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TEMPERED, INVARIANT, POSITIVE-DEFINITE DISTRIBUTIONS ON $SU(1,1)/\{\pm 1\}$

BY

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1. Introduction

Let G denote the group of conformal mappings of the interior of the unit circle, a Lie group which is naturally isomorphic to both $SU(1, 1)/\{\pm 1\}$ and $SL(2,\mathbf{R})/\{\pm 1\}$. In this paper we establish, via the Fourier transform, a bijective correspondence between the collection of tempered, invariant, positive-definite distributions on G and the easily defined class of tempered Bochner measure pairs. Viewed in another way, the result shows that tempered, invariant, positive-definite distributions are merely integrals, in the distributional sense, of characters of the principal and discrete series representations of G.

The major tools used in this work are the various isomorphisms which are obtained via the operator Fourier transform on G. For each $1 \leq p \leq 2$ let $\mathscr{C}^p(G)$ be Harish-Cchandra's L^p-Schwartz space, with $\mathscr{C}(G) = \mathscr{C}^2(G)$. In his Ph.D. dissertation [1] Arthur characterized the image of $\mathscr{C}(G)$ under the Fourier transform for G any semi-simple Lie group of real rank one. However, an invariant, positive-definite distribution is not, in general, tempered; i.e., it does not extend to a continuous linear functional on $\mathscr{C}(G)$. Such distributions extend, instead, onto $\mathscr{C}^1(G)$ [4, §4]. Unfortunately, for $1 \leq p < 2$, the Fourier transform image of $\mathscr{C}^p(G)$ has yet to be determined, even for $SU(1,1)/\{\pm 1\}$. Given the importance of such results for our work, in this paper we will confine ourselves to the tempered distributional case.

In §§4-6 of this paper we state Arthur's Theorem for $SU(1, 1)/\{\pm 1\}$, and develop certain important results concerning spherical function spaces and their images under the Fourier transform. In §7 tempered invariant distributions are examined. It is shown that such a distribution T is determined, via the spherical decomposition of the Fourier transform, by the zonal spherical transform $\hat{T}$ and a unique complex counting measure μ_d (Theorem 7.4). In §8 it is shown that if T is also positive-definite, then $\hat{T}$ is given by a measure μ_c on $\mathbf{R}$, and both μ_c and μ_d are non-negative and of polynomial growth. In fact, there is a bijection between the collection of tempered, invariant, positive-definite distributions and the collection of pairs (μ_c, μ_d) (Theorem 8.2). In §9 this last result is reformulated to show that a tempered, invariant, positive-definite distri-

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bution is, in the distributional sense, merely an integral of principal and discrete series characters of G (Theorem 9.3).

Extensions of this work will depend upon the Fourier transform isomorphism theorems which become available. Arthur has extended his real rank one $\mathscr{C}(G)$ isomorphism theorem to the general case [2],[3]. There are, at present, no isomorphism theorems for $\mathscr{C}^p(G)$, $1 \leq p < 2$, even for particular semi-simple groups. Results for K-finite subspaces of $\mathscr{C}^p(G)$, G of real rank one, have been found by Trombi [11],[12]; these may serve the same role in a general real rank one study of invariant positive-definite distributions that the spherical function isomorphisms from §5 do in this work.

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2. Preliminaries

(a) *General notation.* The standard symbols N, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ shall be used for the sets of non-negative integers, integers, real numbers and complex numbers respectively; $\mathbf{Z}'$ will be the set of nonzero integers. If $z \in \mathbf{C}$, then $\bar{z}$ denotes the complex conjugate of z. If $T \subset S$, and f is a function on S, then $f|_T$ denotes the restriction of f to T.

If S is a topological space, then $C_0(S)$ denotes the space of compactly supported, continuous complex valued functions on S. If S is a topological vector space, then S' denotes its continuous dual.

For M a C^∞ manifold countable at infinity we write $\mathscr{D}(M)$ for the space of compactly supported, C^∞ complex valued functions on M. When $\mathscr{D}(M)$ is given the Schwartz topology, then $\mathscr{D}'(M)$ is the set of distributions on M.

For a Hilbert space $\mathscr{H}$ let $B(\mathscr{H})$ denote the collection of bounded linear operators on $\mathscr{H}$. Fix an orthonormal basis $\{v_m\}$ for $\mathscr{H}$. Then for each $A \in B(\mathscr{H})$ let A_{mn} denote the matrix element (Av_m, v_n).

(b) *The group G.* Let G denote the group of conformal mappings of the interior D of the unit circle. Then G is naturally isomorphic to the group $SU(1, 1)/\{\pm 1\}$, where $SU(1, 1)$ is the collection of all matrices of the form

$$g = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1,$$

and the action of G on D is given by

$$g \cdot \zeta = \frac{\alpha\zeta + \beta}{\bar{\beta}\zeta + \bar{\alpha}} \text{ for } \zeta \in D.$$

Important elements in $\mathfrak{g}$, the Lie algebra of G, are

$$X_0 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad X_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$X_2 = \frac{1}{2}\begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, \quad Y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

Corresponding elements in the group are $u_\theta = \exp(\theta X_0)$, $a_t = \exp(tX_1)$, and $n_\xi = \exp(\xi X_2)$. Matrix forms for elements in G shall be understood as modulo sign throughout the paper.

Particular subgroups of G are defined by

$$K = \{u_\theta : \theta \in \mathbf{R}\}, A = \{a_t : t \in \mathbf{R}\} \quad \text{and} \quad N = \{n_\xi : \xi \in \mathbf{R}\}.$$

The Iwasawa decomposition for G gives $G = KAN$; i.e., each $g \in G$ can be uniquely decomposed into the form $g = u_\theta a_t n_\xi$. We also obtain an action of G on K, $u_\theta \to u_{g\cdot\theta}$, defined by

$$gu_\theta = u_{g\cdot\theta} a_{t(g,\theta)} n_{\xi(g,\theta)}.$$

Define $A^+ = \{a_t : t > 0\}$. The Cartan decomposition for G then gives $G = K\mathscr{C}\ell(A^+)K$; i.e., each $g \in G$ can be decomposed into the form $g = u_\theta a_t u_\psi$. For $g \notin K$ this decomposition is unique; for all g the a_t term is unique. We write

$$t = H(g). \tag{2.1}$$

(c) *Normalizations of measures.* For $a \in G$ let L_a denote the left translation map $g \to ag$ and R_a the right translation map $g \to ga^{-1}$. The groups K, A, N and G have biinvariant Haar measures which we normalize as follows:

$$\begin{aligned} dk &= du_\theta = d\theta/2\pi \quad (0 \le \theta < 2\pi), \\ da &= da_t = dt, \\ dn &= dn_\xi = d\xi, \\ dx &= e^t du_\theta da_t dn_\xi. \end{aligned}$$

Given two $\mathbf{C}$-valued functions f and g on G, define their convolution by

$$(f*g)(y) = \int_G f(x)g(x^{-1}y)dx \quad \text{for all } y \in G$$

whenever the integral exists. Further define the adjoint of f by

$$f^*(x) = \overline{f(x^{-1})} \text{ for all } x \in G.$$

(d) *Differential operators.* The complexified Lie algebra of G, $\mathfrak{g}_c$, can be identified with $\mathfrak{sl}(2,\mathbf{C})$, the set of all 2×2 complex matrices of trace zero. The conjugation $Z \to \tilde{Z}$ in $\mathfrak{g}_c$ is defined by

$$(X + iY)^{\sim} = X - iY \quad \text{for all } X, Y \in \mathfrak{g}.$$

Let U_c denote the universal enveloping algebra of $\mathfrak{g}_c$. There is an isomorphism $A \to L_A$ of U_c with the algebra of all left invariant analytic differential operators on G. This isomorphism is determined by

$$(L_X f)(x) = f(x;X) = \frac{d}{dt} f(x \exp(tX)) \Big|_{t=0}$$

for all $X \in \mathfrak{g}$, $f \in C^\infty(G)$, and $x \in G$. Similarly, an anti-isomorphism with the right invariant operators is determined by

$$(R_X f)(x) = f(X;x) = \frac{d}{dt} f(\exp(tX)x) \Big|_{t=0} .$$

Four specific elements in U_c will be important in subsequent sections:

$$Z_0 = iX_0, Z_+ = -X_1 - iY, Z_- = X_1 - iY, \omega = X_0^2 - X_1^2 - Y^2.$$

3. Irreducible unitary representations

Let $\hat{K}$ denote the collection of equivalence classes of irreducible representations of the compact group K. Then $\hat{K}$ is naturally isomorphic to $\{\chi_n : n \in \mathbf{Z}\}$, where $\chi_n(u_\theta) = e^{in\theta}$.

