_{k=1}^{n}\int_{\{h:|Y_k(h)|>ts_n\}}Y_k^2(h)d\sigma(h) \to 0$ *as* $n \to \infty$ *for all* $t > 0$,

then

$$\sigma\left(h : \frac{S_n(h)}{s_n} \leq x\right) \to \Phi(x) \text{ for every real } x,$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}dy.$$

THEOREM 1B (Feller theorem). *Let* σ *be an independent strategy on* I. *Let* $\{Y_n\}_{n\geq 1}$ *be a sequence of coordinate mappings on* H *such that*

$$\int Y_n(h)d\sigma(h) = 0 \quad \text{and} \quad 0 < \int Y_n^2(h)d\sigma(h) < \infty$$

for all n. *Let* S_n, v_n, *be as in Theorem* 1A. *If*

(i) $\sigma\left(h : \frac{S_n(h)}{s_n} \leq x\right) \to \Phi(x)$ *for every real* x *where* Φ *is as in Theorem* 1A,

and,

(ii) $\max_{1\leq k\leq n} \frac{v_k^2}{s_n^2} \to 0$ *as* $n \to \infty$,

then (L) *holds.*

A sequence $\{Y_n\}$ of coordinate mappings on H is called an *identical sequence of coordinate mappings* if there exists a real valued function f such that $Y_n(h) = f(h_n)$ for all $n \in N$ and $h \in H$, where h_n denotes the n-th coordinate of h.

THEOREM 2. *Let* σ *be an i.i.d. strategy on* I. *Let* $\{Y_n\}$ *be a sequence of identical coordinate mappings on* H *such that*

$$\int Y_n(h)d\sigma(h) = 0 \quad \text{and} \quad 0 < \int Y_n^2(h)d\sigma(h) < \infty$$

for all n. *If* $v^2 = \int Y_n^2(h)d\sigma(h)$, *and* S_n *and* Φ *are as in the previous theorems, then*

$$\sigma\left(h : \frac{S_n}{\sqrt{n}\, v} \leq x\right) \to \Phi(x) \text{ for all real } x.$$

The proofs of Theorems 1A, 1B and 2 are given in Section 2.[2] The main technique used is that of restricting the given strategy to a suitable measurable strategy and using the corresponding known theorems in the countably additive case. This technique of reducing to the countably additive case has been first mentioned in [9]. Later on this has been used by Chen [1] and Halevy and Rao [6] to prove results on almost sure convergence in an independent strategic setting.

For a strategy σ and $p \in I^*$, the *conditional strategy given p*, denoted by $\sigma[p]$, is the strategy defined by $\sigma[p](q) = \sigma(pq)$ for all $q \in I^*$ where pq is the element of I^* whose terms consist of the terms of p followed by the terms of q. If Z is a real valued function on H and σ a measure on $(H, \mathscr{B})$, we shall use de Finetti's convention of denoting $\int Z(h) d\sigma(h)$ by $\sigma(Z)$.

For the rest of the section let σ be a Markov strategy with stationary transitions. For $j \in I$, let $t_{j,1}$ be the function on H defined by

$$t_{j,1}(h) \begin{cases} = n & \text{if } h_n = j \text{ and } h_m \neq j \text{ for } 1 \leq m < n, \\ = \infty & \text{otherwise.} \end{cases}$$

For a real valued function f on I and $j \in I$, let $Z_{1,f,j}$ denote the function on H defined by

$$Z_{1,f,j}(h) = \sum_{m=1}^{t_{j,1}(h)} f(h_m), \quad h \in H.$$

(The subscript 1 does not play any role now but will be helpful later.)

For $i, j \in I$ we say that i weakly leads to j (denoted by $i \overset{w}{\to} j$), if

$$\sigma[i](t_{j,1} < \infty) > 0.$$

An element $i \in I$ is called *positive recurrent* if $m_{ii} = \sigma[i](t_{i,1}) < \infty$. The set I is said to be a *positive recurrent class under* σ if (a) $i \overset{w}{\to} j$ for all $i, j \in I$ and (b) i is positive recurrent for all $i \in I$. It follows from Corollary 5, Section 7 of [10] that if I is a positive recurrent class under σ, and $\sigma[i](Z_{1,|f|,i}) < \infty$ for some $i \in I$, then

$$\frac{\sigma[j](Z_{1,f,j})}{m_{jj}} = M, \quad \text{independent of } j.$$

(In [10], $\sigma[j](Z_{1,f,j})$ is denoted by $\mu_j(f)$. It is easy to see that

$$\mu_j(|f|) = \sigma[i](Z_{1,|f|,i}) < \infty$$

is equivalent to both $\mu_j(f^+)$ and $\mu_j(f^-)$ being finite, since $\mu_j(|f|) = \mu_j(f^+) = \mu_j(f^+) + \mu_j(f^-)$, where f^+ and f^- are the positive and negative parts of f, respectively.)

THEOREM 3. *Let σ be a Markov strategy with stationary transitions and let I be a positive recurrent class under σ. Suppose f is a real valued function on I such that for some $i \in I$,*

(a) $\sigma[i](Z_{1,|f|,i}) < \infty$, *and*

(b) $0 < \sigma[i](Z^2_{1,f-M,i}) < \infty$, *where* $M = \dfrac{\sigma[i](Z_{1,f,i})}{\mathrm{m}_{ii}}$.

Then for every $j \in I$,

$$\sigma[j]\left(h : \frac{S_n(h) - Mn}{\sqrt{B_i n}} \le x \right) \to \Phi(x) \text{ for every real } x,$$

where S_n *and* ϕ *are as in the previous theorems and*

$$B_i = \frac{\sigma[i](Z^2_{1,f-M,i})}{m_{ii}}.$$

The proof of this theorem is given in Section 3. The idea is to reduce it to the i.i.d. strategic situation and use Theorem 2. In Section 4, we prove that B_i is independent of i and make some remarks regarding the assumptions made in Theorem 3.

2. The independent and i.i.d case

We shall need the following theorem proved by Chen [1]. (The proof of a special case is also available in Purves and Sudderth [9].) Let $\{\mathscr{A}_n\}_{n\ge 1}$ be a sequence of σ-fields of subsets of I. A strategy σ on I is said to be a measurable strategy with respect to $\{\mathscr{A}_n\}_{n\ge 1}$ if

(i) $\sigma(p)$ is countably additive when restricted to $\mathscr{A}_n$, whenever p is an element of I^* of length $(n-1)$, *and*

(ii) the map

$$(i_1, \ldots, i_n) \to \sigma(i_1, \ldots, i_n)(A_{n+1})$$

is measurable with respect to $\mathscr{A}_1 \times \ldots. \times \mathscr{A}_n$, the product σ-field, whenever $A_{n+1} \in \mathscr{A}_{n+1}$.

THEOREM. *Let* σ *be a strategy on* I *and* $\{\mathscr{A}_n\}_{n\ge 1}$ *be a sequence of* σ*-fields of subsets of* I. *Suppose* σ *is a measurable strategy with respect to* $\{\mathscr{A}_n\}_{n\ge 1}$. *Then the finitely additive probability* σ *on* $\mathscr{B}$ *is countably additive when restricted to* $\Pi_{n=1}^{\infty}\mathscr{A}_n$, *the product* σ*-field of subsets of* H.

