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EQUIVARIANT BUNDLES OVER A SINGLE ORBIT TYPE

BY

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In this paper we analyze equivariant bundles over a space with a single orbit type. In particular, we reduce the classification of such bundles to a non-equivariant homotopy lifting problem (Corollary 1.12). We have used these ideas to analyze equivariant bundles with abelian structure group [5] and equivariant bundles over semi-free spaces [2]. In a future paper we will analyze bundles over general spaces by reassembling the results given here and replace the equivariant obstruction theory of [3] by another type of lifting problem. In the case that the structure group of the bundle is also a compact Lie group our results are closely related to those of Conner and Floyd [1].

Let $p : E \to X$ be a principal G-A bundle, G compact Lie group, X completely regular. We also assume p is G locally trivial. (For the definition of G locally trivial and the general theory of equivariant bundles we refer the reader to [2].) Let H be a closed subgroup of G and let $x \in X^H$. If $z \in p^{-1}(x)$, then $hz = z\varrho(h)$ for some homomorphism $\varrho : H \to A$ and all $h \in H$. For any other point $z' = za$ over x, $hz' = z'\varrho'(h)$ where $\varrho'(h) = a^{-1}\varrho(h)a$. Thus the A equivalence class of ϱ is well determined by x. We will say that x or more properly the fibre over x belongs to (ϱ). Let

$$X^{(\rho)} = \{x \in X^H \mid x \text{ belongs to } (\varrho)\}.$$

Let R_H be the set of A equivalence classes of homomorphisms of H to A.

LEMMA 1.1. *$X^{(\rho)}$ is open in X^H and $X^H = \coprod X^{(\rho)}$, $(\varrho) \in R_H$.*

Proof. If $x \in X$, then by G local triviality there is a G_x invariant neighborhood U of x and a homomorphism $\lambda: G_x \to A$ such that $p^{-1}(U)$ is G_x equivalent to $U \times A$, G_x acting on A via λ. If $x \in X^{(\rho)}$ then $H \subset G_x$ and $(\lambda | H) = (\varrho)$. It follows that if $x' \in X^H \cap U$, x' belongs to (ϱ). Thus $X^{(\rho)}$ is open in X^H. The second statement follows from this and the above discussion.

Let $E^\rho = \{z \in E \mid hz = z\varrho(h),\ h \in H\}$.

LEMMA 1.2. *E^ρ is an A^ρ bundle over $X^{(\rho)}$, where*

$$A^\rho = \{a \in A \mid a\varrho(h) = \varrho(h)a,\ h \in H\}.$$

Further, $p^{-1}(X^{(\rho)}) \cong E^{(\rho)} X_{A^\rho} A$, as an A-bundle.

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Proof. If $z \in E^\rho$, then $p(z) \in X^{(\rho)}$. Further, $za \in E^\rho$ if and only if $a \in A^\rho$. If $x \in X^{(\rho)}$, then by G local triviality, x has a G_x invariant neighborhood U such that $p^{-1}(U)$ is G_x-A equivalent to $U \times A$, where G_x acts on A through a homomorphism $\lambda: G_x \to A$ such that $(\lambda | H) = (\varrho)$. In fact λ is unique only up to its A equivalence class and we can choose λ so that $\lambda | H = \varrho$. Then

$$p^{-1}(U) \cap E^\rho = U^H \times A^\rho.$$

Thus E^ρ is a locally trivial A^ρ bundle over $X^{(\rho)}$ and

$$p^{-1}(X^{(\rho)}) = E^\rho \times_{A^\rho} A$$

as an A bundle.