Suppose π is an irreducible unitary representation of G on a Hilbert space $\mathscr{H}$. For each $n \in \mathbf{Z}$ define the n-th weight space of π to be

$$\mathscr{H}(n) = \{v \in \mathscr{H} : \pi(u)v = \chi_n(u)v \text{ for all } u \in K\}.$$

The subspace of K-finite vectors of π is $\mathscr{H}_K = \Sigma_{n \in \mathbf{Z}} \mathscr{H}(n)$, and the infinitesimal representation $d\pi$ of U_c on $\mathscr{H}_K$ is defined by

$$d\pi(X)v = \frac{d}{dt} \pi(\exp tX)v \Big|_{t=0} \text{ for all } X \in \mathfrak{g} \text{ and } v \in \mathscr{H}_K.$$

Define the π-classification operations on $\mathscr{H}_K$ by

$$H_0 = d\pi(-Z_0), \quad H_+ = d\pi(-Z_+), \quad H_- = d\pi(-Z_-), \quad \Omega = d\pi(\omega). \quad (3.1)$$

There is a real number $\tilde{q}$, the Casimir scalar of π, such that

$$\Omega v = \tilde{q} v \quad \text{for all } v \in \mathscr{H}_K. \quad (3.2)$$

Define the set of weights of π to be

$$M = \{m \in \mathbf{Z} : \mathscr{H}(m) \text{ is non-trivial}\}.$$

The following classification theorem can be found in [10, §§V. 5-6].

THEOREM 3.1. *Suppose π is an irreducible unitary representation of G on a Hilbert space $\mathscr{H}$ with weight set M and Casimir scalar $\tilde{q}$. Then there exists an orthonormal basis $\{v_m : m \in M\}$ for $\mathscr{H}$ and a set of complex numbers $\{\alpha_m : m \in M\}$ of modulus one such that for each $m \in M$,*

$$\begin{aligned} H_0 v_m &= m v_m, \\ H_+ v_m &= \alpha_{m+1}(\tilde{q} + m(m+1))^{1/2} v_{m+1}, \\ H_- v_m &= \alpha_m^{-1}(\tilde{q} + m(m-1))^{1/2} v_{m-1}, \end{aligned} \tag{3.3}$$

where $v_m = 0$ *if* $m \notin M$. □

A basis for $\mathscr{H}$ as specified in Theorem 3.1 will be called a canonical basis for π.

For any irreducible unitary representation π with Casimir scalar $\tilde{q}$ it will be convenient to define certain scalar constants. For each pair of integers (m, n) define

$$\zeta_{mn} = \begin{cases} \prod_{k=n+1}^{m} (\tilde{q} + k(k-1)) & \text{if } m \geq n, \\ \prod_{k=m+1}^{n} (\tilde{q} + k(k-1)) & \text{if } m < n. \end{cases} \tag{3.4}$$

Let $\hat{G}$ denote the collection of unitary equivalence classes of irreducible unitary representations of G. There are two subcollections of $\hat{G}$ which will be important for our work.

The principal series. Let $\mathscr{H}_c = L^2(K)$. For each $\lambda \in \mathbf{R}$ we can define an irreducible unitary representation π_λ of G on $\mathscr{H}_c$ by

$$[\pi_\lambda(g)\varphi](u_\theta) = \varphi(u_{g^{-1}\theta})\exp(-\frac{1}{2}(1 - i\lambda)t(g^{-1}, \theta)) \tag{3.5}$$

for all $g \in G$, $u_\theta \in K$ and $\varphi \in \mathscr{H}_c$. The representation π_λ has weight set $\mathbf{Z}$ and Casimir scalar

$$\tilde{q} = (1 + \lambda^2)/4. \tag{3.6}$$

A canonical basis for π_λ is given by $\{\varphi_m : m \in \mathbf{Z}\}$, where $\varphi_m(u_\theta) = e^{-im\theta}$. The collection $\{\pi_\lambda : \lambda \in \mathbf{R}\}$ is called the principal series for G.

π_λ and π_δ are unitarily equivalent if and only if $\lambda = \pm\delta$. A unitary intertwining operator $N_\lambda : \mathscr{H}_c \to \mathscr{H}_c$ can be defined by $N_\lambda \varphi_m = \omega_m(\lambda)\varphi_m$ for all $m \in \mathbf{Z}$ where

$$\omega_m(\lambda) = \begin{cases} \prod_{k=n+1}^{m} (k - \frac{1}{2}(1 - i\lambda))/(k - \frac{1}{2}(1 + i\lambda)) & \text{if } m \geq 0, \\ \prod_{k=m+1}^{0} (k - \frac{1}{2}(1 + i\lambda))/(k - \frac{1}{2}(1 - i\lambda)) & \text{if } m < 0, \end{cases}$$

Then

$$N_\lambda \pi_\lambda(x) = \pi_{-\lambda}(x) N_\lambda \quad \text{for all } \lambda \in \mathbf{R} \text{ and } x \in G. \tag{3.7}$$

The matrix coefficients for π_λ are defined by

$$u_{mn}(\lambda, x) = (\pi_\lambda(x)\varphi_n, \varphi_m)$$

for all $m, n \in \mathbf{Z}$ and $x \in G$.

The discrete series. For each $\ell \in \mathbf{Z}'$ there is an irreducible unitary representation ω_ℓ on a Hilbert space $\mathscr{H}_\ell$ with Casimir scalar $\tilde{q} = |\ell|(1 - |\ell|)$ and weight set

$$M(\ell) = \begin{cases} -\ell - N & \text{for } \ell > 0, \\ -\ell + N & \text{for } \ell < 0. \end{cases}$$

The collection $\{\omega_\ell : \ell \in \mathbf{Z}'\}$ is called the discrete series for G. For each $\ell \in \mathbf{Z}'$ fix a canonical basis $\{\psi^\ell_m : m \in M(\ell)\}$ for ω_ℓ. If $(\cdot,\cdot)_\ell$ denotes the inner product of $\mathscr{H}_\ell$, then the matrix coefficients for ω_ℓ are defined by

$$v_{mn}(\ell, x) = (\omega_\ell(x)\psi^\ell_n, \psi^\ell_m)_\ell$$

for all $m, n \in M(\ell)$ and $x \in G$.

4. The Fourier transform

Suppose $f \in C_0(G)$ and π is a representation of G on a Hilbert space $\mathscr{H}$. Define the Fourier transform of f at π as the operator $\mathscr{T}f(\pi) \in B(\mathscr{H})$ given by

$$\mathscr{T}f(\pi) = \int_G f(x)\pi(x^{-1})dx.$$

Let $\mathscr{T}^c$ and $\mathscr{T}^d$ denote the restriction of $\mathscr{T}$ to the representations π_λ and ω_ℓ respectively, where, for each $\lambda \in \mathbf{R}$ and $\ell \in \mathbf{Z}'$, we write

$$\mathscr{T}^c f(\lambda) = \mathscr{T}f(\pi_\lambda), \qquad \mathscr{T}^d f(\ell) = \mathscr{T}f(\omega_\ell). \tag{4.1}$$

The matrix coefficients of $\mathscr{T}^c f(\lambda)$ and $\mathscr{T}^d f(\ell)$ with respect to the canonical bases chosen in §3 will be denoted by $\mathscr{T}^c_{mn} f(\lambda)$ and $\mathscr{T}^d_{mn} f(\ell)$ respectively.

For a fixed pair of integers (m, n) define

$$\ell(m,n) = \begin{cases} -\min\{m,n\} & \text{if } n > 0 \text{ and } m > 0, \\ \min\{-m, -n\} & \text{if } n < 0 \text{ and } m < 0, \\ 0 & \text{if } mn \le 0, \end{cases} \tag{4.2}$$

$$L(m,n) = \begin{cases} \{\ell \in \mathbf{Z} : \ell(m,n) \le \ell \le -1\} & \text{if } n > 0 \text{ and } m > 0, \\ \{\ell \in \mathbf{Z} : 1 \le \ell \le \ell(m,n)\} & \text{if } n < 0 \text{ and } m < 0, \\ \phi & \text{if } mn \le 0. \end{cases} \tag{4.3}$$

Then $\mathscr{T}^d_{mn} f(\ell)$ is defined if and only if $\ell \in L(m, n)$. For convenience we will define all the other symbols $\mathscr{T}^d_{mn} f(\ell)$ to exist and equal zero.