Let σ be an independent strategy and $\{Y_n\}$ a sequence of coordinate mappings on I such that $\int Y_n(h)d\sigma(h) = 0$ and $0 < \int Y_n^2(h)d\sigma(h) < \infty$ for all n. Then for each $n \in N$, we choose a positive integer $M_n \ge n$ such that

$$\int_{\{h:|Y_n(h)| \ge M_n\}} Y_n^2(h)d\sigma(h) < \frac{1}{2^{n+3}}.$$

It is then easy to check that

$$\int_{\{h:|Y_n(h)| \ge M_n\}} |Y_n(h)|d\sigma(h) < \frac{1}{2^{n+3}} \quad \text{and} \quad \sigma\{h : |Y_n(h)| \ge M_n\} < \frac{1}{2^{n+3}}.$$

We now define a new sequence $\{Z_n\}$ of coordinate mappings as follows.

$$Z_n(h)\begin{cases} = -M_n & \text{if } Y_n(h) \leq -M_n, \\ = M_n & \text{if } Y_n(h) \geq M_n, \\ = k + \dfrac{j}{M_n 2^{n+3}} & \text{if } k + \dfrac{j}{M_n 2^{n+3}} \leq Y_n(h) < k + \dfrac{j+1}{M_n 2^{n+3}}, \end{cases}$$

$$k = -M_n, -M_n+1, \ldots, M_n - 1,$$

$$j = 0, 1, \ldots, M_n 2^{n+3} - 1.$$

For each $n \geq 1$, Z_n induces a finite partition on X and let $\mathscr{A}_n$ be the σ-field generated by this partition. Clearly the strategy σ is measurable with respect to $\{\mathscr{A}_n\}$. Therefore by the theorem stated above, σ is countably additive when restricted to $\Pi_{n=1}^{\infty} \mathscr{A}_n$.

Let

$$\mu_n = \int Z_n d\sigma,\ v_n^2 = \int Y_n^2 d\sigma,\ v_n^{*2} = \int Z_n^2 d\sigma - \mu_n^2, s_n^2 = v_1^2 + \ldots + v_n^2$$

and $s_n^{*2} = v_1^{*2} + \ldots + v_n^{*2}$. The following Lemmas 1 and 2 are straightforward to verify, hence their proofs are omitted.

Lemma 1. $|\mu_n| \leq 1/2^{n+1}$ *for all* $n \in N$.

Lemma 2. $|v_n^2 - v_n^{*2}| \leq 1/2^n$ *for all* $n \geq 1$.

Lemma 3. *If Q is a strictly increasing real valued function on N such that $Q(n) \uparrow \infty$ as $n \to \infty$, then for each $\epsilon > 0$,*

$$\sigma \left\{ h : \left| \frac{S_n(h) - S_n^*(h)}{Q(n)} \right| > \epsilon \right\} \to 0 \text{ as } n \to \infty,$$

where $S_n^* = Z_1 + \ldots + Z_n$ for all n. (In other words, $(S_n - S_n^*)/Q(n)$ converges in σ-probability to zero.)

Proof. Given $\epsilon, \delta > 0$, we first choose a positive integer n_0 such that

$$\sum_{n=n_0}^{\infty} \frac{1}{2^{n+3}} < \frac{\delta}{3}$$

and then choose $n_1 > n_0$ such that

(i) $$\sigma \left\{ \left| \frac{S_{n_0}}{Q(n)} \right| > \frac{\epsilon}{3} \right\} < \frac{\delta}{3}$$

(ii) $$\sigma \left\{ \left| \frac{S_{n_0}^*}{Q(n)} \right| > \frac{\epsilon}{3} \right\} < \frac{\delta}{3}, \text{ and}$$

(iii) $\dfrac{1}{Q(n)} < \dfrac{\epsilon}{3}$, for all $n \geq n_1$.

Such a choice of n_1 is possible since $|\sigma(S_{n_0})| < \infty$, $|\sigma(S^*_{n_0})| < \infty$ and $Q(n) \uparrow \infty$. We now observe that for $n \geq n_1$,

$$\left\{ \left| \frac{S_n - S^*_n}{Q(n)} \right| > \epsilon \right\} \subseteq \left\{ \left| \frac{S_{n_0}}{Q(n)} \right| > \frac{\epsilon}{3} \right\}$$

$$\cup \left\{ \left| \frac{S^*_{n_0}}{Q(n)} \right| > \frac{\epsilon}{3} \right\} \cup \left\{ \bigcup_{k=n_0}^{n} \left(|Y_k - Z_k| > \frac{1}{2^k} \right) \right\}.$$

Therefore, for $n \geq n_1$,

$$\sigma \left\{ \left| \frac{S_n - S^*_n}{Q(n)} \right| > \epsilon \right\} \leq \frac{\delta}{3} + \frac{\delta}{3} + \sum_{k=n_0}^{n} \frac{1}{2^{k+3}} < \delta.$$

This proves the lemma.

LEMMA 4. *The condition* (L) *in Theorem* 1A *holds iff the condition* (L′) *stated below holds.*

(L′) $1/s^{*2}_n \sum_{k=1}^n \int_{\{|Z_k - \mu_k| > ts^*_n\}} (Z_k - \mu_k)^2 d\sigma \to 0$ *as* $n \to \infty$ *for all* $t > 0$.

Proof. Suppose (L) holds. Plainly, because of (L), $s^2_n \to \infty$ as $n \to \infty$. Lemma 2 then implies that $s^2_n / s^{*2}_n \to 1$ as $n \to \infty$. It is therefore enough to show that

$$\frac{1}{s^2_n} \sum_{k=1}^{n} \int_{\{|Z_k - \mu_k| > ts^*_n\}} (Z_k - \mu_k)^2 d\sigma \to 0 \text{ as } n \to \infty.$$

Let $t > 0$. Given $\epsilon > 0$, we choose a positive integer n_0 such that for $n \geq n_0$, we have

(a) $\dfrac{1}{s^2_n} < \dfrac{\epsilon}{5}$,

(b) $ts^*_n - \dfrac{t}{2} s_n > 1$,

(c) $\dfrac{1}{s^2_n} \displaystyle\sum_{k=1}^{n} \int_{\{|Y_k| > ts_n/2\}} Y^2_k d\sigma < \dfrac{\epsilon}{5}$.

It easily follows from (b) that if $n \geq n_0$, then for $1 \leq k \leq n$,

$\{h : |Z_k(h) - \mu_k| > ts^*_n\} \subseteq \{|Y_k| > \frac{t}{2} s_n \text{ and } |Y_k| < M_k\} \cup \{|Y_k| \geq M_k\}$.

Therefore for $n \geq n_0$,

$$\frac{1}{s_n^2} \sum_{k=1}^{n} \int_{\{|Z_k-\mu_k|>ts_n^*\}} (Z_k - \mu_k)^2 d\sigma$$

$$\leq \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{\{|Y_k|>ts_n/2 \text{ and } |Y_k|<M_k\}} (Z_k - \mu_k)^2 d\sigma + \int_{\{|Y_k|\geq M_k\}} (Z_k - \mu_k)^2 d\sigma$$

$$\leq \frac{1}{s_n^2} \sum_{k=1}^{n} \left[\int_{\{|Y_k|>ts_n/2\}} 2Y_k^2 + \frac{1}{2^{2k}} \; d\sigma + 2M_k^2 \sigma\{|Y_k| \geq M_k\} + 2\mu_k^2 \right]$$

(by definition of Z_k, and the inequality $(a + b)^2 \leq 2a^2 + 2b^2$)

$\leq 2\epsilon/5 + \epsilon/5 + \epsilon/5 + \epsilon/5 = \epsilon$, by (a), (c) and the choice of M_k.

The if part is also proved similarly.

LEMMA 5. *Let $\{\xi_n\}$ and $\{\eta_n\}$ be two sequences of real valued measurable functions on $(H, \mathscr{B}, \sigma)$ and $\{a_n\}$ a sequence of real numbers such that* (i) $a_n \to 1$, (ii) *$\xi_n - \eta_n$ converges in σ-probability to zero and* (iii) *$\sigma(\xi_n \leq x) \to \Phi(x)$ for all real x. Then*

$$\sigma(a_n\eta_n \leq x) \to \Phi(x)$$

for all real x.

