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POTENTIAL THEORY ON COMPLEX PROJECTIVE SPACE: APPLICATION TO CHARACTERIZATION OF PLURIPOLAR SETS AND GROWTH OF ANALYTIC VARIETIES

BY

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0. Introduction

A set $E \subset \mathbf{P}^n\mathbf{C}$ is said to be locally pluripolar if for each point $p \in E$ there exists a neighborhood U of p and a plurisubharmonic function ψ defined on U such that $E \subset U \quad \{x : \psi(x) = -\infty\}$ and ψ is not identically $-\infty$ on each component of U. A basic problem in function theory of several complex variables is to characterize those sets which are pluripolar. In his paper on projective capacity [1], Alexander gives a characterization of pluripolar sets in $\mathbf{P}^n\mathbf{C}$ in terms of a Tchebycheff constant $\tau(E)$. His theorem says that E is locally pluripolar if and only if $\tau(E) = 0$. The constant $\tau(E)$ is defined in terms of normalized homogeneous polynomials on $\mathbf{P}^n\mathbf{C}$. Another characterization of pluripolar sets was recently given by Bedford and Taylor [3]. Their characterization involves the Monge-Ampere equation and a "balayage" for a set $E \subset \mathbf{C}^n$.

In this paper I give a characterization of locally pluripolar sets in $\mathbf{P}^n\mathbf{C}$ in terms of a singular integral with respect to a probability measure, supported on E; the set in question. The kernel of this singular integral is defined on

$$\mathbf{P}^n\mathbf{C} \times \mathbf{P}(S_{n+1,d})$$

where $S_{n+1,d}$ is the d-fold symmetric tensor product of $\mathbf{C}^{n+1}$; hence the kernel is not symmetric. Explicitly the kernel is given by

$$K_d(Z, a) = \log \frac{|Z|^d}{|a^*(Z)|}$$

where a^* denotes the homogeneous polynomial of degree d dual to a.

The kernel $K_d(Z, a)$ also turns out to play an important role in value distribution theory. If X is an analytic subvariety of $\mathbf{C}^n$ then a basic problem is to relate the growth of X to the growth of intersections of X with algebraic subvarieties of $\mathbf{C}^n$. This was done in [9] in the case where the algebraic subvarieties where hyperplanes. We also remarked in [9] that the growth of X could be related to the growth of $X \cap V^\lambda$ where $\{V^\lambda\}$ was a sufficiently large family of

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algebraic hypersurfaces. The family of algebraic hypersurfaces needed was much larger than in the case of hyperplanes.

Using singular integrals with the kernel $K_d(Z, a)$, I have been able to obtain a result analogous to the hyperplane section growth estimates of [9]. Essentially a family of algebraic hypersurfaces $\{V^t\}$ paramaterized by one real variable t suffices to determine the growth of X in terms of the growth of $X \cap V^t$.

1. Preliminaries

Let $S_{n+1,d}$ denote the d-fold symmetric tensor product of $\mathbf{C}^{n+1}$. This is the space dual to the vector space of homogeneous polynomials of degree d on $\mathbf{C}^{n+1}$. Let $\mathbf{P}(S_{n+1,d})$ be the associated projective space. Let ϕ and $\tilde{\phi}$ denote the Veronese map and the lifted Veronese map respectively. If P is a homogeneous polynomial of degree d on $\mathbf{C}^{n+1}$ then

$$(P, \tilde{\phi}(z)) = P(z)$$

where (,) denotes the dual pairing. The following diagram commutes.

$$\begin{array}{ccc} \mathbf{C}^{n+1} & \xrightarrow{\tilde{\phi}} & S_{n+1,d} \\ \downarrow \pi & & \downarrow \pi \\ \mathbf{P}^n\mathbf{C} & \xrightarrow{\phi} & \mathbf{P}(S_{n+1,d}) \end{array}$$

Here π denotes the usual projection from affine to projective space. Given

$$a \in \mathbf{P}(S_{n+1,d})$$

the projective algebraic variety defined by a is

$$V^a = \{Z \in \mathbf{P}^n\mathbf{C} : a^*(Z) = 0\}$$

where a^* denotes the homogeneous polynomial dual to a. V^a may also be expressed as

$$V^a = \{Z \in \mathbf{P}^n\mathbf{C} : (a, \phi(Z)) = 0\}.$$

Let $|\cdot|$ denote the norm on $\mathbf{C}^{n+1}$ so $|z|^2 = |z_0|^2 + \ldots + |z_n|^2$ and $\|\cdot\|$ denote the norm on $S_{n+1,d}$ induced by $|\cdot|$ on $\mathbf{C}^{n+1}$.

We now define a singular kernel on $\mathbf{P}^n\mathbf{C} \times \mathbf{P}(S_{n+1,d})$; this kernel will then be used to define potential functions.

Let $a \in \mathbf{P}(S_{n+1,d})$ and $Z \in \mathbf{P}^n\mathbf{C}$. Let

(1.1)
$$K_d(Z, a) = \log\,[|Z|^d / |a^*(Z)|] = \log \frac{|Z|^d}{|(a, \phi(Z))|}$$

where $|Z|^2 = |Z_0|^2 + \ldots |Z_n|^2$ and $a = (a_0, \ldots, a_N)$ with $N = \binom{n+d}{d} - 1$. Note that $K_d(Z, a)$ is well defined since $a^*(Z)$ is a homogeneous polynomial in

Z of degree d and the expression for K_d is independent of the representations for Z.