Harish-Chandra's Schwartz space on G is defined by

$$\mathscr{C}(G) = \{f \in C^\infty(G) : \|f\|_{rDE} < \infty \text{ for all } r \in N, \quad D, E \in \mathrm{U}_c\}$$

where

$$\|f\|_{r,D,E} = \sup_{x\in G}|(1 + t^r)e^{t/2}f(E;x;D)| \tag{4.4}$$

and $t = H(x)$ as in 2.1. When topologized by these seminorms $\mathscr{C}(G)$ becomes a Fréchet space with continuous inclusions $\mathscr{D}(G) \subseteq \mathscr{C}(G) \subseteq L^2(G)$. $\mathscr{D}(G)$ is dense in $\mathscr{C}(G)$. Under convolution $\mathscr{C}(G)$ becomes a topological algebra.

Let $\mathscr{C}_c(\hat{G})$ be the collection of all C^∞ operator valued functions $\mathscr{F} : \mathbf{R} \to B(\mathscr{H}_c)$ such that:

(i) $N_\lambda \mathscr{F}(\lambda) = \mathscr{F}(-\lambda)N_\lambda$ for each $\lambda \in \mathbf{R}$;

(ii) $\|\mathscr{F}\|_{r_1,r_2,r_3;r} < \infty$ for all $r_1, r_2, r_3, r \in N$, where

$$\|\mathscr{F}\|_{r_1,r_2,r_3;r} = \sup_{\lambda\in\mathbf{R},m,n\in\mathbf{Z}} \left|\left(\frac{d}{d\lambda}\right)^r \mathscr{F}_{mn}(\lambda)\right|(1 + |\lambda|^{r_1})(1 + |m|^{r_2})(1 + |n|^{r_3}). \tag{4.5}$$

When topologized with these semi-norms, $\mathscr{C}_c(\hat{G})$ becomes a Fréchet space.

Define $\mathscr{C}_d(\hat{G})$ to be the collection of all $F : \mathbf{Z}' \to \Sigma_{\ell\in\mathbf{Z}}B(\mathscr{H}_\ell)$ such that:

(i) $F(\ell) \in B(\mathscr{H}_\ell)$ for each $\ell \in \mathbf{Z}'$;

(ii) $\|F\|_{r_1,r_2,r_3} < \infty$ for all $r_1, r_2, r_3 \in N$, where

$$\|F\|_{r_1,r_2,r_3} = \sup_{\ell\in\mathbf{Z}',m,n\in M(\ell)} |F_{mn}(\ell)|\ (1 + |\ell|^{r_1})(1 + |m|^{r_2})(1 + |n|^{r_3}). \tag{4.6}$$

When topologized by these semi-norms, $\mathscr{C}_d(\hat{G})$ becomes a Fréchet space.

Let $\mathscr{C}(\hat{G}) = \mathscr{C}_c(G) \oplus \mathscr{C}_d(\hat{G})$. Given the obvious topology, $\mathscr{C}(\hat{G})$ is a Fréchet space. For $f \in \mathscr{D}(G)$ let $\mathscr{T}f$ denote $(\mathscr{T}^c f, \mathscr{T}^d f)$. Then $\mathscr{T}$ maps $\mathscr{D}(G)$ into $\mathscr{C}(\hat{G})$.

Theorem 4.1 (Arthur). *The Fourier transform $f \to \mathscr{T}f$ from $\mathscr{D}(G)$ into $\mathscr{C}(\hat{G})$ extends uniquely to a topological isomorphism from $\mathscr{C}(G)$ onto $\mathscr{C}(\hat{G})$. Moreover, the inversion formula, for any $f \in \mathscr{C}(G)$, is given by*

$$f(x) = \frac{1}{8\pi}\sum_{m,n\in\mathbf{Z}} \int_0^\infty \mathscr{T}^c_{mn}f(\lambda)u_{mn}(\lambda,x)\lambda \tanh(\pi\lambda/2)d\lambda + \frac{1}{2\pi}\sum_{m,n\in\mathbf{Z}}\ \sum_{\ell\in L(m,n)} \mathscr{T}^d_{mn}f(\ell)v_{mn}(\ell,x)\left(|\ell| - \frac{1}{2}\right). \quad \square \tag{4.7}$$

Arthur [1] dealt with the Fourier transform of $\mathscr{C}^2(G)$ for G any semi-simple Lie group of real rank one; Theorem 4.1 is his major result when applied to $G = SU(1,1)/\{\pm 1\}$.

Define

$$C_c(G) = \{f \in \mathscr{C}(G) : \mathscr{T}^d f = 0\}, \qquad C_d(G) = \{f \in \mathscr{C}(G) : \mathscr{T}^c f = 0\},$$

where each space is given the relative topology from $C(G)$.

Corollary 4.2. *$\mathscr{C}(G)$ is the direct sum of $\mathscr{C}_c(G)$ and $\mathscr{C}_d(G)$. Moreover, the induced decomposition $f = f_c + f_d$ yields two continuous mappings from $\mathscr{C}(G)$ to $\mathscr{C}_c(G)$ and $\mathscr{C}_d(G)$.*

5. The spherical transforms

For $m, n \in \mathbf{Z}$ define $\mathscr{C}_{mn}$, the space of spherical Schwartz functions of type (m, n), to be the collection of all $f \in \mathscr{C}(G)$ such that

$$f(uxv) = \chi_m(u)f(x)\chi_n(v) \quad \text{for all } x \in G,\ u, v \in K,$$

where $\chi_m(u_\theta) = e^{im\theta}$. Further, define

$$\mathscr{C}_{c,mn} = \mathscr{C}_{mn} \cap \mathscr{C}_c(G), \quad \mathscr{C}_{d,mn} = \mathscr{C}_{mn} \cap \mathscr{C}_d(G).$$

PROPOSITION 5.1. *For each $f \in \mathscr{C}(G)$ there is a unique expansion*

$$f = \sum_{m,n \in \mathbf{Z}} f_{c,mn} + \sum_{m,n \in \mathbf{Z}} f_{d,mn}$$

where $f_{c,mn} \in \mathscr{C}_{c,mn}$ and $f_{d,mn} \in \mathscr{C}_{d,mn}$. The series converges absolutely to f in $\mathscr{C}(G)$, and the mappings $f \to f_{c,mn}$ and $f \to f_{d,mn}$ are continuous.

Proof. Define an operator P_{mn} on $\mathscr{C}(G)$ by

$$P_{mn}f(x) = \iint_{K \times K} \chi_m(u)\chi_n(v)f(u^{-1}xv^{-1})du\,dv.$$

This operator is a continuous projection of $\mathscr{C}(G)$ onto $\mathscr{C}_{mn}$. Moreover, for any $f \in \mathscr{C}(G)$, the series

$$\sum_{m,n \in \mathbf{Z}} P_{mn}f$$

converges absolutely to f in $\mathscr{C}(G)$ [13, p. 161], and is easily seen to be a unique expansion of f into spherical functions. Our result follows by applying the expansion to each term f_c and f_d in the decomposition $f = f_c + f_d$ of Corollary 4.2. □

Let $\|\cdot\|_{HS}$ denote the Hilbert-Schmidt norm.

PROPOSITION 5.2. (i) $\operatorname{tr}\mathscr{F}^c(f * f^*)(\lambda) = \|\mathscr{F}^c f(\lambda)\|_{HS}^2$ *for all $f \in \mathscr{C}(G), \lambda \in \mathbf{R}$.*

(ii) $\operatorname{tr}\mathscr{F}^d(f * f^*)(\ell) = \|\mathscr{F}^d f(\ell)\|_{HS}^2$ *for all $f \in \mathscr{C}(G), \ell \in \mathbf{Z}'$.*

Proof. Using 3.7 it is easy to show that

$$\mathscr{F}^c(f * g)(\lambda) = \mathscr{F}^c g(\lambda)\mathscr{F}^c f(\lambda), \qquad \mathscr{F}^c(f^*)(\lambda) = (\mathscr{F}^c f(\lambda))^*$$

for all $f \in \mathscr{D}(G)$ and $\lambda \in \mathbf{R}$. The density of $\mathscr{D}(G)$ in $\mathscr{C}(G)$, the joint continuity of convolution in $\mathscr{C}(G)$, and the continuity of $\mathscr{F}^c : \mathscr{C}(G) \to \mathscr{C}_c(\hat{G})$ prove these relations valid for all $f, g \in \mathscr{C}(G)$. It is then easy to show that

$$\mathscr{F}^c_{mn}(f * g)(\lambda) = \sum_k \mathscr{F}^c_{mk} f(\lambda)\mathscr{F}^c_{kn} g(\lambda), \tag{5.1}$$

$$\mathscr{F}^c_{mn}(f^*)(\lambda) = (\mathscr{F}^c_{nm} f(\lambda))^- \tag{5.2}$$

for all $m, n \in \mathbf{Z}$ and $\lambda \in \mathbf{R}$. Relation (i) is an easy consequence of 5.1 and 5.2. The discrete case is handled similarly. □

Ehrenpreis and Mautner [5] have characterized the image of $\mathscr{C}_{mn}$ under the spherical transform $\mathscr{T}_{mn} = (\mathscr{T}^c_{mn}, \mathscr{T}^d_{mn})$. We need this result for the case $m = n$. Let $\mathscr{Z}$ be the collection of all C^∞ functions $\Phi : \mathbf{R} \to \mathbf{C}$ such that

(i) $\Phi(-\lambda) = \Phi(\lambda)$ for all $\lambda \in \mathbf{R}$, and

(ii) $\|\Phi\|_{r,s} < \infty$ for all $r, s \in N$, where

$$\|\Phi\|_{r,s} = \sup_{\lambda \in \mathbf{R}} \left| \left(1 + |\lambda|^r\right)\left(\frac{d}{d\lambda}\right)^s \Phi(\lambda) \right|.$$

When topologized by the semi-norms $\|\cdot\|_{r,s}$, $\mathscr{Z}$ becomes a Fréchet space.