Proof. For $\epsilon > 0$, note that for sufficiently large n,

$$\sigma(\xi_n \leq x - \epsilon) - \sigma(|\xi_n - \eta_n| \geq \epsilon) \leq \sigma(a_n\eta_n \leq x)$$

and

$$\sigma(a_n\eta_n \leq x) \leq \sigma(\xi_n \leq x + \epsilon) + \sigma(|\xi_n - \eta_n| \geq \epsilon).$$

By taking lim inf in the first inequality and lim sup in the second and using (ii), we get respectively,

$\Phi(x - \epsilon) \leq \lim_n \inf \sigma(a_n\eta_n \leq x)$ and $\lim_n \sup \sigma(a_n\eta_n \leq x) \leq \Phi(x + \epsilon)$.

Since ϵ is arbitrary, the assertion follows.

Proof of Theorem 1A. If (L) holds, by Lemma 4, so does (L′). Therefore by the classical Lindeberg theorem (page 280, [8]),

$$\sigma\left\{ h : \frac{S_n^* - \sum_{k=1}^{n} \mu_k}{s_n^*} \leq x \right\} \to \phi(x)$$

for all real x. By Lemma 1,

$$\frac{\sum_{k=1}^{n} \mu_k}{s_n^*} \to 0 \text{ as } n \to \infty.$$

Therefore (by Lemma 5),

$$\sigma\left\{\frac{S_n^*}{s_n^*} \le x\right\} \to \Phi(x).$$

By Lemma 3, $S_n^*/s_n^* - S_n/s_n^*$ converges in σ-probability to zero. An application of Lemma 5 with $\xi_n = S_n^*/s_n^*$, $\eta_n = S_n/s_n^*$ and $a_n = s_n^*/s_n$ now completes the proof of the theorem.

LEMMA 6.

$$\operatorname*{Max}_{1\le k\le n} \frac{v_k^2}{s_n^2} \to 0 \quad \text{iff} \quad \operatorname*{Max}_{1\le k\le n} \frac{v_k^{*2}}{s_n^{*2}} \to 0.$$

The assertion is an easy consequence of Lemma 2 and we omit the proof.

Proof of Theorem 1B. The condition (ii) in the hypothesis of the theorem implies that $s_n^2 \uparrow \infty$ and also that

$$\operatorname*{Max}_{1\le k\le n} \frac{v_k^{*2}}{s_n^{*2}} \to 0 \text{ as } n \to \infty$$

(by Lemma 6).

By an application of Lemma 5 (similar to the one made in the proof of Theorem 1A), condition (i) gives us

$$\sigma\left\{\frac{S_n^* - \sum_{k=1}^{n} \mu_k}{s_n^*} \le x\right\} \to \Phi(x) \quad \text{for all } x.$$

It then follows from the classical theorem of Feller ([8], page 280) that (L′) holds. Lemma 4 now completes the proof of the theorem.

Proof of Theorem 2. In view of Theorem 1A, it is enough to verify that

$$\frac{1}{nv^2} \sum_{k=1}^{n} \int_{\{|Y_k| > t\sqrt{n}\, v\}} Y_k^2 d\sigma \to 0 \text{ as } n \to \infty, \text{ for all } t > 0.$$

Since $\{Y_n\}$ is a sequence of identical coordinate mappings, this is equivalent to verifying that

$$\int_{\{|Y_1| > t\sqrt{n}\, v\}} Y_1^2 \to 0 \text{ as } n \to \infty, \quad \text{for all } t > 0.$$

Since $v^2 = \int Y_1^2 d\sigma < \infty$, the last mentioned condition is easily seen to hold. Thus the theorem is proved.

3. The Markov case

An *incomplete stop rule* t is a function on H into $N \cup \{\infty\}$ such that if $t(h) = n$ for some $n \in N$ and h' agrees with h through the first n coordinates, then $t(h') = n$. For example, $t_{j,1}$, the time of first occurence of j, defined in Section 1, is an incomplete stop rule. We begin this section by proving the Strong Markov property.

PROPOSITION 1 (Strong Markov property). *Let σ be a Markov strategy with stationary transitions. Let t be an incomplete stop rule such that for some $i \in I$, $h_{t(h)} = i$ for all $h \in \{t < \infty\}$. Let g be a Borel measurable function on H such that $\sigma[i](g)$ is defined (may be infinite). If f is a function on H defined by*

$$(1) \qquad f(h) \begin{cases} = g(h_{t(h)+1}, h_{t(h)+2}, \ldots) & \textit{for } h \in \{t < \infty\} \\ = 0 & \text{otherwise,} \end{cases}$$

then $\sigma(f)$ exists and $\sigma(f) = \sigma(t < \infty)\sigma[i](g)$, if $\sigma(t < \infty) > 0$.

Proof. If $g = 1_B$, where $B \in \mathscr{B}$, then f is clearly 1_A where $h \in A$ iff $t(h) < \infty$ and

$$(h_{t(h)+1}, h_{t(h)+2}, \ldots) \in B.$$

(In the terminology of [10], this is the same as saying A is conditionally determined given t.) This case is now easily seen to be a restatement of Theorem 4, Section 3 of [10]. The result for a simple function g follows by linearity. If g is a bounded measurable function, then we can get simple functions g_n converging uniformly to g. The assertion is seen to hold for g by just noting that f_n, defined by (1) corresponding to g_n, converges uniformly to f. If g is a nonnegative function,

$$\sigma(f) = \sup_n \sigma(f \Lambda n) = \sup_n \sigma(t < \infty)\sigma(g \Lambda n) = \sigma(t < \infty)\sigma(g).$$

The general case follows by taking positive and negative parts.

We next prove a couple of elementary but useful lemmas on finitely additive integration.

LEMMA 2. *Let P be a finitely additive probability defined on a σ-field $\mathscr{A}$ of subsets of a set X. Let ξ be a nonnegative, measurable, extended real valued function on X such that $P(\xi \geq n) \to 0$ as $n \to \infty$. Then*

$$\operatorname*{Lim}_n \int_{\{\xi < n\}} \xi \, dP \geq \sum_{n=1}^{\infty} P(\xi \geq n).$$

Proof.

$$\operatorname*{Lim}_n \int_{\{\xi<n\}} \xi dP \geq \sum_{k=1}^{\infty} kP(k \leq \xi < k+1)$$

$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P(k \leq \xi < k+1)$$

$$= \sum_{n=1}^{\infty} P(\xi \geq n).$$

The last step follows since $P(\xi \geq n) \to 0$ implies that $P(\xi \geq n) = \Sigma_{k=n}^{\infty} P(k \leq \xi < k+1)$.

LEMMA 3. *Let P be a finitely additive probability defined on a σ-field $\mathscr{A}$ of subsets of a set X. Let ξ be a nonnegative, measurable, extended real valued function on X such that $P(\xi \geq n) \to 0$ as $n \to \infty$. Then*

$$\int \xi dP = \operatorname*{Lim}_n \int_{\{\xi<n\}} \xi dP.$$

Proof. If $\Sigma_{n=1}^{\infty} P(\xi \geq n) = \infty$, then by the previous lemma, $\operatorname{Lim} \int_{\{\xi<n\}} \xi dP = \infty$ and the result is trivially true. If $\Sigma_{n=1}^{\infty} P(\xi \geq n) < \infty$, then $nP(\xi \geq n) \to 0$ as $n \to \infty$. Therefore,

$$\int \xi dP = \operatorname*{Lim}_n \int \xi \wedge n dP = \operatorname*{Lim}_n \left[\int_{\{\xi<n\}} \xi dP + nP(\xi \geq n) \right]$$

$$= \operatorname*{Lim}_n \int_{\{\xi<n\}} \xi dP.$$

Remark. If $\int \xi dP$ is finite, the hypothesis of the lemma is clearly satisfied.