LEMMA 1.3. *Let $\phi : E \to E'$ be a G-A bundle map of the principal G-A bundle $p : E \to X$ into the principal bundle $p' : E' \to X'$ over the G map $f : X \to X'$. Then $\phi^{-1}(E'^\rho) \cap p^{-1}(X^H) = E^\rho$.*

Proof. Clearly, $\phi(E^\rho) \subset E'^\rho$. Now E *may be identified with*

$$f^*E' = \{(x,z') \in X \times E' \,|\, f(x) = p'(z')\}$$

and ϕ corresponds to the projection $(x,z') \to z'$. But if $x \in X^H$ and $z' \in E'^\rho$, then $h(x,z') = (x,z')\varrho(h)$ and $(x,z') \in E^\rho$. So

$$\phi^{-1}(E'^\rho) \cap p^{-1}(X^H) = E^\rho.$$

Let

$$\Lambda^\rho = \{(n,a) \in N(H) \times A \,|\, \varrho(nhn^{-1}) = a\varrho(h)a^{-1}, \text{ all } h \in H\}.$$

Then Λ^ρ is a closed subgroup of $N(H) \times A$.

LEMMA 1.4. *Let $p : E \to X$ and E^ρ be as above. Then*

$$\Lambda^\rho = \{(n,a) \in N(H) \times A \,|\, nE^\rho a^{-1} \subset E^\rho\}.$$

Proof. If $z \in E^\rho$ and $nza^{-1} \in E^\rho$, then $hnza^{-1} = nza^{-1}\varrho(h)$.

But $hnza^{-1} = n(n^{-1}hn)za^{-1} = nz\varrho(n^{-1}hn)a^{-1}$.

Hence $\varrho(n^{-1}hn) = a^{-1}\varrho(h)a$ and $(n,a) \in \Lambda^\rho$.
Conversely, if $(n,a) \in \Lambda^\rho$ and $z \in E^\rho$, then

$$hnza^{-1} = n(n^{-1}hn)za^{-1} = nz\varrho(n^{-1}hn)a^{-1} = nza^{-1}\varrho(h).$$

Hence $nza^{-1} \in E^\rho$.

Let $N_\rho(H) = \{n \in N(H) \,|\, (n,a) \in \Lambda^\rho \text{ for some } a \in A\}$.

LEMMA 1.5. $N_\rho(H) = \{n \in N(H) \,|\, nX^{(\rho)} \subset X^{(\rho)}\}$.

Proof. If $nX^{(\rho)} \cap X^{(\rho)}$, then for $z \in E^{\rho}$, $nz = z'a$, $z' \in E^{\rho}$, $a \in A$. Thus

$$nza^{-1} \in E^{\rho}$$

and as in the first part of the proof of (1.4), $(n,a) \in \Lambda^{\rho}$. Hence $n \in N_{\rho}(H)$.

Conversely, if $n \in N_{\rho}(H)$, $nE^{\rho}a^{-1} \subset E^{\rho}$ for some $a \in A$ by (1.4); and by (1.2), $nX^{(\rho)} \subset X^{(\rho)}$.

Let H^{ρ} be the image of H under the embedding

$$(i,\varrho) : H \to N(H) \times A, \qquad (i,\varrho)(h) = (h, \varrho(h)).$$

Then H^{ρ} is contained in Λ^{ρ} and is a closed normal subgroup of Λ^{ρ}. Let $\Gamma\varrho = \Lambda^{\rho}/H^{\rho}$. We can identify A^{ρ} with $1 \times A^{\rho} \subset \Lambda^{\rho}$. Since $H^{\rho} \subset A^{\rho} = 1$, we can further identify A^{ρ} with the image of $1 \times A^{\rho}$ in Γ^{ρ}. (Since H^{ρ} is a compact Lie group, $\Lambda\varrho \to \Gamma\varrho$ is a locally trivial bundle and $1 \times A\varrho$ maps homeomorphically onto its image in Γ^{ρ}.) A^{ρ} is a normal subgroup of Γ^{ρ}, since $(n,a) \subset \Lambda^{\rho}$ requires that $a \in N(\varrho(H))$ and the centralizer A^{ρ} is normal in the normalizer $N(\varrho(H))$ of $\varrho(H)$.