For each $m \in \mathbf{Z}$, let Z_{mm} be the collection of all functions $\varphi : \mathbf{Z}' \to \mathbf{C}$ such that $\varphi(\ell) = 0$ for all $\ell \notin L(m,m)$. Z_{mm} is a Fréchet space when topologized by the supremum norm

$$\|\varphi\|_m = \sup_{\ell \in L(m,m)} |\varphi(\ell)|.$$

The following result is derived from [5, Theorem 3.1]; it is also a consequence of Arthur's Theorem (Theorem 4.1).

THEOREM 5.3 (Ehrenpreis and Mautner). *Suppose $m \in \mathbf{Z}$.*

(i) *$\mathscr{T}^c_{mm}$ gives a topological isomorphism from $\mathscr{C}_{c,mm}$ onto $\mathscr{Z}$.*

(ii) *$\mathscr{T}^d_{mm}$ gives a topological isomorphism from $\mathscr{C}_{d,mm}$ onto $\mathscr{Z}_{mm}$.* □

6. Differential operators and spherical functions

PROPOSITION 6.1. *Suppose $f, g \in \mathscr{C}(G)$, $\lambda \in \mathbf{R}$, and $m, n \in \mathbf{Z}$. Then:*

(i) $\mathscr{T}^c_{mm}(L_{Z_+}f)(\lambda) = -\alpha_n(\tilde{q} + n(n-1))^{1/2}\mathscr{T}^c_{m,n-1}f(\lambda)$.

(ii) $\mathscr{T}^c_{m,n-1}(L_{Z_-}f)(\lambda) = -\alpha_n^{-1}(\tilde{q} + n(n-1))^{1/2}\mathscr{T}^c_{mn}f(\lambda)$.

(iii) $\mathscr{T}^c_{m-1,n}(R_{Z_+}f)(\lambda) = -\alpha_m(\tilde{q} + m(m-1))^{1/2}\mathscr{T}^c_{mn}f(\lambda)$.

(iv) $\mathscr{T}^c_{mn}(R_{Z_-}f)(\lambda) = -\alpha_m^{-1}(\tilde{q} + m(m-1))^{1/2}\mathscr{T}^c_{m-1,n}f(\lambda)$.

The same equations are valid for $\mathscr{T}^d_{mn}f(\ell)$, $\ell \in \mathbf{Z}'$, with λ replaced by ℓ.

Proof. Take $f \in \mathscr{D}(G)$. Since $\mathscr{T}^c f(\lambda)$ maps $\mathscr{H}_c$ into $\mathscr{H}_\infty$, the space of C^∞ vectors for π_λ [10, Prop. 5.10], then the equations

$$\mathscr{T}^c(L_Z f)(\lambda)v = d\pi_\lambda(Z)\mathscr{T}^c f(\lambda)v,$$

$$\mathscr{T}^c(R_Z f)(\lambda)v = \mathscr{T}^c f(\lambda)d\pi_\lambda(Z)v,$$

are easily verified for all $Z \in \mathfrak{g}_c$ and $v \in \mathscr{H}_\infty$. Moreover,

$$(d\pi_\lambda(Z)u, v) = (u, -d\pi_\lambda(\tilde{Z})v)$$

for all $\lambda \in \mathbf{R}, Z \in \mathfrak{g}_c$ and $u, v \in \mathcal{H}_\infty$. Equations (i) thru (iv) for $f \in \mathcal{D}(G)$ now follow from 3.1 and 3.3. The density of $\mathcal{D}(G)$ in $\mathcal{C}(G)$, along with Theorem 4.1, prove the equations true for $f \in \mathcal{C}(G)$. The proof for the discrete case follows in a similar manner. □

For any integer r define the differential operator ϵ_r by

$$\epsilon_r = \begin{cases} R_{Z_-}^r L_{Z_+}^r & \text{if } r \geq 0 \\ R_{Z_+}^{-r} L_{Z_-}^{-r} & \text{if } r < 0. \end{cases}$$

These operators were first introduced by Ehrenpreis and Mautner in [5, p. 439]. Throughout this section let m, n be fixed integers, with $r = m - n$.

THEOREM 6.2. *The mapping $f \to \epsilon_r f$ restricts to a topological isomorphism of $\mathcal{C}_{c,nn}$ onto $\mathcal{C}_{c,mm}$.*

Proof. Given $h \in \mathcal{C}_{c,mm}$, define $\mathcal{H}_{mm} = \mathcal{T}_{mm}^c h$. Then $\mathcal{H}_{mm} \in \mathcal{Z}$ by Theorem 5.3. Further, define $\mathcal{F}_{nn} = \mathcal{H}_{mm}/\zeta_{nm}$. From 3.4 and 3.6 we see that $\zeta_{nm}(\lambda)$ is a polynomial in λ^2 which is uniformly bounded away from zero. It is straightforward to show that $\mathcal{F}_{nn} \in \mathcal{Z}$. Hence by Theorem 5.3 there exists $f \in \mathcal{C}_{c,nn}$ such that $\mathcal{T}_{nn}^c f = \mathcal{F}_{nn}$. By Proposition 6.1 we have

$$\mathcal{T}_{mm}^c(\epsilon_r f) = \zeta_{nm} \mathcal{T}_{nn}^c f = \mathcal{H}_{mm} = \mathcal{T}_{mm}^c h.$$

However, since h is in $\mathcal{C}_{c,mm}$ by assumption, and $\epsilon_r f$ is in $\mathcal{C}_{c,mm}$ by Proposition 6.1, we have $\epsilon_r f = h$ by Theorem 5.3. This proves surjectivity. For injectivity assume $\epsilon_r f = 0$ for some $f \in \mathcal{C}_{c,nn}$. Then $\zeta_{nm} \mathcal{T}_{nn}^c f = 0$ by Proposition 6.1. Since $\zeta_{nm} \neq 0$, then $\mathcal{T}_{nn}^c f = 0$. Theorem 5.3 then gives $f = 0$. Clearly, ϵ_r is continuous between the two Fréchet spaces $\mathcal{C}_{c,nn}$ and $\mathcal{C}_{c,mm}$; thus ϵ_r is a topological isomorphism by the Open Mapping Theorem. □

THEOREM 6.3. *The mapping $f \to \epsilon_r f$ restricts to a continuous map of $\mathcal{C}_{d,nn}$ into $\mathcal{C}_{d,mm}$. This mapping is* (i) *surjective if and only if $0 \leq m \leq n$ or $n \leq m \leq 0$, and* (ii) *injective if and only if $0 \leq n \leq m$ or $m \leq n \leq 0$.*

Proof. Proposition 6.1 shows that ϵ_r maps $\mathcal{C}_{d,nn}$ into $\mathcal{C}_{d,mm}$; it is clearly continuous. Suppose the mapping is surjective. Then from Theorem 5.3 and Proposition 6.1, for each $H_{mm} \in Z_{mm}$ there exists $F_{nn} \in Z_{nn}$ such that

$$\zeta_{nm}(\ell) F_{nn}(\ell) = H_{mm}(\ell) \tag{6.1}$$

for all $\ell \in \mathbf{Z}'$. In particular, take $H_{mm}(\ell) = 1$ when $\ell \in L(m, m)$ (cf. 4.3) and zero otherwise. Then 6.1 shows that $F_{nn}(\ell)$ must be non-zero for $\ell \in L(m, m)$; however, $F_{nn}(\ell)$ can be non-zero only when $\ell \in L(n, n)$. Thus $L(m, m) \subseteq L(n, n)$ when ϵ_r is surjective.

Suppose the mapping is injective. Then from Theorem 5.3 and Proposition 6.1 this injectivity is equivalent to: if $F_{nn} \in Z_{nn}$ is such that $\zeta_{nm}(\ell) F_{nn}(\ell) = 0$ for all $\ell \in L(m,m)$, then $F_{nn}(\ell) = 0$ for all $\ell \in L(n,n)$. This easily shows $L(n,n) \subseteq L(m,m)$ if ϵ_r is injective.