LEMMA 4. *Let $\tau = \gamma^N$ be an i.i.d. strategy on X and $\{\xi_n\}$ a sequence of nonnegative identical coordinate mappings on X^N such that $|\tau(\xi_n)| < \infty$ for all n. Let D be a subset of X and let θ be the first hitting time of D defined by*

$$\theta(\omega) \begin{cases} = k & \text{if } \omega_m \notin D,\ 1 \leq m < k,\ \omega_k \in D, \\ = \infty & \text{if no such } k \text{ exists.} \end{cases}$$

Let $p > 0$. Assume $\tau(\theta^p) < \infty$ and $\tau(\theta^{p+1}) < \infty$. Then $\tau(\theta^p \Sigma_{m=1}^{\theta} \xi_m) < \infty$.

Proof. There is no harm in assuming that $\xi_m \geq 1$ for all m. Since $|\tau(\xi_m)| < \infty$ and $\tau(\theta_p) < \infty$, it is easily seen that

$$\tau \left\{ \left(\theta^p \sum_{m=1}^{\theta} \xi_m \right) \geq n \right\} \to 0 \quad \text{as } n \to \infty.$$

Given $\epsilon > 0$, we can first choose n_0 such that $\tau(\theta > n_0) < \epsilon/2$ and then choose n_1 such that $\tau(\xi_1 > n_1) < \epsilon/2n_0$. Then

$$\tau\left\{\left(\theta^p \sum_{m=1}^{\theta} \xi_m\right) > n_0^{p+1} n_1\right\} \le \tau(\theta > n_0) + \sum_{m=1}^{n_0} \tau(\xi_m > n_1) < \epsilon.$$

Therefore by Lemma 3,

$$\tau\left(\theta^p \sum_{m=1}^{\theta} \xi_m\right) = \sum_{m=1}^{\infty} \tau\left(1_{(\theta=n)}\theta^p \sum_{m=1}^{\infty} \xi_m\right) \quad \left(\text{since } \theta^p \sum_{m=1}^{\theta} \xi_m < n^p \text{ implies } \theta < n\right)$$

$$= \sum_{n=1}^{\infty} n^p \tau\left(\sum_{m=1}^{n} \xi_m 1_{\{\theta=n\}}\right)$$

For $1 \le m < n$,

$$\tau(\xi_m 1_{(\theta=n)}) = \tau(\xi_1 1_{(\theta\neq 1)})\{\gamma(D^c)\}^{n-2}\gamma(D).$$

For $m = n$,

$$\tau(\xi_m 1_{\theta=n}) = \tau(\xi_1 1_{(\theta=1)})\{\gamma(D^c)\}^{n-1}.$$

Further observing that $\tau(\theta = n) = \{\gamma(D^c)\}^{n-1}\gamma(D)$ and using the finiteness of $\tau(\xi_1)$, we have,

$$\tau\left(\theta^p \sum_{m=1}^{\theta} \xi_m\right) = \sum_{n=1}^{\infty} n^p\{(n-1)k_1 + k_2\}\tau(\theta = n),$$

where k_1, k_2 are constants. The right side of the above expression is finite because

$$\tau(\theta^p) < \infty \quad \text{and} \quad \tau(\theta^{p+1}) < \infty.$$

This proves the lemma.

Let σ be a Markov strategy under which I is a positive recurrent class (defined in Section 1). For $i \in I$, we shall denote by $t_{i,n}$ the incomplete stop rule corresponding to the n^{th} occurrence of i, and by G_i the set of $h \in H$ with infinitely many coordinates equal to i. We state below some of the results from [10] we shall be using.

THEOREM. *For all i, j in the positive recurrent class I, $\sigma[j](G_i) = 1$.* (This assertion is a part of Theorem 9, Section 4 of [10].)

For an $i \in I$, let F be the set of all nonempty finite sequences of elements of I whose last coordinate is i and none of the other coordinates is i, (F will be called the set of i-blocks). Let $\Omega = F^N$ be equipped with the product of discrete topologies and let $\mathscr{F}$ be the Borel σ-field on Ω. On G_i, we define a sequence $\{\beta_n\}$ of functions into F called the i-block variables as follows:

$$\beta_1(h) = (h_1, \ldots, h_{t_{i,1}(h)}), \text{ and } \beta_{n+1}(h) = (h_{t_{i,n}(h)+1}, \ldots, h_{t_{i,n+1}(h)})$$

for all $h \in G_i$ and $n \in N$. Let Ψ be the mapping on G_i in Ω defined by

$$\Psi(h) = (\beta_1(h), \beta_2(h), \ldots, \ldots).$$

It is easily checked that Ψ is 1-1, onto and a (topological) homeomorphism.

Let γ be a measure on F defined by

$$\gamma(D) = \sigma[i](\beta_1^{-1}(D)) \quad \text{for all } D \subseteq F.$$

Let π be the i.i.d. strategy induced by γ on Ω.

Blocks Theorem (Theorem 2, Section 5 of [10]). *For each* $B \in \mathscr{F}$, $\pi(B) = \sigma[i](\Psi^{-1}(B))$ (i.e., Ψ *is a measure isomorphism of* $(H, \mathscr{B}, j[i])$ *and* $(\Omega, \mathscr{F}, \pi)$).

For g a real valued function on I, let

$$Z_{n+1,g,i}(h) \begin{cases} = \displaystyle\sum_{m=t_{i,n}(h)+1}^{t_{i,n+1}(h)} g(h_m) & \text{for } h \in G_i, \\ = 0 & \text{for } h \notin G_i. \end{cases}$$

Since $\sigma[j](G_i) = 1$ for all $j \in I$, on a set of $\sigma[j]$-measure one, $Z_{n,g,i}$ is the sum of g values in the n^{th} i-block. Let g be a real valued function on I such that for some $i \in I$, $\sigma[i](|Z_{1,g,i}|^p) < \infty$, where p is a positive number. For a $j \neq i$ and $h \in H$, let

$$\theta^{**}(h) \begin{cases} = n & \text{if the } n^{\text{th}}\ i\text{-block contains the } 2^{\text{nd}}\ j \text{ in } h, \\ = \infty & \text{if no such } n \text{ exists.} \end{cases}$$

Lemma 5.

$$\sigma[i]\left(\Big|\sum_{k=1}^{\theta^{**}} Z_{k,g,i}\Big|^p\right) < \infty.$$

Proof. Let $\theta(h)$ be the smallest n such that the n^{th} i-block contains a j (if such an n exists), and ∞ if no such n exists; and let $\theta^*(h)$ be the smallest $n \geq 1$ such that the $(n+\theta)^{\text{th}}$ i-block contains j (if such an n exists), and ∞ if no such n exists. It is easy to see that

$$\theta^{**}(h) \leq \theta(h) + \theta^*(h)$$

for all $h \in H$. Therefore the lemma would be proved if we showed that

$$\sigma[i]\left\{\left(\sum_{k=1}^{\theta+\theta^*} |Z_{k,g,i}|^p\right)\right\} < \infty.$$

It follows by a use of the elementary inequality $(\Sigma_{s=1}^m x_s)^p \leq m^p \Sigma_{s=1}^m x_s^p$, for all $p > 0$, $m \geq 1$ and non-negative real numbers $x_1, \ldots, x_m$, that it is sufficient to prove that

$$\sigma[i]\left\{\left(\sum_{k=1}^{\theta} |Z_{k,g,i}|\right)^p\right\} < \infty \quad \text{and} \quad \sigma[i]\left\{\left(\sum_{k=\theta+1}^{\theta+\theta^*} |Z_{k,g,i}|\right)^p\right\} < \infty.$$

We can use the strong Markov property to the incomplete stop rule $t_{i,\theta}$ to observe that

$$\sigma[i]\left\{\left(\sum_{k=\theta+1}^{\theta+\theta^*} |Z_{k,g,i}|\right)^p\right\} = \sigma[i]\left\{\left(\sum_{k=1}^{\theta} |Z_{k,g,i}|\right)^p\right\}.$$