Now consider the $N_{\rho}(H)$ trivial $N_{\rho}(H)$-A bundle $E = N_{\rho}(H) \times {}_{H}A$ over the orbit $\overline{N}_{\rho}(\mathrm{H}) = N_{\rho}(H)/H$, where H acts on A via ϱ. Then Γ^{ρ} may be identified with E^{ρ} under the map $[n,a] \to [n,a^{-1}]$, which extends to the $N_{\rho}(H)$ equivalence

$$(N_{\rho}(H) \times A)/H^{\rho} \to N_{\rho}(H) \times {}_{H}A.$$

Further the homeomorphism $\Gamma^{\rho}/A^{\rho} \cong E^{\rho}/A^{\rho} \cong E/A \cong N_{\rho}(H)/H$ is induced by the homomorphism $[n,a] \to [n]$ of Γ^{ρ} onto $N_{\rho}(H)/\mathrm{H}$ by passage to the quotient. Thus Γ^{ρ}/A^{ρ} is isomorphic to $N_{\rho}(H)/H$ as a topological group.

Note that E^{ρ} above can be considered a principal Γ^{ρ} bundle over a point (under the right action $z \to n^{-1}za$). This generalizes:

Proposition 1.6. *Let $p:E \to X$ be a principal G-A bundle. The action $z(n,a) = n^{-1}za$ of Λ^{ρ} on E^{ρ} induces a right action of Γ^{ρ} on E^{ρ}, extending the A^{ρ} action. If X has a single orbit type (H), E^{ρ} is a principal Γ^{ρ} bundle over $\overline{X}^{\rho} = X^{(\rho)}/\overline{N}_{\rho}(H)$,*

Proof. By the definition of E^{ρ}, the above action of Λ^{ρ} restricted to H^{ρ} is trivial and induces a Γ^{ρ} action. Now $p^{-1}(X^{(\rho)})$ is an $N_{\rho}(H)$ locally trivial bundle. To show E is a locally trivial Γ^{ρ} bundle when X has a single orbit type (H) it is sufficient to consider for $x \in X^{(\rho)}$ a slice V in $X^{(\rho)}$ such that

$$p^{-1}(N_{\rho}(H)V) \cong N_{\rho}(H) \times {}_{H}(V \times A),$$

H acting on A via ϱ. Then by (1.3),

$$p^{-1}(N_{\rho}(H)V)^{\rho} = \pi^{-1}(N_{\rho}(H) \times {}_{H}A)^{\rho},$$

where $\pi : N_{\rho}(H) \times {}_{H}(V \times A) \to N_{\rho}(H) \times {}_{H}A$ is the projection. Since π induces a Γ^{ρ} map of $p^{-1}(N_{\rho}(H)V)^{\rho}$ onto $(N_{\rho}(H) \times {}_{H}A)^{\rho} \cong \Gamma^{\rho}$, $p^{-1}(N_{\rho}(H)V)^{\rho}$ is a trivial Γ^{ρ} bundle over V; and E^{ρ} is a locally trivial Γ^{ρ} bundle over $\overline{\mathrm{X}}^{\rho}$. (Note that

$$E^\rho/\Gamma^\rho = (E^\rho/A^\rho)/(\Gamma^\rho/A^\rho) = X^\rho/\overline{N}_\rho(H),$$

$\overline{N}_\rho(H)$ acting on the right of X^ρ by $x\overline{n} = \overline{n}^{-1}x$ and hence $E^\rho/\Gamma^\rho = \overline{X^\rho}$.)

Let $R_{(H)}$ denote the family of G-A equivalence classes of homomorphisms $\varrho : H \to A$; i.e., $\varrho : H \to A$ is equivalent to $\varrho' : H' \to A$, $H' = gHg^{-1}$, if

$$\varrho'(ghg^{-1}) = a\varrho(h)a^{-1} \quad \text{for some } a \in A \text{ and all } h \in H.$$

Note that this is the same as the $N(H)$-A equivalence classes. From (1.5) we have:

LEMMA 1.7. *Let $p : E \to X$ be a G-A bundle. Then*

$$X^H = \coprod_{(\varrho)\in R_{(H)}} N(H) \times_{N_\rho(H)} X^{(\rho)} = \coprod_{(\varrho)\in R_{(H)}} \overline{N}(H) \times_{N_\rho(H)} X^{(\rho)}.$$

If X has a single orbit type (H), then

(a) $X = \coprod G/H \times_{N_\rho(H)} X^{(\rho)}, (\varrho) \in R_{(H)}$,

(b) $\overline{X} = X/G = X^H/\overline{N}(H) = \quad X^{(\rho)}/\overline{N}_\rho(H),\ (\varrho) \in R_{(H)}$.