Suppose $L(m,m) \subseteq L(n,n)$. This is equivalent to having $0 \leq m \leq n$ or $n \leq m \leq 0$. In both of these situations $\zeta_{nm}(\ell)$ is non-zero for $\ell \in L(m,m)$. This follows from 3.4 with $\tilde{q} = |\ell|(1 - |\ell|)$. Take any $h \in \mathscr{C}_{d,mm}$ and define

$$F_{nn}(\ell) = \mathscr{T}^d_{mm}(\ell)/\zeta_{nm}(\ell) \quad \text{for all } \ell \in L(m,m),$$

and zero otherwise. Then $F_{nn} \in Z_{nn}$, and hence there exists $f \in \mathscr{C}_{d,nn}$ such that $\mathscr{T}^d_{nn}F = F_{nn}$ by Theorem 5.3. Thus, as in the proof of Theorem 6.2, $\mathscr{T}^d_{mm}(\epsilon_r f) = \mathscr{T}^d_{mm}h$ on $\mathbf{Z}'$, and $\epsilon_r f = h$, proving ϵ_r surjective.

Suppose $L(n,n) \subseteq L(m,m)$. Assume $\epsilon_r f = 0$ for some $f \in \mathscr{C}_{d,nn}$. Then

$$\zeta_{nm}(\ell)\mathscr{T}^d_{nn}f(\ell) = 0 \quad \text{for all } \ell \in \mathbf{Z}'.$$

But, as shown above, $\zeta_{nm}(\ell) \neq 0$ when $\ell \in L(n,n)$, and hence $\mathscr{T}^d_{nn}f(\ell) = 0$ for all $\ell \in L(n,n)$. Thus $\mathscr{T}^d_{nn}f(\ell) = 0$ for all ℓ, proving $f = 0$. This shows that ϵ_r is injective. □

For any integer r define the differential operator $\sigma_r = L_-^{|\frac{r}{2}|}L_+^{|\frac{r}{2}|}$. The following result is an easy consequence of Proposition 6.1,

PROPOSITION 6.4. *Suppose $f \in C(G)$, $\lambda \in \mathbf{R}$, and $m,n \in \mathbf{Z}$. Then*

$$\mathscr{T}^c_{nn}(\sigma_r f)(\lambda) = \zeta_{nm}\mathscr{T}^c_{nn}f(\lambda) = \mathscr{T}^c_{mm}(\epsilon_r f)(\lambda).$$

The same equations are valid for $\mathscr{T}^d_{nn}(\sigma_r f)(\ell)$ with $\ell \in \mathbf{Z}'$ replacing λ. □

From the Ehrenpreis-Mautner theorem we know that all the spaces $\mathscr{C}_{c,mm}$ are isomorphic via the Fourier transform with the space $\mathscr{Z}$. This gives natural isomorphisms between the $\mathscr{C}_{c,mm}$ spaces which can be concretely realized via the ϵ and σ operators as in the next result.

PROPOSITION 6.5. *There is a topological isomorphism $\mathscr{B}_{mn} : \mathscr{C}_{c,mm} \to \mathscr{C}_{c,nn}$ given by*

$$\epsilon_r f \to \sigma_r f \quad \textit{for all } f \in \mathscr{C}_{c,nn}$$

such that

$$\mathscr{B}_{mn} = (\mathscr{T}^c_{nn})^{-1} \circ \mathscr{T}^c_{mm}. \tag{6.2}$$

Proof. From Proposition 6.1 we see that σ_r maps $\mathscr{C}_{c,nn}$ into itself; Theorem 6.2 shows $\mathscr{B}_{mn}$ is a well-defined mapping of $\mathscr{C}_{c,mm}$ into $\mathscr{C}_{c,nn}$. Equation 6.2 follows directly from Proposition 6.4, and in turn verifies the remainder of the proposition. □

For the discrete series analogue of the preceding result, suppose m and n are such that $0 \leq m \leq$ n or $n \leq m \leq 0$. Then $Z_{mm} \subseteq Z_{nn}$, and, via the inverse Fourier transform, this sets up a natural injection of $\mathscr{C}_{d,mm}$ into $\mathscr{C}_{d,nn}$ as concretely realized in the next result.

PROPOSITION 6.6. *There is a continuous linear injection* $B_{mn} : \mathscr{C}_{d,mm} \to \mathscr{C}_{d,nn}$ *given by*

$$\epsilon_r f \to \sigma_r f \quad \text{for all } f \in \mathscr{C}_{d,nn}$$

such that

$$B_{mn} = (\mathscr{F}^d_{nn})^{-1} \circ i_{mn} \circ \mathscr{F}^d_{mm} \tag{6.3}$$

where i_{mn} *is the natural inclusion map of* Z_{mm} *into* Z_{nn}.

Proof. From Proposition 6.1 we see that σ_r maps $\mathscr{C}_{d,nn}$ into itself; B_{mn} will then be a well-defined map of $\mathscr{C}_{d,mm}$ into $\mathscr{C}_{d,nn}$ once we show $\sigma_r f = 0$ for any $f \in \mathscr{C}_{d,nn}$ such that $\epsilon_r f = 0$. This is, however, easily seen from Proposition 6.4 and Theorem 5.3(ii). Equation 6.3 follows from Proposition 6.4, and yields the rest of our result from Theorem 5.3(ii). □

7. Tempered, invariant distributions

A distribution T on G is called tempered, if it extends to a continuous linear functional on the Schwartz space $\mathscr{C}(G)$, i.e., $T \in \mathscr{C}'(G)$. Given such a T, for each pair of integers m, n define

$$T_{c,mn}[f] = T[f_{c,mn}], \qquad T_{d,mn}[f] = T[f_{d,mn}]$$

for all $f \in \mathscr{C}(G)$, where $f_{c,mn}$ and $f_{d,mn}$ are as defined in Proposition 5.1. The following result is immediate from Proposition 5.1.

PROPOSITION 7.1. *Suppose* $T \in \mathscr{C}'(G)$. *Then*

$$T = \sum_{m,n \in \mathbf{Z}} T_{c,mn} + \sum_{m,n \in \mathbf{Z}'} T_{d,mn}, \tag{7.1}$$

where the series converges absolutely to T *in the weak topology of* $\mathscr{C}'(G)$. □

A tempered distribution T is said to be invariant (or central) if $T[f^a] = T[f]$ for all $f \in \mathscr{C}(G)$ and $a \in G$, where $f^a(x) = f(a^{-1}xa)$.

PROPOSITION 7.2. *Suppose T is an invariant, tempered distribution.*

(i) $T_{c,mn} = 0$ *and* $T_{d,mn} = 0$ *unless* $m = n$.

(ii) $T[L_Z f] = T[R_Z f]$ *for all* $Z \in \mathfrak{g}_c$ and $f \in \mathscr{C}(G)$.

(iii) $T[\epsilon_r f] = T[\sigma_r f]$ *for all* $f \in \mathscr{C}(G)$ *and* $r \in \mathbf{Z}$.

Proof. (i) It is easily seen that

$$T_{c,mn}[f] = \chi_m(u^{-1})\chi_n(u) T_{c,mn}[f]$$

for all $f \in \mathscr{C}_{c,mn}$ and $u \in K$. Part (i) then follows for $T_{c,mn}$, and similarly for $T_{d,mn}$.

(ii) Suppose $\varphi \in \mathcal{D}(G)$ and $X \in \mathfrak{g}$. Define $\alpha(t) = \exp tX$ for all t, and

$$\psi_t(x) = (\varphi(x\alpha(t)) - \varphi(x))/t \quad \text{for } t \neq 0, \tag{7.2}$$

and

$$\psi(x) = \frac{d}{dt}\varphi(x\alpha(t))\Big|_{t=0}. \tag{7.3}$$

Then $L_X\varphi = \psi$, and from Lemma 7.3, proven below, we know ψ_t converges to ψ in $\mathcal{D}(G)$ as $t \to 0$. Thus

$$T[L_X\varphi] = \lim_{t\to 0} (T_{[x]}[\varphi(x \exp tX)] - T[\varphi])/t. \tag{7.4}$$

However, the invariance of T shows

$$T_{[x]}[\varphi(x \exp tX)] = T_{[x]}[\varphi \exp tX\cdot x)],$$

and the analogue of Lemma 7.3 for $\varphi(\alpha(t)x)$, along with 7.4, then yields $T[L_X\varphi] = T[R_X\varphi]$. The density of $\mathcal{D}(G)$ in $\mathcal{C}(G)$, and the linearity of $Z \to L_Z$ and $Z \to R_Z$ on $\mathfrak{g}_c$ prove (ii). Part (iii) is a consequence of (ii). □

Suppose $\varphi \in \mathcal{D}(G)$, $\alpha(t)$ a C^∞ curve in G with $\alpha(0) = e$, and ψ_t, ψ defined as in 7.2 and 7.3.