If $E \subset \mathbf{P}^n\mathbf{C}$ is a Borel measurable set let $\mathscr{P}(E)$ denote the probability measures supported on E, that is the positive Borel measures of unit mass supported on E. Similarly if $F \subset \mathbf{P}(S_{n+1,d})$ is Borel measurable let $\mathscr{P}(F)$ denote the probability measures supported on F. Let

$$\mu \in \mathscr{P}(S_{n+1,d}) \text{ and } \nu \in \mathscr{P}(\mathbf{P}^n\mathbf{C}).$$

Define

$$U_{d,\mu}(Z) = \int_{S_{n+1,d}} K_d(Z, a)d\mu(a) \tag{1.2}$$

and

$$V_{d,\nu}(a) = \frac{1}{d}\int_{\mathbf{P}^n\mathbf{C}} K_d(Z, a)d\nu(Z). \tag{1.3}$$

If $E \subset \mathbf{P}(S_{n+1,d})$ is compact, define

$$U_d(E) = \inf_{\mu \in \mathscr{P}(E)} \sup_{Z \in \mathbf{P}^n\mathbf{C}} U_{d,\mu}(Z). \tag{1.4}$$

If $F \subset \mathbf{P}^n\mathbf{C}$ is compact, define

$$V_d(F) = \inf_{\nu \in \mathscr{P}(F)} \sup_{a^* \in \mathscr{N}^d} V_{d,\theta}(a). \tag{1.5}$$

where $\mathscr{N}_d$ denotes the normalized polynomials of degree d as defined by Alexander [1].

A homogeneous polynomial f on $\mathbf{C}^{n+1}$ is said to be normalized if $\deg f = d$ and

$$\int_S \log |f| d\sigma = d\int_S \log|z_0| d\sigma \tag{1.6}$$

where S denotes the unit sphere in $\mathbf{C}^{n+1}$ and $d\sigma$ denotes the normalized unitarily invariant measure on S and $z = (z_0, \ldots, z_n) \in \mathbf{C}^{n+1}$. The quantities $U_d(E)$ and $V_d(E)$ may of course take on the value $+\infty$. If $\mu(\nu)$ is a probability measure with the property that

$$U_d(E) = \sup_{Z \in \mathbf{P}^n\mathbf{C}} U_{d,\mu}(Z) \tag{1.6}$$

and

$$\left[V_d(F) = \sup_{a^* \in \mathscr{N}d} V_{d,\nu}(a)\right] \tag{1.7}$$

then we call $\mu(\nu)$ an *equilibrium* measure for the set $E(F)$.

The potential function $V_{d,\nu}(a)$ will be used to give a characterization of pluripolar sets in $\mathbf{P}^n\mathbf{C}$ and the potential function $U_{d,\mu}(Z)$ will be used to make a growth estimate for a problem in value distribution theory. We remark that one could define capacity functions on subsets of $\mathbf{P}(S_{n+1,d})$ or $\mathbf{P}^n\mathbf{C}$ as the reciprocal of U_d or V_d respectively. We will state our results here however in terms of the set functions U_d and V_d.

2. Pluripolar subsets of P^nC

We say a set $E \subset P^nC$ is locally pluripolar if for each point $p \in E$ there exist a neighborhood U of p in P^nC and a plurisubharmonic function ψ defined on U such that ψ is not identically $-\infty$ on any component of E and

$$E \cap U \subset \{\psi = -\infty\}.$$

Alexander gives in [1] a criterion, in terms of Tchebycheff polynomials, that a set $E \subset P^nC$ be pluripolar. We will give here a necessary and sufficient condition that a set be pluripolar in terms of the potentials $V_{d,\mu}(a)$. This will be done by relating $V_{d,\mu}(a)$ to the Tchebycheff constant of Alexander.

We first define some Tchebycheff constants closely related to the potential function $V_{d,\mu}(a)$. For d and k positive integers and $E \quad P^nC$ a compact set let

$$(2.1) \qquad r_{d,k}(E) = \inf_{\{a_i^*\}_{1,k} \subset \mathcal{N}_d} \sup_{Z \in E} \left[\prod_{i=1}^{k} \frac{|(a_i, \phi(Z))|}{|Z|^d} \right]^{1/kd}.$$

Let

$$(2.2) \qquad r_d(E) = \lim_{k \to \infty} r_{d,k}(E)$$

and

$$(2.3) \qquad r(E) = \lim_{d \to \infty} r_d(E).$$

The proof that these limits exist follows from the same arguments used in the classical definition of the Tchebycheff constant. See, for example, [1].

PROPOSITION 2.1. *Let $E \subset P^nC$ be compact. Then for all positive integers d we have the inequality $e^{-V_d(E)} \leq r_d(E)$.*

Proof. First fix d. Suppose $F \subset P(S_{n+1,d})$ is compact. We may identify F with $K \subset S \subset C^n$ where S is the unit sphere in C^n. Define

$$(2.4) \qquad \tilde{V}(F) = \inf_{\mu \in \mathcal{P}(K)} \sup_{a^* \in \mathcal{N}_d} \int \log \frac{1}{|(a,X)|} d\mu(X),$$

$$\tilde{r}_k(F) = \inf_{\{a_i\}_{1,k} \subset \mathcal{N}_d} \sup_{X \in K} \left[\prod_{l}^{k} |(a_i, X)| \right]^{1/k}$$

and

$$(2.5) \qquad \tilde{r}(F) = \lim_{k \to \infty} \tilde{r}_k(F).$$

Let $F = \phi(E)$ so $F \quad P(S_{n+1,d})$. Given a probability measure μ on E the push forward $\phi_* \mu$ gives a probability measure on $\phi(E) = F$. It follows that

$$(2.7) \qquad \inf_{\nu \in P(F)} \sup_{a^* \in N_d} \int_{X \in F} \log \frac{|a| \ |X|}{|(a,X)|} d\nu(X) \leq$$

$$\inf_{\mu \in P(E)} \sup_{a^* \in N_d} \int_{Z \in E} \log \frac{|a| \, |\phi(Z)|}{|(a, \phi(Z))|} \, d\mu(Z).$$