From (1.2) and (1.6) we have:

LEMMA 1.8. *Let $p:E \to X$ be a G-A bundle. Then*

$$p^{-1}(X^H) = \coprod N(H) \times_{N_\rho(H)} E^\rho \times_{A_\rho} A, \quad (\varrho) \in R_{(H)}$$

If X has a single orbit type (H), *then*

$$E = \coprod_{(\varrho)\in R_{(H)}} G \times_{N_\rho(H)} E^\rho \times_{A_\rho} A.$$

The Γ^ρ structure of E^ρ determines the $N_\rho(H)$-A structure of $E^\rho \times_{A_\rho} A$ by the formula $n[z,a] = [z[n^{-1},a_o^{-1}],a_oa]$, $z \in E^\rho$, $a \in A$, $a_o \in A$ such that $[n,a_o] \in \Gamma^\rho$, and hence determines the G-A structure of E.

DEFINITION. Let X be a G-space with a single orbit type (H), and assume $G = N_\rho(H)$ for some homomorphism $\varrho : H \to A$. Let $\overline{G} = G/H$ and $\overline{X} = X/G = X/\overline{G}$. A principal Γ^ρ bundle $p : E \to \overline{X}$ *extends* X if there is a $\overline{G}$ equivalence $\psi : E/A^\rho \to X$ over $\overline{X}$ (after switching the right $\overline{G} = \Gamma^\rho/A^\rho$ action on E/A^ρ to a left $\overline{G}$ action). Two Γ^ρ extensions of X, (E,p,ψ) and (E',p',ψ') are equivalent if there exists a Γ^ρ bundle equivalence $\phi : E' \to E$ such that $\psi(\phi/A^\rho)$ is $\overline{G}$ isotopic to ψ' over $\overline{X}$, where $\phi/A^\rho : E'/A^\rho \to E/A^\rho$ is the $\overline{G}$ equivalence induced by ϕ.

Now let X be a paracompact G-space with the single orbit type (H) and $G = N_\rho(H)$:

THEOREM 1.9. *With X as above, the following are in bijective correspondence:*

(a) *Equivalence classes of G-A bundles over X with all fibres in* (ϱ).
(b) *Equivalence classes of* Γ^ρ *bundles over* $\overline{X}$ *extending X.*
(c) *Homotopy classes of lifts to* $B\Gamma^\rho$ *of a fixed classifying map*

$$\overline{f} : \overline{X} \to B\overline{G}$$

for the $\overline{G}$ *bundle X.* ($B\Gamma^\rho$ *is considered here as a bundle*

$$\overline{\partial} : B\Gamma^\rho \to B\overline{G} = B(\Gamma^\rho / A_\varrho)$$

with fibre BA^ρ.)
(d) *Equivariant homotopy classes of* $\overline{G}$ *maps of X into* B^ρ, *where*

$$B^\rho = E\Gamma^\rho / A^\rho$$

as a $\overline{G} = \Gamma^\rho / A^\rho$ *space.*

Proof. We construct surjective maps (a)→(b)→(c)→(d)→(a) and show the composition is the identity.

(a)→(b). If $p : E \to X$ is a G-A bundle with all $p^{-1}(x)$ in (ϱ), then $X = X^{(\rho)}$ and E^ρ is a Γ^ρ bundle over $\overline{X}$ by (1.6). Let $\psi : E_\rho / A_\rho \to E/A = X$ be the map induced by inclusion; then ψ is a $\overline{G}$ equivalence by (1.2) and (1.6). Thus

$$(E_\varrho, q(p \,|\, E_\varrho), \psi), \quad q : X \to \overline{X} \text{ the quotient map,}$$

is a Γ^ρ extension of X defined by p.