LEMMA 7.3. *ψ_t converges to ψ in $\mathcal{D}(G)$as $t \to 0$.*

Proof. There exists a compact set C which contains the supports of ψ and ψ_t for all $|t| \leq 1$. A Taylor expansion on $t \mapsto \varphi(x\alpha(t))$ will show

$$\sup_{x\in C} |DE(\psi - \psi_t)(x)| = |t/2| \sup_{x\in C} \left| DE\left(\frac{d^2}{dt^2}\psi(x\alpha(t))\Big|_{t=t_x}\right)\right| \tag{7.5}$$

for some $|t_x| \leq |t|$ and D (resp. E) any left (resp. right) invariant differential operator on G. The lemma is an easy consequence of 7.5.□

Suppose $T \in \mathcal{C}'(G)$. Then for each pair $m, n \in \mathbf{Z}$ define the (m, n)-spherical transforms of T, $\mathcal{T}^c_{mn}T$ and $\mathcal{T}^d_{mn}T$, by

$$\mathcal{T}^c_{mn}T[\mathcal{T}^c_{mn}f] = \mathcal{T}_{c,mn}[f], \qquad \mathcal{T}^d_{mn}T[\mathcal{T}^d_{mn}f] = T_{d,mn}[f]$$

for all $f \in \mathcal{C}(G)$. To show $\mathcal{T}^c_{mn}T$ well-defined we need only show $f_{c,mn} = 0$ whenever $\mathcal{T}^c_{mn}f = 0$. This, however, follows easily from inversion formula 4.7. In a similar fashion, $\mathcal{T}^d_{mn}T$ is shown well-defined.

Consider $f \in \mathcal{C}_{00}$. Then $\mathcal{T}^d f = 0$, and, as a consequence of [10, §V.9], for each $\lambda \in \mathbf{R}$ we have

$$\mathcal{T}^c f(\lambda)\varphi_k = \begin{cases} \hat{f}(\lambda)\varphi_0 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here $\{\varphi_k : k \in \mathbf{Z}\}$ is the canonical basis of $\mathcal{H}_c$, and $\hat{f}$ is the zonal spherical transform of f as defined in [10, §V.9]. Thus $\mathcal{T}^c_{00}f = \hat{f}$ for all $f \in \mathcal{C}(G)$ and $\mathcal{T}^c_{00}T = \hat{T}$, where $\hat{T}$ is the zonal spherical transform of T, defined by

$$\hat{T}[\hat{f}] = T[f] \quad \text{for all } f \in \mathscr{C}_{00}.$$

THEOREM 7.4. *For each invariant, tempered distribution T there is a unique complex counting measure μ_d defined on $\mathbf{Z}'$ such that, with $f \in \mathscr{C}(G)$, $\mathscr{F}_{mm} = \mathscr{T}^c_{mm} f$ and $F_{mm} = \mathscr{T}^d_{mm} f$, $T[f]$ can be expanded by*

$$T[f] = \sum_{m \in \mathbf{Z}} \hat{T}[\mathscr{F}_{mm}] + \sum_{m \in \mathbf{Z}'} \left(\sum_{\ell \in L(m,m)} F_{mm}(\ell)\mu_d(\ell) \right). \tag{7.6}$$

Proof. From Proposition 7.1 and Proposition 7.2(i) we obtain

$$T[f] = \sum_{m \in \mathbf{Z}} \mathscr{T}^c_{mm} T[\mathscr{F}_{mm}] + \sum_{m \in \mathbf{Z}'} \mathscr{T}^d_{mm} T[F_{mm}], \tag{7.7}$$

for all $f \in \mathscr{C}(G)$, where $\mathscr{F}_{mm} = \mathscr{T}^c_{mm} f$ and $F_{mm} = \mathscr{T}^d_{mm} f$. Moreover, Theorem 5.3 shows $\mathscr{T}^c_{mm} T \in \mathscr{Z}'$ and $\mathscr{T}^d_{mm} T \in Z'_{mm}$ for each $m \in \mathbf{Z}$.

LEMMA 7.5. (i) $\mathscr{T}^c_{nn} T = \mathscr{T}^c_{mm} T$ *for all* $m, n \in \mathbf{Z}$.

(ii) $\mathscr{T}^d_{mm} T = \mathscr{T}^d_{nn} T \big|_{Z_{mm}}$ *for all* $0 \le m \le n$ *or* $n \le m \le 0$.

Proof. For $f \in \mathscr{C}_{c,nn}$ and $r = m - n$, we have $\epsilon_r f \in \mathscr{C}_{c,mm}$ and $\sigma_r f \in \mathscr{C}_{c,nn}$. Hence from Proposition 7.2,

$$T_{c,mm}[\epsilon_r f] = T[\epsilon_r f] = T[\sigma_r f] = T_{c,nn}[\sigma_r f].$$

In the notation of Proposition 6.5 this shows

$$T_{c,mm} = T_{c,nn} \circ \mathscr{B}_{mn},$$

and Proposition 6.5 then yields $\mathscr{T}^c_{mm} T = \mathscr{T}^c_{nn} T$, proving (i).

The discrete case follows in a similar way using Theorem 6.3 and Proposition 6.6. □

Returning to the proof of Theorem 7.4, we see, from Lemma 7.5(i), that $\mathscr{T}^c_{mm} T = \hat{T}$ for all $m \in \mathbf{Z}$. For the discrete half of 7.6, observe that Z_{mm} is isomorphic to $\mathbf{C}^{|m|}$, and $\mathscr{T}^d_{mm} T \in Z'_{mm}$. Hence there exists a unique set $\{a^m_\ell \in \mathbf{C} : \ell \in L(m,m)\}$ such that

$$\mathscr{T}^d_{mm} T[F_{mm}] = \sum_{\ell} F_{mm}(\ell) a^m_\ell \quad \text{for all } F_{mm} \in Z_{mm}.$$

From Lemma 7.5(ii) we then have, for $0 \le m \le n$ *or* $n \le m \le 0$,

$$\sum_{\ell \in L(m,m)} F_{mm}(\ell) a^m_\ell = \sum_{\ell \in L(n,n)} F_{mm}(\ell) a^n_\ell$$

for all $F_{mm} \in Z_{mm}$. This proves $a^m_\ell = a^n_\ell$ for all $m, n \in M(\ell)$, and allows us to define a complex counting measure μ_d on $\mathbf{Z}'$ by $\mu_d(\ell) = a^m_\ell$ for any $m \in M(\ell)$. Combined with 7.7, this finishes the verification of 7.6. □

8. Tempered, invariant, positive-definite distributions

DEFINITION. (μ_c, μ_d) is a tempered Bochner measure pair if:

(i) μ_c is a non-negative Baire measure on $\mathbf{R}$ which is symmetric and of polynomial growth; i.e.,

$$d\mu_c(-\lambda) = d\mu_c(\lambda) \quad \text{for all } \lambda \in \mathbf{R},$$

and

$$\int_{\mathbf{R}} \frac{d\mu_c(\lambda)}{1 + |\lambda|^r} < \infty \quad \text{for some } r \geq 0.$$

(ii) μ_d is a non-negative counting measure on $\mathbf{Z}' = \mathbf{Z} - \{0\}$ which is of polynomial growth; i.e.,

$$\sum_{\ell \in \mathbf{Z}'} \frac{\mu_d(\ell)}{1 + |\ell|^r} < \infty \quad \text{for some } r \geq 0.$$

DEFINITION. A distribution T on G is said to be positive-definite if $T[f*f^*] \geq 0$ for all $f \in \mathscr{D}(G)$.

THEOREM 8.1. *Suppose (μ_c, μ_d) is a tempered Bochner measure pair. Define $T : \mathscr{C}(G) \to \mathbf{C}$ by $T = T_c + T_d$ where*

$$T_c[f] = \int_{\mathbf{R}} \operatorname{tr} \mathscr{F}^c f(\lambda) d\mu_c(\lambda) \quad \textit{and} \quad T_d[f] = \sum_{\ell \in \mathbf{Z}'} \operatorname{tr} \mathscr{F}^d f(\ell) \mu_d(\ell) \tag{8.1}$$

for all $f \in \mathscr{C}(G)$. Then T_c, T_d and T are tempered invariant, positive-definite distributions.