Again with $F = \phi(E)$ we have

$$(2.8) \qquad \inf_{\{a_i^*\}_{1,k} \cap N_d} \sup_{X \in F} \left[\prod_{i=1}^{k} \frac{|(a_i, X)|}{|a_i| \, |X|} \right]^{1/k} \leq$$

$$\inf_{\{a_i^*\}_{1,k} \cap N_d} \sup_{Z \in E} \left[\prod_{i=1}^{k} \frac{|(a_i, \phi(Z))|}{|a_i| \, |\phi(Z)|} \right]^{1/k}$$

Now using (2.6), (2.7) and (2.8) and taking d-th roots we get the result. ■

Our next result will compare $r_d(E)$ with the Tchebycheff constant defined by Alexander in [1].

The Tchebycheff constant $\tau(E)$ (denoted by cap(E) in [1]) is defined for $E \in \mathbf{P}^n\mathbf{C}$ compact, as

$$\tau(E) = \lim_{d \to \infty} m_d(E)$$

where

$$m_d(E) = \inf_f \sup_{Z \in E} \left[\frac{|f(Z)|}{|Z|^d} \right]^{1/d}$$

where the infimum is taken over normalized homogeneous polynomials of deg d. The Tchebycheff constant, $m_d(E)$ may be expressed as

$$(2.10) \qquad m_d(E) = \inf_{a^* \in N_d} \sup_{Z \in E} \left[\frac{|(a, \phi(Z))|}{|a| \, |Z|^d} \right]^{1/d}.$$

PROPOSITION 2.2. *Let $E \subset \mathbf{P}^n\mathbf{C}$ be compact. Then for all positive integers d we have $r_d(E) \leq m_d(E)$.*

Proof. We have by definition

$$r_{d,k}(E) = \inf_{\{a_i^*\}_{1,k} N_d} \sup_{Z \in E} \left[\prod_{i=1}^{k} \frac{|(a_i, \phi(Z))|}{|Z|^d} \right]^{1/kd}$$

$$\leq \inf_{a^* \in N_d} \sup_{Z \in E} \left[\frac{|(a, \phi(Z))|}{|Z|^d} \right]^{1/d}$$

$$= m_d(E)$$

by (2.10). Taking the limit as $k \to \infty$ on the left-hand side gives the result. ■

We now present a result which gives a lower bound on $e^{-V_d(E)}$ for $E \subset \mathbf{P}^n\mathbf{C}$ compact. Letting $\pi : \mathbf{C}^{n+1} \to \mathbf{P}^n\mathbf{C}$ as before, let $S \subset \mathbf{C}^{n+1}$ be the unit sphere and $K = \pi^{-1}(E) \cap S$. Then K is a compact circled subset of S. We let $\hat{K}$ denote the polynomially convex hull of K.

PROPOSITION 2.3. *Suppose $E \subset \mathbf{P}^n\mathbf{C}$ is compact, K and $\hat{K}$ as above. If $\hat{K}$ contains a neighborhood of 0 in $\mathbf{C}^{n+1}$ then there exists a constant M such that $V_d(E) \leq M$ for all d sufficiently large.*

Proof. Let $\mathscr{H}_d$ denote the normalized homogeneous polynomials on $\mathbf{C}^{n+1}$ which factor as a product of homogeneous polynomials of degree $\leq d$. Note that if P and Q are elements of $\mathscr{H}_d$ then $P \cdot Q$ is an element of $\mathscr{H}_d$. Let X denote the $\mathscr{H}_d$ hull of K in $\mathbf{C}^{n+1}$, that is,

$$X = \{z \in \mathbf{C}^{n+1} : |P(z)| \leq \sup_K |P(x)| \text{ for all } P \in \mathscr{H}_d.$$

Then $\hat{K} \subset X$. By an extension of Bishop's theorem on Jensen measures given by Alexander (see [2]) there exists for each probability measure μ on X a probability measure ν on K such that

$$\int_X \log|P|\,d\mu \leq \int_K \log|P|\,d\nu \tag{2.11}$$

for all $P \in \mathscr{H}_d$.

By the assumptions of the proposition there exists a $\delta > 0$ independent of d such that $B_\delta \subset \hat{K} \subset X$ where B_δ denotes the closed ball of radius δ in $\mathbf{C}^{n+1}$. Let σ_δ denote the normalized unitarily invariant measure on ∂B_δ and $\sigma = \sigma_1$. Now apply Bishop's theorem with $a^* \in \mathscr{N}_d$. We have

$$\int_{\partial B_\delta} \log\,|a^*(Z)|\,d\sigma_\delta \leq \int_K \log\,|a^*(Z)|\,d\nu \tag{2.12}$$

since $\partial B_\delta \subset X$ and the measure ν is a probability measure on K. Let $\eta = \pi_* \nu$ which is a probability measure on E. Then (2.12) becomes

$$\int_{\partial B} \log\,|a^*(\delta Z)|\,d\sigma \leq \int_E \log \frac{|a^*(Z)|}{|Z|^d}\,d\eta(Z)$$

with $Z = [Z_0 : \cdots : Z_n] \in E \subset \mathbf{P}^n\mathbf{C}$. Since $a^* \in \mathscr{N}_d$ is a normalized polynomial in the sense of (1.6) we obtain

$$\log[\delta^d] + \int_{\partial B} \log\,|z_0|^d d\sigma \leq \int_E \log \frac{|a^*(Z)|}{|Z|^d}\,d\eta(Z)$$

Hence

$$d \cdot V_d(E) \leq d\left[\log\frac{1}{\delta} - \int_S \log|z_0|\,d\sigma\right],$$

and the proposition follows by letting

$$M = \log\frac{1}{\delta} - \int_S \log|z_0|\,d\sigma. \blacksquare$$

We now state two results concerning locally pluripolar subsets of $\mathbf{P}^n\mathbf{C}$ due to Alexander [1].