Another use of the elementary inequality above shows that it suffices to prove that

$$\sigma[i]\left(\theta^p \sum_{k=1}^{\theta} |Z_{k,g,i}|^p\right) < \infty.$$

Let D be the set of elements of F (where F is the set of i-blocks defined before the blocks theorem) in which no coordinate is j. Clearly for each n,

$$\Psi^{-1}(D^{n-1} \times D^c \times \Omega) = \{\theta = n\},$$

where Ψ, and Ω are as defined earlier. So, by the Blocks theorem,

$$\begin{aligned}\sigma[i]\{\theta = n\} &= \pi(D^{n-1} \times D^c \times \Omega)\\ &= \{\gamma(D)\}^{n-1} \times \gamma(D^c) \quad \text{(since } \pi \text{ is i.i.d.)}\\ &= \{\sigma[i](t_{i,1} < t_{j,1})\}^{n-1}\sigma[i](t_{j,i} < t_{i,1}),\ n \geq 1\\ &\qquad\qquad \text{(by definition of } \gamma).\end{aligned}$$

Lemma 3, Section 6 of [10] shows that for i,j in a positive recurrent class,

$$\sigma[i](t_{j,1} < t_{i,1}) > 0$$

(the hypothesis $i \xrightarrow{\omega} j$ of Lemma 3, Section 6 of [10] is satisfied by definition of positive recurrent class and t, $t(1)$ in [10] are $t_{j,1}$ and $t_{i,1}$ respectively in our notation). So θ has a geometric distribution and hence $\sigma[i](\theta^p) < \infty$ for all $p > 0$. Since Ψ is a measure isomorphism between $(h, \mathscr{B}, \sigma[i])$ and $(\Omega, \mathscr{F}, \pi)$, to prove

$$\sigma[i]\left(\theta^p \sum_{k=1}^{\theta} |Z_{k,g,i}|^p\right) < \infty$$

is equivalent to proving

$$\pi\left[(\theta \circ \Psi^{-1})^p \sum_{k=1}^{\theta\circ\Psi^{-1}} |Z_{k,g,i} \circ \Psi^{-1}|^p\right] < \infty.$$

This follows from Lemma 4 (since $\sigma[i](\theta^p) < \infty$ for all $p > 0$ is equivalent to $\pi[(\theta \circ \psi^{-1})^p] < \infty$ for all $p > 0$). This completes the proof of the lemma.

For $h \in G_i$, let $\ell(n)(h)$ denote the number of coordinates equal to i among $h_1, \ldots, h_n, n \geq 1$.

LEMMA 6. *For every $\epsilon > 0$ and $j \in I$, there exists an integer k such that*

$$\sigma[j](n - t_{i,\ell(n)} \geq k) < \epsilon$$

for all $n \in N$. (If $\ell(n) = 0, t_{i,\ell(n)}$ is taken to be zero.)

Proof. $\sigma[j](n - t_{i,\ell(n)} \geq k)$

$$= \sum_{s=k}^{n} \sigma[j](n - t_{i,\ell(n)} = s)$$

$= \sum_{s=k}^{n} \sigma[j]\{(n-s)^{\text{th}}$ coordinate is i and $(n-s+1)^{\text{th}}$ through n^{th} coordinates are different from $i\}$

$$= \sum_{s=k}^{n} \sigma[j](h : h_{n-s} = i)\, \sigma[i](t_{i,1} > s)$$

(Use the strong Markov property for each term—for the s^{th} term with $t \equiv n - s$)

$$\leq \sum_{s=k}^{n} \sigma[i](t_{i,1} > s) \quad \text{for all } n \in N.$$

The result now follows because $\Sigma_{s=k}^{\infty} \sigma[i](t_{i,1} > s) \leq \sigma[i](t_{i,1}) < \infty$, as the state space is assumed to be positive recurrent. (Here we use the fact that for a nonnegative integer valued random variable ξ, the integral with respect to a finitely additive probability P, $\int \xi dP$, is $\Sigma_{n=1}^{\infty} P(\xi \geq n)$. This is Lemma 1, Section 6 of [10]. We could also use Lemma 2 and the remark after Lemma 3.)

From now onwards let f be a real valued function on I such that

$$\sigma[i](Z_{1,|f|,i}) < \infty.$$

Let

$$Y'(n)(h) \begin{cases} = \displaystyle\sum_{k=1}^{t_{i,1}(h)} f(h_k) & \text{if } t_{i,1}(h) \leq n, \\ = 0 & \text{otherwise.} \end{cases}$$

LEMMA 7. *For all $j \in I$, $Y'(n)/\sqrt{n}$ converges to zero in $\sigma[j]$-probability.*

Proof. We first observe that $\Sigma_{s=t_{j,2}+1}^{t_{i,\theta^{**}}} |f(h_s)| \leq \Sigma_{k=1}^{\theta^{**}} Z_{k,|f|,i}$. Therefore by Lemma 5 applied with $p = 1$, $\sigma[i](\Sigma_{s=t_{j,2}+1}^{t_{i,\theta^{**}}} |f(h_s)|) < \infty$. However, this, by the strong Markov property (applied to $t_{j,2}$) is easily seen to be equal to $\sigma[j](Z_{1,|f|,i})$. Further

$$|Y'(n)| \leq Z_{1,|f|,i}$$

for all n. Hence the result would follow if we show that $\sigma[j](Z_{1,|f|,i} \geq k)$ tends to zero as $k \to \infty$. This is true because $\sigma[j](Z_{1,|f|,i}) < \infty$.

For $n \in N$, let $Y''(n) = \Sigma_{k=t_{i,\ell(n)}+1}^{n} f(h_k)$, $h \in H$.

LEMMA 8. *For all $j \in I$, $Y''(n)/\sqrt{n}$ converges to zero in $\sigma[j]$-probability.*

Proof. For any positive integers k and n, $k \leq n$, let t' be the incomplete stop rule corresponding to the first occurrence of i after the $(n-k)^{\text{th}}$ coordinate, and t'' be the incomplete stop rule corresponding to the $(k+1)^{\text{st}}$ occurence of i after the $(n-k)^{\text{th}}$ coordinate. On the set for which $n - t_{i,\ell(n)} < k$,

$$|Y''(n)| \leq \sum_{s=t'+1}^{t''} |f(h_s)|.$$

By the strong Markov property (applied to t'), we see that the $\sigma[j]$-distribution of $\Sigma_{s=t'+1}^{t''}|f(h_s)|$ and the $\sigma[i]$-distribution of $\Sigma_{s=1}^{k} Z_{s,|f|,i}$ are the same. Hence,

$$\sigma[j]\left(\frac{|Y''(n)|}{\sqrt{n}} > \epsilon\right) \leq \sigma[j](n - t_{i,\ell(n)} \geq k) + \sigma[i]\left(\frac{\sum_{s=1}^{k} Z_{s,|f|,i}}{\sqrt{n}}\right) > \epsilon .$$

We now apply Lemma 6, to complete the proof.

From now onwards we shall assume that f satisfies the hypotheses of Theorem 3 (stated in Section 1). The next few lemmas are aimed at showing that

$$\sigma[j]\left(h : \frac{\sum_{m=2}^{\ell(n)} Z_{m,f-M,i}}{\sqrt{B_i n}} \leq x\right) \to \Phi(x) \text{ as } n \to \infty,$$

for all real x and for all j, where M, B_i, Φ are defined earlier.