Proof. Each $\mathscr{F}(\lambda)$, for $\mathscr{F} \in \mathscr{C}_c(\hat{G})$ and $\lambda \in \mathbf{R}$, is an operator of trace class. Moreover, using 4.5, there exists $M < \infty$ such that, with r as in the definition of μ_c,

$$\int_{\mathbf{R}} |\operatorname{tr}\mathscr{F}(\lambda)| d\mu_c(\lambda) \leq M \|\mathscr{F}\|_{r,2,0;0} \quad \textit{for all } \mathscr{F} \in \mathscr{C}_c(\hat{G}),$$

proving the map

$$\mathscr{F} \to \int_{\mathbf{R}} \operatorname{tr}\mathscr{F}(\lambda) d\mu_c(\lambda)$$

continuous from $\mathscr{C}_c(\hat{G})$ into $\mathbf{C}$. Theorem 4.1 then shows T_c to be a tempered distribution. Arguing in a similar manner proves T_d to be a tempered distribution.

To prove T_c invariant it suffices to show

$$\operatorname{tr} \mathscr{F}^c(f^a)(\lambda) = \operatorname{tr} \mathscr{F}^c f(\lambda) \tag{8.2}$$

for all $f \in \mathscr{C}(G)$, $\lambda \in \mathbf{R}$ and $a \in G$; this is easily verified since each π_λ is unitary. T_d is handled similarly.

The positive-definiteness of T_c and T_d is a direct consequence of Proposition 5.2. □

THEOREM 8.2. *Every tempered, invariant, positive-definite distribution arises from a unique tempered Bochner measure pair as in Theorem 8.1.*

Proof. Suppose T is a tempered invariant, positive-definite distribution. From Theorem 7.4, there exists a unique complex counting measure μ_d defined on $\mathbf{Z}'$ such that $T = T_c + T_d$ where, for each $f \in \mathscr{C}(G)$,

$$T_c[f] = \sum_{m \in \mathbf{Z}} \hat{T}[\mathscr{T}^c_{mm} f] \tag{8.3}$$

and

$$T_d[f] = \sum_{m \in \mathbf{Z}'} \left(\sum_{\ell \in L(m,m)} \mathscr{T}^d_{mm} f(\ell) \mu_d(\ell) \right). \tag{8.4}$$

Since T_c and T_d represent the first and second terms in 7.1 respectively, then Proposition 7.1 shows both to be tempered distributions.

From the spherical Bochner theorem ([4], Theorems 4.5 and 5.5; also see [9], Theorem 2 and [8], Theorem 2) there exists a unique non-negative Baire measure μ_c of polynomial growth on $\mathbf{R}$ which is symmetric and generates $\hat{T}$ according to the formula

$$\hat{T}[\Phi] = \int_{\mathbf{R}} \Phi(\lambda) d\mu_c(\lambda) \quad \text{for all } \Phi \in \mathscr{Z}.$$

Thus 8.3 becomes

$$T_c[f] = \sum_{m \in \mathbf{Z}} \int_{\mathbf{R}} \mathscr{T}^c_{mm} f(\lambda) d\mu_c(\lambda) \tag{8.5}$$

for all $f \in \mathscr{C}(G)$. By using the semi-norms 4.5 of $\mathscr{C}_c(\hat{G})$, and the polynomial growth of μ_c, it is easy to see that the function

$$\lambda \to \sum_{m \in \mathbf{Z}} |\mathscr{T}^c_{mm} f(\lambda)|, \quad \lambda \in \mathbf{R},$$

is in $L^1(\mu_c)$. Dominated convergence then changes 8.5 into

$$T_c[f] = \int_{\mathbf{R}} \operatorname{tr} \mathscr{T}^c f(\lambda) d\mu_c(\lambda). \tag{8.6}$$

We now show that μ_d is non-negative and of polynomial growth.

For each $m \in \mathbf{Z}'$ consider $f \in \mathscr{C}_{d,mm}$. Then $T[f * f^*] \geq 0$, so by the discrete series analogues of 5.1 and 5.2 we obtain

$$0 \leq \sum_{\ell \in L(m,m)} |\mathscr{T}^d_{mm} f(\ell)|^2 \mu_d(\ell).$$

However, from Theorem 5.3(ii) we can choose $f_m \in \mathscr{C}_{d,mm}$ such that

$$\mathcal{F}^d_{mm} f_m(\ell) = \begin{cases} 1 & \text{if } \ell = -m \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mu_d(-m) \geq 0$ for each $m \in \mathbf{Z}'$.

For any $f \in \mathscr{C}(G)$, m, $\ell \in \mathbf{Z}'$, we know $\mathcal{F}^d_{mm}(f*f^*)(\ell) \geq 0$ from the discrete series analogues of 5.1 and 5.2. Thus, using $f*f^*$ in 8.4 yields

$$T_d\,[f*f^*] = \sum_{\ell \in \mathbf{Z}''} \operatorname{tr} \mathcal{F}^d(f*f^*)(\ell)\mu_d(\ell). \tag{8.7}$$

where the switch in order of summation is legal since this is a sum of non-negative terms with a finite limit. We use this equation to prove that μ_d is of polynomial growth.

Define $\mathcal{F}^d T : C_d(\hat{G}) \to \mathbf{C}$ by

$$\mathcal{F}^d T[\mathcal{F}^d f] = T_d[f] \quad \text{for all } f \in \mathscr{C}_d(G).$$

Since T_d is a tempered distribution, Theorem 4.1 shows $\mathcal{F}^d T$ to be a well-defined, continuous linear operator on $\mathscr{C}_d(\hat{G})$. Thus there exist $r_1, r_2, r_3 \in N$ and $M < \infty$ such that

$$|\mathcal{F}^d T[H]| \leq M\|H\|_{r_1, r_2, r_3}$$

for all $H \in \mathscr{C}_d(\hat{G})$ (cf., 4.6). Hence

$$|T_d[h]| \leq M\|\mathcal{F}^d h\|_{r_1, r_2, r_3} \tag{8.8}$$

for all $h \in \mathscr{C}_d(G)$. Let $r = r_1 + r_2 + r_3$.

For each $\beta > 0$ define $F^\beta \in \mathscr{C}_d(\hat{G})$ by $F^\beta_{\ell\ell}(-\ell) = (1 + |\ell|^r)^{-1/2}$ for all $1 \leq |\ell| \leq \beta$, and $F^\beta_{mm}(\ell) = 0$ otherwise. Let $h_\beta = f_\beta * f^*_\beta$, where $f_\beta \in \mathscr{C}_d(G)$ and $\mathcal{F}^d f_\beta = F^\beta$. Then 8.7, combined with 8.8, yields

$$\sum_{1 \leq |\ell| \leq \beta} \frac{\mu_d(\ell)}{1 + |\ell|^r} \leq M \sup_{1 \leq |\ell| \leq \beta} |(1 + |\ell|^r)^{-1}(1 + |\ell|^{r_1})(1 + |\ell|^{r_2})(1 + |\ell|^{r_3})|.$$

Since the right side of this inequality is bounded above as a function of β, we have shown μ_d to be of polynomial growth on $\mathbf{Z}'$.

Return to 8.4. As in the proof of Theorem 8.1 we can now show, since μ_d is of polynomial growth, that the function $\ell \to \Sigma_{m \in M(\ell)} |\mathcal{F}^d_{mm} f(\ell)|$ is in $L^1(\mu_d)$ by appealing to the defining semi-norms 4.6 of $\mathscr{C}_d(\hat{G})$. Hence the summations in 8.4 can be reversed and we obtain 8.1. The proof of Theorem 8.2 is thus complete. □

9. The tempered invariant Bochner theorem

The distributional character of an irreducible unitary representation π is that invariant, positive-definite distribution Ⓗ defined by

$$Ⓗ\,[f] = \operatorname{tr} \int_G f(x)\pi(x)dx \text{ for all } f \in \mathscr{D}(G).$$

Such characters can be realized as invariant, locally summable functions on

G, which we will denote by the same symbols as the distributions themselves.

Let $D(x)$ be the coefficient of $s-1$ in the expansion of $\det(s - Ad(x))$ in powers of $s-1$. Then D is invariant, and

$$D(a_t) = -(e^{t/2} - e^{-t/2})^2, \qquad D(u_\theta) = -(e^{i\theta/2} - e^{-i\theta/2})^2 \tag{9.1}$$

for all $t, \theta \in \mathbf{R}$ [7, §IV.2]. Let dg_A be any G-invariant measure on G/A. The next result follows from [7, Theorem IV. 1.5] and the proof of Step I in [7, §IV.2].