PROPOSITION 2.4. *Let $E \subset \mathbf{P}^n\mathbf{C}$ be compact, K and $\hat{K}$ as above. Then the following statements hold:*

(1) *E is not locally pluripolar if and only if $\hat{K}$ contains a neighborhood of* 0 in $\mathbf{C}^{n+1}$.

(2) *If E is locally pluripolar then $\tau(E) = 0$.*

We now give the characterization of locally pluripolar subsets of $\mathbf{P}^n\mathbf{C}$ in terms of the potential $V(E)$.

THEOREM 2.5. *Let $E \subset \mathbf{P}^n\mathbf{C}$ be compact. Then E is locally pluripolar if and only if*

$$\lim_{d \to \infty} V_d(E) = +\infty.$$

Proof. First suppose $\lim_{d\to\infty} V_d(E) = +\infty$. Suppose E is not locally pluripolar. Then by Proposition 2.4, $\hat{K}$, the polynomially convex hull of

$$K = \pi^{-1}(E) \cap S,$$

contains a neighborhood of $0 \in \mathbf{C}^{n+1}$. By Proposition 2.3, $V_d(E)$ must be bounded, a contradiction.

Now suppose E is locally pluripolar. By Proposition 2.4, $\tau(E) = 0$. Using the inequalities of Propositions 2.1 and 2.2, and letting $d \to \infty$, we conclude $V_d(E) \to +\infty$. ■

3. Growth estimates for affine analytic varieties

We now turn to a problem in value distribution theory related to the potential functions $U_{d,\mu}(Z)$. In an earlier paper [9], growth estimates for an affine analytic variety $X \subset \mathbf{C}^n$ were given in terms of the growth of $X \cap H^\lambda$ where $\{H^\lambda\}$ was a family of hyperplanes. A family $\{H^\lambda\}$ parameterized by $\lambda \in [0,1]$ sufficed to obtain the growth estimate for X. In this paper we also remarked that the growth of X could be expressed in terms of $X \cap gS$ where $\{gS\}$ consisted of the family of algebraic hypersurfaces obtained by letting $g \in G\ell(n+1,\mathbf{C})$ act on the algebraic hypersurface S. A set E of g's in $G\ell(n+1,\mathbf{C})$ of positive volume was required to obtain the growth estimate in contrast to the case where S was a hyperplane.

Using the potential functions $U_{d,\mu}(Z)$ we can now show that in fact a much smaller family of algebraic hypersurfaces suffices to estimate the growth of X in terms of intersection with the hypersurfaces.

We will first recall some necessary notation from value distribution theory.

Suppose $X \subset \mathbf{C}^n$ is an analytic subvariety of pure dimension $s \geq 1$. We want to consider the intersection of X with algebraic varieties. We will regard $\mathbf{C}^n$ as projective space minus the hyperplane at ∞ so $\mathbf{C}^n \sim \mathbf{P}^n\mathbf{C} - H_\infty$. If

$$Z = [Z_0 : \ldots : Z_n]$$

are homogeneous coordinates on $\mathbf{P}^n\mathbf{C}$ then on $\mathbf{P}^n\mathbf{C} - H_\infty$, $Z = [1 : x_1 : \ldots : x_n]$ and a point in $\mathbf{C}^n$ is identified as

$$(x_1, \ldots, x_n) \sim [1 : x_1 : \ldots : x_n].$$

For $x \in \mathbf{C}^n$ write $|x|^2 = |x_1|^2 + \ldots + |x_n|^2$. Write

$$X[r] = \{x \in X : |x| \leq r\},$$

$$X<r> = \{x \in X : |x| = r\},$$

$$X[r_o, r_1] = \{x \in X : r_0 \leq |x| \leq r_1\}.$$

Recall that for $a \in \mathbf{P}(S_{n+1,d})$ a projective algebraic hypersurface is defined by

$$V^a = \{Z \in \mathbf{P}^n\mathbf{C} : a^*(Z) = 0\}.$$

V^a may be regarded as an affine variety and is then given by

$$V^a = \{x \in \mathbf{C}^n : a^*(1, x_1, \ldots, x_n) = 0\}.$$

When we consider the intersection of X with V^a we will be considering V^a as the affine variety.

For most $a \in \mathbf{P}(S_{n+1,d})$, $V_a \cap X$ will have dimension $s - 1$. Precisely, let

(3.1) $$D_r = \{a \in \mathbf{P}(S_{n+1,d}) : X[r] \cap V^a = \emptyset \text{ or } \dim_p X \cap V^a = s - 1 \text{ for all } p \in X[r] \cap V^a\}.$$

Then D_r is a nonempty open set in $\mathbf{P}(S_{n+1,d})$. For $a \in D_r$, the set

$$X \cap V^a \cap \{|x| \leq r\}$$

is a pure $(s-1)$-dimensional subvariety of the open ball $\{|x| < r\}$ or it is empty.

Growth of Analytic Varieties. On $\mathbf{C}^n$ define the following differential forms.