LEMMA 10. *Let τ be an independent strategy on X. Let $\{Y_n\}$ be a sequence of coordinate mappings on X^N such that $\tau(|Y_n|) < \infty$ for all $n \geq 1$. If $\tau\{S_n/n \to \mu\} = 1$, where μ is a real number and $S_n = \Sigma_{k=1}^{n} Y_k$, then given $\epsilon > 0, \eta > 0$, there exists an integer N_0 such that*

$$\tau\left\{\left|\frac{S_n}{n} - \mu\right| < \epsilon \textit{ for all } n \geq N_0\right\} \geq 1 - \eta.$$

Proof. Assume ϵ, η are given. We choose a sequence of positive integers $\{M_k\}$ such that $\int_{\{|Y_k| > M_k\}} |Y_k| d\tau < 1/2^k$, $k \geq 1$. We then define Z_k exactly as in Section 2 so that Z_k assumes only finitely many values and $|Y_k - Z_k| < 1/2^k$, whenever $|Y_k| < M_k$, k ≥ 1. Let

$$S_n^* = \sum_{k=1}^{n} Z_k.$$

We first choose n_0 such that

$$\prod_{n=n_0}^{\infty} \left[1 - \frac{1}{2^n}\right] > 1 - \frac{\eta}{8}.$$

We then choose $n_1 \geq n_0$ (using the fact that $\tau(|S_{n_0}|) < \infty$) such that

$$\tau\left(\left|\frac{S_{n_0}}{n}\right| > \frac{\epsilon}{8}\right) < \frac{\eta}{8}, \quad \tau\left(\left|\frac{S^*_{n_0}}{n}\right| > \frac{\epsilon}{8}\right) < \frac{\eta}{8}$$

and $1/n < \epsilon/8$ for all $n \geq n_1$. Since

$$\tau\left(\left|\frac{S_n}{n} - \mu\right| < \frac{\epsilon}{8} \text{ for all sufficiently large } n\right) = 1,$$

and

$$\left\{\left|\frac{S_n}{n} - \mu\right| < \frac{\epsilon}{8}\right\} \cap \left\{\left|\frac{S_{n_0}}{n}\right| < \frac{\epsilon}{8}\right\} \cap \left\{\left|\frac{S^*_{n_0}}{n}\right| < \frac{\epsilon}{8}\right\}$$
$$\cap \left\{|Y_k - Z^k| < \frac{1}{2^k} \text{ for all } n_0 \leq k \leq n\right\}$$
$$\subseteq \left\{\left|\frac{S^*_n}{n} - \mu\right| < \frac{4\epsilon}{8}\right\} \quad \text{for all } n \geq n_1,$$

it follows that

$$\tau\left\{\left|\frac{S_n}{n} - \mu\right| < \frac{4\epsilon}{8} \quad \text{for all sufficiently large } n\right\} > 1 - \frac{3\eta}{8}.$$

By the theorem of Chen stated at the beginning of Section 2, there exists an integer N_0 such that

$$\tau\left\{\left|\frac{S^*_n}{n} - \mu\right| < \frac{4\epsilon}{8} \quad \text{for all } n \geq N_0\right\} > 1 - \frac{4\eta}{8}.$$

We may assume $N_0 \geq n_1$. It is now easy to check that

$$\tau\left\{\left|\frac{S_n}{n} - \mu\right| < \epsilon \text{ for all } n \geq N_0\right\} > 1 - \eta,$$

since

$$\left\{\left|\frac{S^*_n}{n} - \mu\right| < \frac{4\epsilon}{8}\right\} \cap \left\{\left|\frac{S_{n_0}}{n}\right| < \frac{\epsilon}{8}\right\} \cap \left\{\left|\frac{S^*_{n_0}}{n}\right| < \frac{\epsilon}{8}\right\}$$
$$\cap \left\{|Y_k - Z_k| < \frac{1}{2^k}, \, n_0 \leq k \leq n\right\}$$
$$\subseteq \left\{\left|\frac{S_n}{n} - \mu\right| < \epsilon\right\} \quad \text{for all } n \geq n_1,$$

This completes the proof of the lemma.

Let F be the set of i-blocks as defined before and let Ω, $\mathscr{F}$, β_1, Ψ also be as defined earlier before blocks theorem. Let $j \in I$. Let γ_1, γ_2 be the finitely additive probabilities defined on all subsets of F, defined by

$$\gamma_1(D) = \sigma[j](\beta_1^{-1}(D)), \quad \gamma_2(D) = \sigma[i](\beta_1^{-1}(D)), \quad D \subseteq F.$$

Let $\gamma_1 \times \gamma_2^N$ be the independent strategy on F which associates γ_1 with the empty sequence and γ_2 with every other finite sequence.

LEMMA 11. *For every* $B \in \mathscr{F}$, $\sigma[j](\Psi^{-1}(B)) = \gamma_1 \times \gamma_2^N(B)$.

Proof. For $B \in \mathscr{F}$, by the basic integration formula [9, page 265],

$$\gamma_1 \times \gamma_2^N(B) = \int \gamma_2^N(Bx) d\gamma_1(x) \quad \text{where } Bx = \{w : xw \in B\}.$$

By the blocks theorem, (since γ_2^N is same as π defined earlier),

$$\gamma_2^N(Bx) = \sigma[i](\Psi^{-1}(Bx)), \quad x \in F.$$

Therefore

$$\begin{aligned}\gamma_1 \times \gamma_2^N(B) &= \int \sigma[i](\Psi^{-1}(Bx)) d\gamma_1(x) \\ &= \int_{\{t_{i,1} < \infty\}} \sigma[i](\Psi^{-1}(B < h_1, \ldots, h_{t_{i,1}} >) d\sigma[i](h)\end{aligned}$$

by the change of variable theorem where

$$B < h_1, \ldots, h_{t_{i,1}} > = \{h' \in H : (h_1, \ldots, h_{t_{i,1}}, h') \in B\}.$$

Then, this equals $\sigma[i](\Psi^{-1}(B))$, by Proposition 3, Section 3 of [10]. Hence the lemma is proved.

Let $m_{ii} = \sigma[i](t_{i,1})$. Fix an ϵ such that $0 < \epsilon < 1$. Let $\phi_1(n)$ be the integral part of

$$\frac{n}{m_{ii}} (1 - \epsilon^3),$$

$\phi_2(n)$ the integral part of

$$\frac{n}{m_{ii}} (1 + \epsilon^3)$$

and $\phi^*(n)$ the smallest integer larger than n/m_{ii}.

LEMMA 12. *For the given* ϵ, *there exists a positive integer* n_0 *such that*

$$\sigma[j]\{h : \phi_1(n) < \ell(n)(h) < \phi_2(n) \text{ for all } n \geq n_0\} > 1 - \epsilon.$$

Proof. Let $0 < \delta < 1$. On $\Omega = F^N$, let λ_k be the length of the k^{th} coordinate, $k \geq 1$. By Lemma 11 proved above and by Corollary 8, Section 7 of [10], which asserts that

$$\sigma[j]\left\{h : \frac{\ell(n)}{n} \to \frac{1}{m_{ii}}\right\} = 1,$$

it follows that

$$\gamma_1 \times \gamma_2^N \left\{ w : \frac{\sum_{k=1}^{n} \lambda_k(w)}{n} \to m_{ii} \right\} = 1.$$

By Lemma 10, there exists a positive integer $N(\delta)$ such that

$$\gamma_1 \times \gamma_2^N \left\{ w : \left| \frac{\sum_{k=1}^{n} \lambda_k(w)}{n} - m_{ii} \right| < \delta \text{ for all } n \geq N(\delta) \right\} \geq 1 - \frac{\epsilon}{2}.$$