PROPOSITION 9.1. *For $f \in C_0(G)$ define*

$$\Lambda_f(a_t) = |D(a_t)|^{1/2} \int_{G/A} f(ga_t g^{-1}) dg_A \quad \textit{for } t \in \mathbf{R}.$$

Then Λ_f is a bounded function on A which vanishes outside of a compact subset of A. □

We will also need another technical result, this one a consequence of [6, Lemma 12.1 and Corollary 13.1].

PROPOSITION 9.2. *Let S be a locally summable invariant function on G for which there exists numbers $C_0, m \geq 0$ such that*

$$|D(a_t)|^{1/2} |S(a_t)| \leq C_0(1 + t^m) \quad \textit{for almost all } t \geq 0, \tag{9.2a}$$

and

$$|D(u_\theta)|^{1/2} |S(u_\theta)| \leq C_0 \quad \textit{for almost all } \theta. \tag{9.2b}$$

Then S yields an invariant tempered distribution according to the formula

$$S[f] = \int_G f(x) S(x) dx \quad \text{for all } f \in \mathscr{C}(G). \qquad \square$$

From [10, §V.7] we have the following formulas for Φ^λ and Θ^ℓ, the characters of π_λ and ω_ℓ respectively:

$$\Phi^\lambda(ga_t g^{-1}) = (e^{i\lambda t/2} + e^{-i\lambda t/2}) / |e^{t/2} - e^{-t/2}|, \quad t \neq 0, \tag{9.3}$$

$$\Theta^\ell(ga_t g^{-1}) = e^{(\frac{1}{2} - |\ell|)|t|} \,/\, |e^{t/2} - e^{-t/2}|, \quad t \neq 0,$$

$$\Theta^\ell(gu_\theta g^{-1}) = \operatorname{sgn}(\ell) e^{i \operatorname{sgn}(\ell)(\frac{1}{2} - |\ell|)\theta} \,/\, (e^{i\theta/2} - e^{-i\theta/2}), \quad \theta/2\pi \notin \mathbf{Z}.$$

All other values of these functions are zero. Hence, from Proposition 9.2, for all $f \in \mathscr{C}(G)$ we have

$$\Phi^\lambda[f] = \int_G f(x) \Phi^\lambda(x) dx, \; \Theta^\ell[f] = \int_G f(x) \Theta^\ell(x) dx. \tag{9.4}$$

THEOREM 9.3. *There is a natural one-to-one correspondence between tempered invariant positive-definite distributions T and tempered Bochner measure pairs (μ_c, μ_d). This correspondence is given by*

$$T = \lim_{n \to \infty} \left(\int_{-n}^{n} \Phi^\lambda d\mu_c(\lambda) + \sum_{1 \leq |\ell| \leq n} \Theta^\ell \mu_d(\ell) \right),$$

the limit understood in the tempered distributional sense.

Proof. Suppose T is a tempered invariant positive-definite distribution on G corresponding to the measure pair (μ_c, μ_d). From Theorem 8.1 and equations 9.4 we have, for all $f \in C(G)$,

$$T_d[f] = \sum_{\ell \in \mathbf{Z}'} \left(\int_G f(x) \Theta^\ell(x) dx \right) \mu_d(\ell),$$

and

$$T_c[f] = \int_{\mathbf{R}} \left(\int_G f(x) \Phi^\lambda(x) dx \right) d\mu_c(\lambda).$$

For each $n \geq 0$ define $T_{c,n} : \mathscr{C}(G) \to \mathbf{C}$ by

$$T_{c,n}[f] = \int_{-n}^{n} \left(\int_G f(x) \Phi^\lambda(x) dx \right) d\mu_c(\lambda). \tag{9.5}$$

By Theorem 8.1, $T_{c,n}$ is a tempered, invariant, positive-definite distribution. We show that the order of integration may be reversed in 9.5.

First restrict to $f \in \mathscr{D}(G)$. By 9.1, 9.3, and [10, Proposition V.7.13] we have

$$\int_{-n}^{n} \left(\int_G f(x) \Phi^\lambda(x) \, | dx \right) d\mu_c(\lambda)$$

$$= \int_{-n}^{n} \left(\int_{A^+} |D(a_t)| \cdot |\Phi^\lambda(a_t)| \int_{G/A} |f(g a_t g^{-1})| \, dg_A da_t \right) d\mu_c(\lambda).$$

Here dg_A is an appropriately normalized G-invariant measure on G/A. However,

$$|D(a_t)| \cdot |\Phi^\lambda(a_t)| = 2 |\cos(\lambda t/2)| \cdot |D(a_t)|^{1/2}.$$

Let $\Lambda = \Lambda_{|f|}$ be as defined in Proposition 9.1. Then

$$\int_{-n}^{n} \left(\int_G |f(x) \phi^\lambda(x)| \, dx \right) d\mu_c(\lambda) \leq 2 \int_{-n}^{n} \left(\int_{A^+} \Lambda(a_t) da_t \right) d\mu_c(\lambda).$$

From Proposition 9.1 we know the last iterated integral is finite. Thus Fubini's Theorem applies to 9.5 when $f \in \mathscr{D}(G)$.

For each $n > 0$ define $S_n : G \to \mathbf{C}$ by

$$S_n(x) = \int_{-n}^{n} \Phi^\lambda(x) d\mu_c(\lambda).$$

From above we see that S_n is a locally summable invariant function which equals the distribution $T_{c,n}$ on $\mathscr{D}(G)$. Moreover, $S_n(u_\theta) = 0$ for all θ, and

$$|S_n(a_t)| \leq 2 |D(a_t)|^{-1/2} \mu_c([-n, n]) \quad \text{for } t > 0.$$

Hence Proposition 9.2 shows each S_n gives a tempered distribution. This proves that

$$T_{c,n}[f] = \int_G f(x) \left(\int_{-n}^{n} \Phi^\lambda(x) d\mu_c(\lambda) \right) dx \quad \text{for all } f \in \mathscr{C}(G). \tag{9.6}$$

From 9.5 it is easy to see by dominated convergence that T_c is the tempered distributional limit of the $T_{c,n}$; thus 9.6 shows

$$T_c = \lim_{n\to\infty} \int_{-n}^{n} \Phi^\lambda d\mu_c(\lambda) \tag{9.7}$$

as tempered distributions.

For T_d the procedure is similar. For each $n > 0$ define $T_{d,n} : \mathscr{C}(G) \to \mathbf{C}$ by

$$T_{d,n}[f] = \sum_{1\le|\ell|\le n} \left(\int_G f(x)\Theta^\ell(x)dx\right) \mu_d(\ell).$$

Since the sum is finite there is no problem in bringing it inside of the integral. We will then obtain

$$T_d = \lim_{n\to\infty} \sum_{1\le|\ell|\le n} \Theta^\ell \mu_d(\ell) \tag{9.8}$$

as tempered distributions.

Equations 9.7 and 9.8, when combined with Theorem 8.2, prove our theorem. □

References

1. J. G. Arthur, *Harmonic analysis of tempered distributions on semi-simple Lie groups of real rank one*, Ph.D. Dissertation, Yale University, 1970.
2. ———, *Harmonic analysis of the Schwartz space on a reductive Lie group I, preprint.*
3. ———, *Harmonic analysis of the Schwartz space on a reductive Lie group II, preprint.*
4. W. H. Barker, *The spherical Bochner theorem on semi-simple Lie groups*, J. Functional Anal., vol. 20 (1975), pp. 179–207.
5. L. Ehrenpreis and F. I. Mautner, *Some properties of the Fourier transform on semi-simple Lie groups III*, Trans. Amer. Math. Soc., vol. 90 (1959), pp. 431–484.
6. Harish-Chandra, *Harmonic analysis on real reductive groups I: the theory of the constant term*, J. Functional Anal., vol. 19 (1975), pp. 104–204.
7. S. Helgason, *Analysis on Lie groups and Homogeneous Spaces*, Regional Conference Series in Math., No. 14, Amer. Math. Soc., 1972.
8. Y. Muta, *Positive definite spherical distributions on a semi-simple Lie group*, Mem. Fac. Sci. Kyushu Univ., vol. 26 (1972), pp. 263–273.
9. M. Nichanian, *Transformées de Fourier des distributions de type positif sur SL*(2, **R**), C. R. Acad. Sci. Paris., vol. 278 (1974), pp. 17–19.
10. M. Sugiura, *Unitary Representations and Harmonic Analysis*, Kodansha, Tokyo, 1975.
11. P. C. Trombi, *Harmonic analysis of* $C^p(G{:}F)$ $(1 \le p < 2)$, J. Functional Anal., vol. 40 (1981), pp. 84–125.
12. ———, *Invariant harmonic analysis on split rank one groups with applications*, Pacific J. Math., vol. 101 (1982), pp. 223–245.
13. G. Warner, *Harmonic Analysis on Semi-simple Lie Groups II*, Springer-Verlag, New York, 1972.

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