(3.2) $$\alpha = \frac{1}{4\pi} dd^c \log |x|^2, \qquad \beta = \frac{1}{4\pi} dd^c |x|^2, \qquad \gamma = \frac{1}{2\pi} d^c \log |x|^2 \wedge \alpha^{s-1}.$$

The growth of X is then defined by

(3.3)
$$n(X, r) = \frac{1}{r^{2s}} \int_{X[r]} \beta^s = \frac{1}{r\pi} \int_{X<r>} d^c \log |x|^2 \wedge \alpha^{s-1} = \int_{X[r]} \alpha^s + n(X, 0)$$

where $n(X, 0)$ is the Lelong number of X at 0. The integrated growth function of X is, for $s \geq 1$, (3.4) $$N(X,r) = \int_{r_0}^{r} n(X,t)d\log t = \int_X \tau_r\alpha^s + \tau_r(0)n(X,0)$$

where

(3.5) $$\tau_r = \begin{cases} 0 & \text{if } |x| \geq r \\ \log(r/|x|) & \text{if } r_0 \leq |x| \leq r \\ \log(r/r_0) & \text{if } |x| \leq r_0. \end{cases}$$

Note: If $s = 0$, that is, $\dim X = 0$, then $N(X,r) = \Sigma_{x\in X}\tau_r(x)$.

We will be interested in computing the growth of $X \cap V^a$. This will be done by means of Jensen's formula. Suppose D is a pure $(s-1)$-dimensional subvariety of the s-dimensional variety X so $D = \text{divisor}(f)$. Then

(3.6) $$N(D,r) = \frac{1}{2\pi}\int_{X<r_0>} \log|f|\, d^c\log|x| \wedge \alpha^{s-1} - \int_{X[r_0,r]} \log|f|\alpha^s$$
$$-\frac{1}{2\pi}\int_{X<r_0>} \log|f| d^c \log|x| \wedge \alpha^{s-1}.$$

Let $\Gamma : D_r \to \mathbf{R}$ be defined by

(3.7) $$\Gamma(a) = N(X \cap V^a, r).$$

Then Γ is continuous and bounded on D_r (see for example [11] or [16]).

We now turn to the connection with the potential function $U_{d,\mu}(Z)$ defined by (1.2). Define a function $f_d^a(x)$ for $a \in \mathbf{P}(S_{n+1,d})$ and $x \in \mathbf{C}^n$ by

(3.8) $$f_d^a(x) = \frac{a^*(1,x_1,\ldots,x_n)}{\|a\|}.$$

Note that $f_d^a(x)$ is well defined since it is independent of the representation chosen for $a \in \mathbf{P}(S_{n+1,d})$. By Jensen's formula we have

(3.9)
$$N(X \cap V^a, r) = \int_{X<r>} \log|f_d^a|\gamma - \int_{X[r_0,r]} \log|f_d^a|\alpha^s - \int_{X<r_0>} \log|f_d^a|\gamma.$$

Let $E \subset \mathbf{P}(S_{n+1,d})$ be compact and $\mu \in \mathscr{P}(E)$. Let

(3.10) $$\tilde{U}_{d,\mu}(x) = \int_{a\in E} \log|f_d^a(x)|\,d\mu(a).$$

We will integrate equation (3.9) with respect to a measure μ on $E \subset \mathbf{P}(S_{n+1,d})$ and then estimate the resulting integrals on the right-hand side. For this we need the following:

LEMMA 3.1. *Suppose $v(x)$ is a C^∞ plurisubharmonic function on $\mathbf{C}^n$ such that for some constant M and an integer d,*

(3.11) $$d\cdot\log(1+|x|^2) - M \leq v(x) \leq d\cdot\log(1+|x|^2).$$

Then there exist constants k_1 and k_2 depending only on M such that

$$r^2 \int_{X[r]} dd^c v \wedge \beta^{s-1} \geq k_1 \int_{X[k_2 r]} \beta^s. \tag{3.12}$$

Proof. The lemma is a straightforward extension of a result due to Gruman [8]. We present a proof for the convenience of the reader.

Let

$$X'[r] = \{x \in X : v(x) \leq d \cdot \log(1 + r^2)\}$$

and

$$y(x) = d \cdot \log(1 + r^2) - v(x).$$

If $x \in X'[r]$ then $|x|^2 \leq (e^{M/d} - 1) + e^{M/d} r^2$. Hence if r is sufficiently large then $x \in X'[r]$ implies $|x|^2 \leq k_1 r^2$ for some constant k_1. Hence

$$X[r] \subset X'[r] \subset X[k_1 r]. \tag{3.13}$$

Then

$$\begin{aligned} k_1^2 r^2 \int_{X'[r]} dd^c v \wedge \beta^{s-1} &= k_1^2 r^2 \int_{\partial X'[r]} d^c v \wedge \beta^{s-1} \\ &\geq \int_{\partial X'[r]} |x|^2 d^c v \wedge \beta^{s-1} \\ &\geq \int_{X'[r]} d|x|^2 \wedge d^c v \wedge \beta^{s-1} \\ &= \int_{X'[r]} dv \wedge d^c |x|^2 \wedge \beta^{s-1}. \end{aligned} \tag{3.14}$$

We also have

(3.15)

$$\begin{aligned} \int_{X'[r]} dv \wedge d^c|x|^2 \wedge \beta^{s-1} &= -\int_{X'[r]} dy \wedge d^c|x|^2 \wedge \beta^{s-1} \\ &= \int_{X'[r]} y dd^c|x|^2 \wedge \beta^{s-1} - \int_{\partial X'[r]} y d^c|x|^2 \wedge \beta^{s-1} \\ &= 4\pi \int_{X'[r]} y \beta^s \end{aligned}$$

since $y = 0$ on $\partial X'[r]$.