We can arrange matters so that

$$(m_{ii} + \delta)\frac{N(\delta) + 1}{N(\delta)} < m_{ii} + 2\delta.$$

We then choose a positive integer $N_1(\delta)$ such that

$$\sigma[j](\ell(N_1(\delta)) \geq N(\delta)) \geq 1 - \frac{1}{2}\epsilon.$$

To make this choice we may first choose integers m_1, m_2 such that

$$\sigma[j](t_{i,1} \leq m_1) \geq 1 - \frac{\epsilon}{2N(\delta)} \quad \text{and} \quad \sigma[i](t_{i,1}) \leq m_2) \geq 1 - \frac{\epsilon}{2N(\delta)}$$

and then set $N_1(\delta) = m_1 + (N(\delta) - 1)m_2$. The choices of m_1, m_2 exists because of Theorem 7, Section 6 of [10] which asserts that $\sigma[j](t_{i,1}) < \infty$ for all i, j in a positive recurrent class. The above choice of $N_1(\delta)$ satisfies our requirement since $\{\ell(N_1(\delta)) \geq N(\delta)\}$

$$\supset \{t_{i,1} \leq m_1\} \cap \{m_1 + (k-2)m_2 + 1 \leq t_{i,k} \leq m_1 + (k-1)m_2, \text{ for all } 2 \leq k \leq N(\delta)\};$$

consequently, $\sigma[j]\{\ell(N_1(\delta) \geq N(\delta)\}$

$$\geq \sigma[j][\{t_{i,1} \leq m_1\} \cap \{m_1 + (k-2)m_2 + 1 \leq t_{i,k} \leq m_1 + (k-1)m_2, \text{ for all } 2 \leq k \leq N(\delta)\}]$$

$$\geq \sigma[j](t_{i,1} \leq m_1)\{\sigma[i](t_{i,1} \leq m_2)\}^{N(\delta)-1} \quad \text{(by the strong Markov property)}$$

$$\geq 1 - \frac{\epsilon}{2}.$$

Since for $h \in G_i$,

$$\frac{t_{i,\ell(n)}}{\ell(n)} \leq \frac{n}{\ell(n)} \leq \frac{t_{i,\ell(n)+1}}{\ell(n)},$$

it follows that

$$\sigma[j]\left\{ h : \left| \frac{n}{\ell(n)} - m_{ii} \right| < 2\delta \text{ for all } n \geq N_1(\delta) \right\} \geq 1 - \epsilon.$$

This implies that

$$\sigma[j]\left\{h : \frac{n}{m_{ii}}\left(1 - \frac{2\delta}{m_{ii} - 2\delta}\right) < \ell(n) < \frac{n}{m_{ii}}\left(1 + \frac{2\delta}{m_{ii} - 2\delta}\right) \text{ for all } n \geq N_1(\delta)\right\} \geq 1 - \epsilon.$$

If we now choose $0 < \delta_0 < 1$ small enough such that

$$\frac{2\delta_0}{m_{ii} - 2\delta_0} < \frac{\epsilon^3}{2}$$

and take n_0 to be the larger of $N_1(\delta_0)$ and $2m_{ii}/\epsilon^3$, then it is easily seen that n_0 satisfies the assertion in the lemma.

LEMMA 13 (Kolmogorov's inequality). *If τ is an independent strategy and $\{Y_k\}$ a sequence of coordinate mappings such that $\tau(Y_k) = 0$ for all $k \geq 1$ and $\tau(Y_k^2) < \infty$ for all k, then for $\epsilon > 0$, and all n,*

$$\tau(\max_{k \leq n} |S_k| \geq \epsilon) \leq \sum_{k=1}^{n} \tau(Y_k^2)/\epsilon^2.$$

Proof. The same reasoning as in the countably additive theory goes through and hence we omit the details (see [8], page 235 for a proof in the countably additive case).

LEMMA 14. *For all $j \in I$ and all real x,*

$$\sigma[j]\left\{h : \frac{\sum_{m=2}^{\ell(n)(h)} Z_{m,f-M,i}}{\sqrt{B_i n}} \leq x\right\} \to \Phi(x) \text{ as } n \to \infty.$$

Proof. Let $0 < \epsilon < 1$ and $\phi_1(n)$, $\phi_2(n)$ and $\phi^*(n)$ be as defined before. By Lemma 12, there exists n_0 such that $n_0 > 2m_{ii}/\epsilon^3$ and $\phi_1(n) < \phi^*(n) < \phi_2(n)$ for $n \geq n_0$, and

$$\sigma[j](C) \geq 1 - \epsilon \quad \text{where } C = \{h : \phi_1(n) < \ell(n) < \phi_2(n) \text{ for all } n \geq n_0\}.$$

If $h \in C$, then for all $n \geq n_0$,

$$\left| \sum_{m=2}^{\ell(n)} Z_{m,f-M,i} - \sum_{m=2}^{\phi^*(n)} Z_{m,f-M,i} \right| \leq 2 \max_{\phi_1(n)+1 \leq k \leq \phi_2(n)} \left| \sum_{s=\phi_1(n)+1}^{k} Z_{s,f-M,i} \right|.$$

Now by Kolmogorov's inequality,

$$\sigma[j]\left\{2 \operatorname*{Max}_{\phi_1(n)+1\le k\le \phi_2(n)} \left| \sum_{s=\phi_1(n)+1}^{k} Z_{s,f-M,i} \right| > \epsilon v_i\sqrt{\phi^*(n)}\right\}$$

$$\le \frac{4\{\phi_2(n)-\phi_1(n)\}v_i^2}{\epsilon^2 v_i^2 \phi^*(n)} \quad (\text{where } v_i^2 = \sigma[i](Z_{1,f-M,i}^2))$$

$$\le 10\epsilon.$$

Therefore

$$\sigma[j]\left\{\left| \sum_{m=2}^{\ell(n)} Z_{m,f-M,i} - \sum_{m=2}^{\phi^*(n)} Z_{m,f-M,i}\right| > \epsilon v_i\sqrt{\phi^*(n)}\right\} \le 11\epsilon.$$

Consequently,

$$\frac{1}{v_i\sqrt{\phi^*(n)}}\sum_{m=2}^{\ell(n)} Z_{m,f-M,i} - \frac{1}{v_i\sqrt{\phi^*(n)}}\sum_{m=2}^{\phi^*(n)} Z_{m,f-M,i}$$

converges to zero in $\sigma[j]$-probability. By the strong Markov property,

$$\sigma[j]\left\{\frac{1}{v_i\sqrt{\phi^*(n)}}\sum_{m=2}^{\phi^*(n)} Z_{m,f-M,i} \le x\right\}$$

$$= \sigma[i]\left\{\frac{1}{v_i\sqrt{\phi^*(n)}}\sum_{m=1}^{\phi^*(n)-1} Z_{m,f-M,i} \le x\right\}$$

which converges to $\Phi(x)$ by Theorem 2. It only remains to observe that

$$\lim_{n\to\infty}\frac{v_i\sqrt{\phi^*(n)}}{\sqrt{B_i n}} = 1,$$

to complete the proof of the lemma, using Lemma 5 of Section 2.

Proof of Theorem 3. If $S_n(h) = f(h_1) + \ldots + f(h_n)$ for $n \ge 1$,

$$S_n - Mn \begin{cases} = Y'(n) + \sum_{m=2}^{\ell(n)} Z_{m,f-M,i} + Y''(n) - M(n - t_{i,\ell(n)} + t_{i,1}) & \text{if } t_{i,1} \le n, \\ = Y''(n) - Mn & \text{if } t_{i,1} > n \end{cases}$$

Since $\sigma[j](t_{i,1} \ge k) \to 0$, the assertion follows by using Lemmas 6,7,8 and 14 and Lemma 6 of Section 2.

4. Remarks

In the countably additive case, the Central Limit Theorem for positive recurrent Markov chains holds for f such that

(a) $|\sigma[i](Z_{1,f,i})| < \infty$ and

(b) $0 < \sigma[i](Z^2_{1,f-M,i}) < \infty$ (see [2]).

The following example shows that these assumptions are not sufficient in the finitely additive case.