If $p' : E' \to X$ is G-A equivalent to p, say $\phi : E' \to E$, then $\phi | E'^\rho : E'^\rho \to E^\rho$ is a Γ^ρ bundle equivalence over $\overline{X}$ and $(\phi | E'^\rho)/A^\rho : E'^\rho / A^\rho \to E^\rho / A^\rho$ is a $\overline{G}$ equivalence such that $\psi(\phi | E'^\rho)/A^\rho = \psi'$. Thus $p \to (E^\rho, q(p | E^\rho), \psi)$ is well defined on equivalence classes.

Now if (E_o, p_o, ψ_o) is a Γ^ρ extension of X, $E = E_o \times_{A^\rho} A$ is a G-A bundle over X, defining the G action by the formula in (1.8) and identifying $E/A = E_o / A^\rho$ with X via ψ_o. But then $E_o = E^\rho$ and $p_o = q(p | E_o)$ and by definition $E_o / A^\rho \to E/A = X$ is ψ_o. Thus the map is surjective.

(b)→(c). Pick a fixed $\overline{G}$ map $\theta : X \to E\overline{G}$ covering $\overline{f}$. If $f : \overline{X} \to B\Gamma^\rho$ covers $\overline{f}$, there is a uniquely defined $\overline{G}$ map $\theta_f : X \to E\Gamma^\rho / A^\rho$ over f such that $\partial\theta_f = \theta$, where $\partial : E\Gamma^\rho / A^\rho \to E\overline{G}$ is a fixed $\overline{G}$ map over $\overline{\partial}$. Note that if $\theta' : X \to E\overline{G}$ is any $\overline{G}$ map, then θ' is $\overline{G}$ homotopic to θ by the universality of $E\overline{G}$, and this homotopy covers a homotopy $\overline{f_t}$, $\overline{f_1} = \overline{f}$. If θ_t is *any* $\overline{G}$ homotopy of θ' over $\overline{f_t}$, then θ is $\overline{G}$ homotopic over $\overline{f_1}$ to θ_1 and $\theta = \theta_1 \lambda$, $\lambda : X \to X$ a $\overline{G}$ equivalence $\overline{G}$ isotopic to the identity over $\overline{X}$.

Now given a Γ^ρ extension (E, p, ψ) of X, let $f_o : \overline{X} \to B\Gamma^\rho$ be a classifying map for E. Since $\overline{\partial} f_o$ is covered by the $\overline{G}$ map

$$\theta' = \partial(\hat{f}_o / A^\rho)\psi^{-1},$$

where $\hat{f}_o : E \to E\Gamma^\rho$ is a Γ^ρ bundle map over f_o, the $\overline{G}$ homotopy of θ' to θ covers a homotopy $\overline{f_t}$ of $\overline{\partial} f_o$ to $\overline{f}$. Let $f_t : \overline{X} \to B\Gamma^\rho$ be a homotopy of f_o covering $\overline{f_t}$ and $\hat{f}_t$ a Γ^ρ homotopy of $\hat{f}_o$ over f_t. Then

$$\partial(\hat{f}_t/A^\rho)\psi^{-1} : X \to E\overline{G}$$

covers $\overline{f_t}$ and $\theta = \partial(f_1/A\varrho)\psi^{-1}\lambda$, $\lambda : X \to X$ as above. Setting $f = f_1$, we have

$$(*) \qquad \hat{f}/A^\rho\psi^{-1}\lambda = \theta_f.$$

Thus, we can assign to (E,p,ψ) a classifying map $f : \overline{X} \to B\Gamma^\rho$ covering $\overline{f}$ and satisfying (*) for some Γ^ρ bundle map $\hat{f} : E \to \to E\Gamma^\rho$ covering f.