Let $0 < k_2 < 1$. Then

$$X'[k_2 r] = \{x \in X : v(x) - d \cdot \log(1 + k_2^2 r^2) \leq 0\}$$

and there exists a constant k_3 such that $y(x) \geq k_3$ if $x \in X'[k_2 r]$. Using (3.14) and (3.15) we have

$$k_1^2 r^2 \int_{X'[r]} dd^c v \wedge \beta^{s-1} \geq 4\pi k_3 \int_{X'[k_2 r]} \beta_s. \tag{3.16}$$

Finally using (3.13) and relabeling the constants we get

$$r^2 \int_{X[r]} dd^c v \wedge \beta^{s-1} \geq k_1 \int_{X[k_2 r]} \beta^s. \blacksquare$$

We may now give the main growth estimate for $X \cap V^a$ for a family $\{V^a\}$ of algebraic hypersurfaces.

THEOREM 3.2. *Let X be an analytic subvariety of $\mathbf{C}^n$ of pure dimension $s \geq 1$. Let*

$$E \subset \mathbf{P}(S_{n+1,d})$$

be a compact set such that $U_d(E) \leq M < +\infty$ for some constant M. Then there exist positive constants k_i, $i = 1, \ldots, 4$ depending only on M such that

$$(3.17) \qquad k_1 N(X, k_2 r) \leq \int_{\alpha \in \mathbf{P}(S_{n+1,d})} N(X \cap V^a, r) d\mu(a) \leq k_3 N(X, k_4 r)$$

where μ is an equilibrium measure for E.

Proof. The proof follows the idea of the proof of Theorem 3.1 of [9]. The first inequality involves some extra technical difficulty because of the non-symmetric nature of the kernel $K_d(Z, a)$. The second inequality however follows by the same argument used in [9] and we therefore only present the proof of the first inequality.

By Jensen's formula,

(3.18)

$$N(X \cap V^a, r) = \int_{X<r>} \log |f_d^a(x)| \gamma - \int_{X[r_0, r]} \log |f_d^a(x)| \alpha^s - \int_{X<r_0>} \log |f_d^a(x)| \gamma.$$

With $N = \binom{n+d}{d} - 1$ as before, let $H \subset U(N+1, \mathbf{C})$ be subgroup of $U(N+1, \mathbf{C})$ defined as follows. Let $Y = \tilde{\phi}(\mathbf{C}^{n+1}) \quad S_{n+1,d}$. An element of the unitary group $U(N+1, \mathbf{C})$ is determined by its action on Y since Y contains a set of $\binom{n+d}{d}$ linearly independent (over $\mathbf{C}$) elements of $S_{n+1,d}$. If $g \in U(n+1, \mathbf{C})$, let

$$\sigma(g) \in U(N+1, \mathbf{C})$$

such that

$$\sigma(g)\tilde{\phi}(Z) = \tilde{\phi}(gZ)$$

where $Z \in \mathbf{C}^{n+1}$. Then $\sigma(g)$ determines a unique element of $U(N+1, \mathbf{C})$. Let

$$H = \sigma(U(n+1, \mathbf{C})).$$

Let dh denote the normalized Haar measure on H and let $\{\psi_\epsilon\}$ be a C^∞ approximate identity on H so

$$\int \psi_\epsilon(h) dh = 1$$

and the support of ψ_ϵ decreases to $id = \sigma(1) \in H$. Define the sequence of measures μ_k by

$$\mu_k = \int_H \psi_{1/k}(h) h_* \mu dh.$$

Define potentials U_{d,μ_k} by letting $K_d'(Z,a) = \log[|Z|^d \|a\| / |(a, \phi(Z))|]$ and

(3.20)

$$\begin{aligned} U_{d,\mu_k}(Z) &= \int_{a \in \mathbf{P}(S_{n+1,d})} K_d'(Z,a) d\mu_k(a) = \int_H \int_{\mathbf{P}(S_{n+1,d})} K_d'(Z, ha) \psi_{1/k}(h) d\mu(a) dh \\ &= \int_H \int_{\mathbf{P}(S_{n+1,d})} \log \frac{\|ha\| \ \|\phi(Z)\|}{|(ha, \phi(Z))|} \psi_{1/k}(h) d\mu(a) dh \\ &= \int_H \int_{\mathbf{P}(S_{n+1,d})} \log \frac{\|a\| \ \|h^{-1}\phi(Z)\|}{|(a, h^{-1}\phi(Z))|} \psi_{1/k}(h) du(a) dh \\ &= \int_{U(n+1,\mathbf{C})} \int_{\mathbf{P}(S_{n+1,d})} \log \frac{\|a\| \ \|\phi(g^{-1}Z)\|}{|(a, \phi(g^{-1}Z))|} \tilde{\psi}_{1/k}(g) d\mu(a) dg \\ &= \int_{U(n+1,\mathbf{C})} U_{d,\mu}(g^{-1}Z) \tilde{\psi}_{1/k}(g) dg \end{aligned}$$

where dg is Haar measure on $U(n+1, \mathbf{C})$ and $\tilde{\psi}_\epsilon(g) = \psi_\epsilon(\sigma(g))$. By assumption,

$$0 \leq U_{d,\mu}(Z) \leq M \tag{3.21}$$

for all $Z \in \mathbf{P}^n\mathbf{C}$. It follows that $U_{d,\mu_k} \in C^\infty(\mathbf{P}^n\mathbf{C})$ and is bounded by M.