Example. Let I be the set of all integers. Let σ be the Markov strategy with transitions defined by $\sigma(0) = \Sigma_{n=1}^{\infty} p_n \cdot \delta_n + \gamma/2$ where p_n are (strictly) positive numbers adding up to 1/2 and $\Sigma_{n=1}^{\infty} p_{2n} = 1/4$, δ_n is the point mass at n, and γ is a diffuse probability on I such that $\gamma(N) = 1$ and γ(even numbers) = γ(odd numbers); (clearly, by choice, $\sigma(0)$(even numbers) = $\sigma(0)$(odd numbers)). $\sigma(n) = \delta_{-n}$, for $n \geq 1$ and $\sigma(n) = \delta_0$ for $n \leq -1$. Let f be the function on I defined by

$$f(2n)\begin{cases} = 2n & \text{if } n \geq 0, \\ = 2n-1 & \text{if } n \leq -1, \end{cases}$$

$$f(2n+1)\begin{cases} = 2n+1 & \text{if } n \geq 0, \\ = 2n & \text{if } n \leq -1. \end{cases}$$

It is easily checked that I is a positive recurrent class under σ. Further $Z_{1,f,0} = -1$ on a set of $\sigma[0]$-measure 1/2 (all sequences which are extensions of $\langle 2n, -2n, 0\rangle$) and $Z_{1,f,0} = 1$ on a set of $\sigma[0]$-measure 1/2 (all sequences which are extensions of $\langle 2n+1, -2n-1, 0\rangle$). Consequently

$$\sigma[0](Z_{1,f,0}) = 0 \quad \text{and} \quad \sigma[0](Z^2_{1,f,0}) = 1.$$

However

$$\sigma[0]\left\{\frac{S_{3n+1}}{\sqrt{3n+1}} < x\right\} \geq \sigma(0)\{k : k > x\sqrt{3n+1} + n\}$$
$$\geq \frac{1}{2} \quad \text{for all } n \geq 1 \text{ and all real } x.$$

Therefore the CLT does not hold for f.

The rest of the section is devoted to showing that B_i is independent of i.

LEMMA 1. *If f is a function such that $\sigma[i](|Z_{1,f,i}|^p) < \infty$, then*

$$\sigma[j](|Z_{1,f,j}|^p) < \infty$$

for all $j \in I$ where $p > 0$.

Proof. Let $j \neq i$. By lemma 5 of Section 3, $\sigma[i]\left(|\Sigma_{m=1}^{\theta^{**}} Z_{m,f,i}|^p\right) < \infty$. Let

$$Y_1(h) = \sum_{k=1}^{t_{j,1}(h)} f(h_k), \quad Y_2(h) = \sum_{k=t_{j,1}(h)}^{t_{j,2}(h)} f(h_k) \quad \text{and}$$

$$Y_3(h) = \sum_{k=t_{j,2}(h)+1}^{t_{i,\theta^{**}(h)}} f(h_k) \quad h \in H.$$

By the strong Markov property $\sigma[i](|Y_2|^p) = \sigma[j](|Z_{1,f,j}|^p)$. So the lemma is equivalent to proving that if $\sigma[i](|Y_1 + Y_2 + Y_3|^p)$ is finite then

$$\sigma[i](|Y_2|^p) < \infty.$$

We shall first show that $\sigma[i](|Y_1 + Y_2|^p) < \infty$. First, it is not possible that

$$\sigma[i](|Y_3|^p \geq k) = 1 \quad \text{for all } k \geq 1,$$

since then $\sigma[i]\{|Y_3|^p \geq 2^p|Y_1 + Y_2|^p + k\} = 1$ for all $k \geq 1$. (To see the last assertion we can apply the strong Markov property to the incomplete stop rule $t_{j,2}$.) This would imply

$$\sigma[i](2^p|Y_1 + Y_2 + Y_3|^p \geq k) = 1 \quad \text{for all } k$$

(since $2^p|Y_1 + Y_2 + Y_3|^p \geq |Y_3|^p - 2^p|Y_1 + Y_2|^p$) and this is impossible since

$$\sigma[i](|Y_1 + Y_2 + Y_3|^p) < \infty.$$

Let k be such that $\sigma[i](2^p|Y_3|^p < k) > 0$. Let $\sigma[i](|Y_1 + Y_2|^p)$ be infinite if possible. We can then choose an integer N_0 such that

$$\sum_{n=k+1}^{k+N_0} \sigma[i](|Y_1 + Y_2|^p \geq n) > \frac{\sigma[i](2^p|Y_1 + Y_2 + Y_3|^p)}{\sigma[i](2^p|Y_3|^p < k)}.$$

The last step uses the fact that a random variable ξ has a finite integral with respect to a finitely additive probability P iff $\Sigma_{n=1}^{\infty} P(|\xi| \geq n) < \infty$. An easy way to see this is to observe that $[|\xi|] \leq |\xi| \leq [|\xi|] + 1$, where $[|\xi|]$ is the integral part of $|\xi|$, and note that

$$\int [|\xi|]dP = \sum_{n=1}^{\infty} P([|\xi|] \geq n) = \sum_{n=1}^{\infty} P(|\xi| \geq n).$$

We then have

$$\begin{aligned}
\sigma[i](2^p|Y_1 + Y_2 + Y_3|^p) &< \sum_{n=k+1}^{k+N_0} \sigma[i](|Y_1 + Y_2|^p \geq n) \cdot \sigma[i](2^p|Y_3|^p < k) \\
&= \sum_{n=k+1}^{k+N_0} \sigma[i]\{|Y_1 + Y_2|^p \geq n, 2^p|Y_3|^p < k\} \\
&\qquad \text{(by the strong Markov property)} \\
&\leq \sum_{n=1}^{N_0} \sigma[i]\{2^p|Y_1 + Y_2 + Y_3|^p \geq n\}.
\end{aligned}$$

This is a contradiction by Lemma 2 of Section 3. Therefore $\sigma[i](|Y_1 + Y_2|^p) < \infty$. A similar argument as above shows that $\sigma[i](|Y_2|^p) < \infty$. This completes the proof.

LEMMA 2. *If* $0 < B_i < \infty$ *for some i, then* $0 < B_j < \infty$ *for all* $j \in I$.

Proof. By the previous lemma applied to $f - M$ with $p = 2$, it follows that $B_j < \infty$ for all $j \in I$. If $B_j = 0$ for some j, it is easily seen that $(S_n - Mn)/\sqrt{n}$ converges to zero in $\sigma[j]$-probability, which would contradict Theorem 3. This proves the lemma.

PROPOSITION 3. *Let I be a positive recurrent class under* σ. *Then* B_i *is independent of i, where* B_i *is as defined before.*

Proof. If B_i is infinite for some i, then it is so for all i, because of Lemma 1. If $B_i = 0$ for some i, B_j cannot be positive for any j because of Lemma 2. If $0 < B_j < \infty$ for all $j \in I$, by Theorem 3 it follows that $B_i = B_j$ for all $i, j \in I$. This proves the proposition.

Acknowledgements. Several discussions with Professors A. Maitra, B.V. Rao and R.L. Karandikar were of great help during the preparation of this paper. Some thoughtful comments were also provided by the referee.

REFERENCES

1. R. CHEN, *A finitely additive version of Kolmogorov's law of iterated logarithm*, Israel J. Math., vol. 23(1976), pp. 209–220.
2. K.L. CHUNG, *Markov chains with stationary transition probabilities*, 2nd edition, Springer, Berlin, 1967.
3. L.E. DUBINS, *On Lebesgue-like extensions of finitely additive measures*, Ann. Probability, vol. 2(1974), pp. 456–463.
4. L.E. DUBINS and L.J. SAVAGE, *Inequalities for stochastic processes: How to gamble if you must*, Dover, New York, 1976.
5. N. DUNFORD and J.T. SCHWARTZ, *Linear operators*, Part 1, Interscience, New York, 1958.
6. A. HALVEY and M.B. RAO, *On an analogue of Komlos' theorem for strategies*, Ann. Probability, vol. 7(1979), pp. 1073–1077.
7. R.L. KARANDIKAR, *A general principle for limit theorems in finitely additive probability*, Trans. Amer. Math. Soc., to appear.
8. M. LOEVE, *Probability theory*, 3rd edition, Van Nostrand, Princeton, 1963.
9. R.A. PURVES and W.D. SUDDERTH, *Some finitely additive probability*, Ann. Probability, vol. 4(1976), vol. 259–276.
10. S. RAMAKRISHNAN, *Finitely additive Markov chains*, Trans Amer. Math. Soc., vol. 265(1981), pp. 247–272.

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