Let (E',p',ψ') be equivalent to (E,p,ψ) and suppose we have chosen f' covering $\overline{f}$ and $\hat{f}'$ covering f' such that $\hat{f}'/A^\rho\psi'^{-1}\lambda' = \theta_{f'}$. Let $\phi : E' \to E$ be the Γ^ρ equivalence such that $\psi(\phi/A^\rho)\psi'^{-1}$ is $\overline{G}$ isotopic to the identity over $\overline{X}$. Now f' is Γ^ρ homotopic to $\hat{f}\phi$ by say $\hat{f}_t'$, and $\theta_{f'}$ is $\overline{G}$ isotopic to

$$(\hat{f}/A^\rho)(\phi/A^\rho)\psi'^{-1}\lambda' = \hat{f}/A^\rho\psi^{-1}\psi(\phi/A^\rho)\psi'^{-1}\lambda'$$

which is $\overline{G}$ isotopic to $\hat{f}/A^\rho\psi^{-1}\lambda'$ and hence to $\hat{f}/A^\rho\psi^{-1}\lambda = \theta_f$. The isotopy of $\theta_{f'}$ to θ_f covers a $\overline{G}$ isotopy θ_t of $\theta' = \partial\theta_{f'}$ to $\theta = \partial\theta_f$ and an isotopy f_t of f' to f, which in turn covers an isotopy $\overline{f_t}$ of $\overline{f}$ to itself. Since $E\overline{G}$ is universal, θ_t is $\overline{G}$ homotopic to the constant map rel endpoints, and $\overline{f_t}$ must be homotopic to the constant map rel endpoints. But then f_t is homotopic rel endpoints to a homotopy of f' to f over $\overline{f}$.

Thus the assignment of a classifying map f covering $\overline{f}$ and satisfying (*) to (E,p,ψ) gives a well defined homotopy class of lifts of $\overline{f}$ to each equivalence class of Γ^ρ extensions of X.

Now if $f : \overline{X} \to B\Gamma^\rho$ is any lift of $\overline{f}$, $f^*(E\Gamma^\rho)$ is a Γ^ρ bundle over $\overline{X}$, and since

$$(f^*E\Gamma^\rho)/A^\rho \xrightarrow{\rho f/A^\rho} E\Gamma^\rho/A^\rho \to E\overline{G}$$

covers $\partial f = \overline{f}$ (where $\hat{f} : f^*E\Gamma^\rho \to E^\rho$ is the cononical projection) there is a well defined $\overline{G}$ equivalence

$$\psi : f^*E\Gamma^\rho/A^\rho \to X$$

over $\overline{X}$ such that $\partial(\hat{f}/A^\rho)\psi^{-1} = \theta$. But this means that we may assign the lift f to $(f^*E\Gamma^\rho, p, \psi)$, and this shows (b)$\to$(c) is surjective.

(c)$\to$(d). By the remarks at the beginning of the previous step of the proof, the assignment $f \to \theta_f$ sends a homotopy class of lifts of $\overline{f}$ to an equivariant homotopy class of maps of X to $E\Gamma^\rho/A^\rho$.

Now given any equivariant map $\psi : X \to E\Gamma^\rho/A^\rho$, $\partial\psi : X \to E\overline{G}$ is $\overline{G}$ isotopic to θ. Thus if ψ covers $\overline{\psi} : \overline{X} \to B\Gamma^\rho$, $\overline{\partial\psi}$ is homotopic to $\overline{f}$ by a homotopy $\overline{f_t}$. If $\overline{\psi_t}$ covers $\overline{f_t}$ and ψ_t is a $\overline{G}$ homotopy of ψ covering $\overline{\psi_t}$, then $\partial\psi_1$ is $\overline{G}$ isotopic to θ over $\overline{f}$ and $\theta = \partial\psi_1\lambda$, $\lambda : X \to X$ a $\overline{G}$ equivalence $\overline{G}$ isotopic to the identity. Let $f = \overline{\psi_1}$. Then $\theta_f = \psi_1\lambda$ which is $\overline{G}$ isotopic to ψ. Thus (c)$\to$(d) is surjective.