Now define $\tilde{U}_{d,k} \in C^\infty(\mathbf{C}^n)$ by

$$\begin{aligned} \tilde{U}_{d,k}(x) &= d \cdot \log(1 + |x|^2) - U_{d,\mu_k}([x]) \\ &= \int \log |f_d^a(x)| d\mu_k(a) \end{aligned} \tag{3.22}$$

where $x = (x_1, \cdots, x_n)$ and $[x] = [1 : x_1 : \cdots : x_n]$. Then we have the inequality

$$d \cdot \log(1 + |x|^2) - M \leq \tilde{U}_{d,k}(x) \leq d \cdot \log(1 + |x|^2). \tag{3.23}$$

We have also the equation of currents

$$V^a = \frac{1}{2\pi} dd^c \log |f^a|. \tag{3.24}$$

Assuming $0 \notin X$ we use the expression (3.3) for $n(X, r)$, equations (3.23) and (3.24) to obtain

$$\int n(X \cap V^a, r) d\mu_k(a) = \frac{1}{2\pi} \cdot \frac{2}{r^{2s-2}} \int_{X[r]} dd^c \tilde{U}_{d,k} \wedge \beta^{s-1}. \tag{3.25}$$

It follows from Lemma 3.1 that

$$r^2 \int_{X[r]} dd^c \tilde{U}_{d,k} \wedge \beta^{s-1} \geq 4\pi k_1 \int_{X[k_2 r]} \beta^s \tag{3.26}$$

for some constants k_1, k_2 depending only on M. It then follows from (3.25) and (3.26) that

$$\int n(X \cap V^a, r)\, d\mu_k(a) \geq \tilde{k}_1 n(X, \tilde{k}_2 r). \tag{3.27}$$

Integrating gives

$$\int N(X \cap V^a, r)\, d\mu_k(a) \geq k_1 N(X, k_2 r) \tag{3.28}$$

(after relabeling the constants).

Now if $0 \in X$ then we replace X by $X_t = X + tb$ and let $t \to 0$. (This is a standard argument, see for example [9].) Hence equation (3.28) holds in general.

The first inequality then follows by letting $k \to \infty$ in (3.28) since

$$\int N(X \cap V^a, r)\, d\mu_k(a) \to \int N(X \cap V^a, r)\, d\mu(a) \text{ as } k \to \infty. \blacksquare$$

We now give, as an application of the above theorem, a sufficient condition that an affine analytic variety be algebraic.

THEOREM 3.3. *Let X be an analytic subvariety of $\mathbf{C}^n$ of pure dimension s. Suppose $E \subset \mathbf{P}(S_{n+1,d})$ is a compact set such that $U_d(E)$ is finite and suppose $X \cap V^a$ is algebraic for all $a \in$ E. Then X is algebraic.*

Proof. We may assume $\dim(X \cap V^a) = s - 1$ for all $a \in E$ since in fact

$$\dim(X \cap V^a) = s - 1$$

for μ almost all $a \in E$ where μ is an equilibrium measure supported on E. To see this fix $p \in X$ $\mathbf{C}^n$. Let

$$Q_p = \{a \in \mathbf{P}(S_{n+1,d}) : a^*(p) = 0\}.$$

Since $U_{d,\mu}(E)$ is finite, $\mu(Q_p) = 0$. Now choose points $p_1, p^2, \ldots$ in each irreducible component of X and let $Q = U_i Q_{p_i}$. Then $\mu(Q) = 0$ and $\dim(X \cap V^a) = s - 1$ for all $a \notin Q$.

Now let

$$E_m = \{a \in E : N(X \cap V^a, r) \leq m \log r \text{ for } r \geq 2\}.$$

A theorem of Stoll says that if $Y \subset \mathbf{C}^n$ is an affine analytic variety of pure dimension then Y is algebraic if and only if $N(Y, r) = O(\log r)$.

Hence $N(X \cap V^a, r) = O(\log r)$ for all $a \in E$ and $E = U_m E_m$. Since $U_d(E)$ is finite it follows that $U_d(E_m)$ is finite for some m, say m_0. (This follows by the usual argument that a countable union of sets of capacity zero has capacity zero; see, for example, [9].) By Theorem 3.2 it then follows that

$$N(X, r) = O(\log r)$$

which implies that X is algebraic. $\blacksquare$

One could at this point present a whole sequence of results concerning growth estimates for X in terms of the growth estimates for $X \cap V^a$ for families of algebraic hypersurfaces V^a following the results of [9]. We will leave this to the reader. We will however give a result concerning growth estimates when the family of hypersurfaces V^a is parameterized by $t \in [0,1]$. The nondegeneracy condition in the present context is not so simple as in [9]. We state the result in terms of intersection with codimension s algebraic varieties although what we have is essentially a codimension one result.

We will say that a curve $\sigma : [0,1] \to G\ell(n+1, \mathbf{C})$ is algebraically non-degenerate if the matrix entries $g_{ij}(t)$ of $g(t)$ are algebraically independent.

THEOREM 3.4. *Let $\sigma : [0,1] \to G\ell(n+1,\mathbf{C})$ be an analytic algebraically non-degenerate arc. Let $X \subset \mathbf{C}^n$ be an analytic variety of pure dimension s and $V \subset C^n$ an algebraic variety of pure dimension $n-s$ such that $V = Y_1 \cap \cdots \cap Y_{n-s}$ is a complete intersection of algebraic hypersurfaces. Let*

$$t = (t_1, \cdots, t_{n-s}) \in I^{n-s}$$

with $I = [0,1]$ and V^t the family of varieties

$$V^t = g(t_1)Y_1 \cap \cdots \cap g(t_{n-s})Y_{n-s}.$$

If $V^t \cap X$ is finite for all $t \in I^{n-s}$ then X is algebraic.

Proof. The proof will follow from Theorem 3.3 and a series of lemmas. We first give a class of sets $E \subset \mathbf{P}(S_{n+1,d})$ such that $U_d(E)$ is finite.

LEMMA 3.5. *Let $\sigma : [0,1] \to \mathbf{P}(S_{n+1,d})$ be a linearly nondegenerate analytic curve; that is, assume $E = \sigma([0,1])$ is not contained in a hyperplane of $\mathbf{P}(S_{n+1,d})$. Then $U_d(E) \le M < +\infty$ for some constant M.*

Proof. Theorem 2.1 of [9] tells us that

$$\inf_{\mu \in \mathscr{P}(E)} \sup_{b \in \mathbf{P}(S_{n+1})} \int_{a \in E} \log \frac{\|a\| \ \|b\|}{|(a,b)|} d\mu(a) \le M$$

for some constant M. Let μ be a measure such that

$$\sup_{b \in \mathbf{P}(S_{n+1,d})} \int_{a \in E} \log \frac{\|a\| \ \|b\|}{|(a,b)|} d\mu(a) \le M.$$

$$U_{d,\mu}(Z) = \int_{a \in E} \log \frac{\|a\| \ \|\phi(Z)\|}{|(a,\phi(Z))|} d\mu(a) \le M. \ \blacksquare$$

LEMMA 3.6. *Let*

$$V = \{a^*(Z) = 0\}$$

be an algebraic hypersurface of degree d in $\mathbf{P}^n\mathbf{C}$ so $a \in S_{n+1,d}$. Let

$$\sigma : [0,1] \to G\ell(n+1, \mathbf{C})$$

be an algebraically nondegenerate real analytic arc. Then the set $E \subset \mathbf{P}(S_{n+1,d})$ corresponding to the set of algebraic hypersurfaces $\{g(t)V : t \in [0,1]\}$ has the property that $U_d(E)$ is finite.

Proof. By the previous lemma it suffices to show that E is the image of a map

$$\tilde{\sigma} : [0,1] \rightarrow \mathbf{P}(S_{n+1,d})$$

and that E is not contained in a hypersurface in $\mathbf{P}(S_{n+1,d})$. With

$$V = \{Z \in \mathbf{P}^n\mathbf{C} : a^*(Z) = 0\}$$

we have

$$g(t)V = \{Z \in \mathbf{P}^n\mathbf{C} : a^*(g(t)Z) = 0\} = \{Z : (a, \phi(g(t)Z)) = 0\}.$$

Write $(g(t)Z)_i = g_{ij}(t)Z_j$, $i,j = 0,\dots,n$, where we use the summation convention. Since the $g_{ij}(t)$ are algebraically independent, the linear polynomials $(g(t)Z)_i \in \mathbf{C}[(Z_j)]$ are algebraically independent. Now if $W \in \mathbf{P}^n\mathbf{C}$ then write

$$\phi(W) = [\phi_0(W);\dots;\phi_N(W)]$$

where the $\phi_\nu(W)$ are linearly independent polynomials of degree d in the W_i, $i = 0,n$; $\nu = 0N$. Hence it follows that $\phi_\nu(g(t)Z)$ are linearly independent polynomials of degree d in $g_{ij}(t)$ with coefficients which are polynomials of degree d in Z_i. In fact the polynomials $\phi_\nu(g(t)Z)$ are linearly independent over $\mathbf{C}[(Z_i)]$, $i = 0,n$. Considering $\phi_\nu(g(t)Z)$ as a polynomial with coefficients in the $g_{ij}(t)$, $\phi_\nu(g(t)Z)$ is homogeneous in the Z_i and of degree d and the $\phi_\nu(g(t)Z)$ are linearly independent over $\mathbf{C}$. We may write

$$\phi_\nu(g(t)Z) = h_{\nu\mu}(t)\psi_\mu(Z)$$

where ν, $\mu = 0,N$ and ψ_μ is a homogeneous polynomial of degree d. It follows that the functions $h_{\mu\nu}(t)$ are linearly independent over $\mathbf{C}$ and

$$E = \{[a_\nu h_{\nu 0}(t): \cdots :a_\nu h_{\nu N}(t)]\}_{t \in [0,1]} \subset \mathbf{P}(S_{n+1,d}).$$

Since the $h_{\nu\mu}(t)$ are linearly independent over $\mathbf{C}$, E does not lie in a hyperplane of $\mathbf{P}(S_{n+1,d})$. ∎

To prove the theorem write

$$X \cap V^t = (X \cap g(t_1)Y_1 \cap \cdots \cap g(t_{n-s-1})Y_{n-s-1}) \cap g(t_{n-s})Y_{n-s}.$$

It follows by the two lemmas above and Theorem 3.3 that

$$X \cap g(t_1)Y_1 \cap \cdots \cap g(t_{n-s-1})Y_{n-s-1}$$

is algebraic and hence the result follows by induction. ∎

A simple example shows that some sort of nondegeneracy condition is needed in Theorem 3.4. Suppose

$$g(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in G\ell(4, \mathbf{C}), \quad t \in [0,1].$$

Let $V \subset \mathbf{P}^4\mathbf{C}$ be given by

$$V = \{Z : Z_0^2 Z_3 - Z_0^3 - Z_2^3 = 0\}.$$

Then

$$V^t = \{Z : Z_0^2 Z_3 - Z_0^3 - (1+t)^3 Z_2^3 = 0\}.$$

The affine algebraic variety associated with V^t is

$$V^t = \{z \in C^3 : z_3 = 1 + (1+t)^3 z_2^3\}.$$

Let $X \subset \mathbf{C}^3$ be the analytic curve

$$X = \{z : z_1 = e^{z_2}, z_2^2 = z_3\}.$$

Consider $X \cap V^t$. For each t there are three points in the intersection; however, X is not algebraic.

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