(d)$\to$(a). $E\Gamma^\rho \times_{A^\rho} A$ is a G-A bundle over $(E\Gamma^\rho)/A^\rho$, using the formula of (1.8). (Note that the Γ^ρ local triviality of $E\Gamma^\rho$ implies the G-A local triviality of $E\Gamma^\rho \times_{A^\rho} A$, since

$$(U \times \Gamma^\rho) \times_{A^\rho} A = U \times (G \times_H A) = G \times_H (U \times A),$$

U any trivializing open set in $B\Gamma^\rho$, by the arguement preceding (1.6).) Hence equivariant homotopy classes of G maps of X into $(E\Gamma^\rho)/A^\rho$ pull back G-A equivalence classes of G-A bundles over X. But since H acts trivially on X, G maps are the same as $\overline{G}$ maps.

Finally to prove (d)→(a) is surjective and that all the maps are bijective, it is sufficient to prove that (a)→(b)→(c)→(d)→(a) is the identity: Let $f : \overline{X} \to B\Gamma^\rho$ cover $\overline{f}$. There is a uniquely defined $\psi : (f^*E\Gamma^\rho)/A^\rho \to X$ such that

$$\theta\psi = \partial\pi/A^\rho \text{ or } \pi/A^{\rho\pi^{-1}} = \theta_f$$

where $\pi : f^*E\Gamma^\rho \to E\Gamma^\rho$ is the projection. Then

$$[\pi,1] : f^*E\Gamma^\rho \times_{A^\rho} A \to E\Gamma^\rho \times_{A^\rho} A \quad \text{and} \quad \psi q : f^*E\Gamma^\rho \times_{A^\rho} A \to x$$

satisfy $\theta_f\psi q = \pi/A^\rho q = q[\pi,1]$ and hence define a G-A equivalence $f^*E\Gamma^\rho \times_{A^\rho} A \to \theta_{f_*}(E\Gamma_\varrho \times_{A^\rho} A)$ with G action given by (1.8), q the quotient under A.

Now let $p:E \to X$ be a G-A bundle of type (ϱ). Then E is G-A equivalent to $E^\rho \times_{A^\rho} A$ with *G action defined by (1.8). As in (b)*→(c), let

$$f : \overline{X} \to B\Gamma^\rho$$

cover $\overline{f}$ and be covered by a Γ^ρ bundle map $\hat{f} : E^\rho \to E\Gamma^\rho$ such that $\hat{f}/A^\rho\lambda = \phi_f$, where $E^\rho/A^\rho = E/A$ is identified with X via p. E^ρ is equivalent to $f^*E\Gamma^\rho$ by an equivalence $\phi : E^\rho \to f^*E\Gamma^\rho$ such that $\hat{f} = \pi\psi$. Then

$$\pi/A^\rho\phi/A^\rho\lambda = \hat{f}/A^\rho\lambda = \theta_f.$$

Hence with $\psi = \lambda^{-1}(\phi/A^\rho)^{-1}$ we see from the paragraph above that

$$\theta q : f^*E\Gamma^\rho \times_{A^\rho} A \to X$$

is equivalent to

$$\theta_{f_*}(E\Gamma^\rho \times_{A^\rho} A) \to X.$$

But (E,p) is G-A equivalent to $(E^\rho \times_{A^\rho} A, q)$ and $[\phi,1]$ is a G-A bundle map of this last to $(f^*E\Gamma^\rho \times_{A^\rho} A, q)$ over λ^{-1}. Since λ is $\overline{G}$ homotopic to the identity, it follows from the equivariant covering homotopy property that E is G-A equivalent to $f^*E\Gamma^\rho \times_{A^\rho} A$ and hence to $\theta_{f_*}(E\Gamma^\rho \times_{A^\rho} A)$. So the cycle of maps is the identity, proving the theorem.

COROLLARY 1.10. *For any closed subgroup $H \cap G$, the universal G-A bundle for spaces of orbit type (H) is*

$$E(H) = \coprod_{(\varrho)\in R_{(H)}} G \times_{N_\rho(H)} E\Gamma^\rho \times_{A^\rho} A,$$

$$B(H) = \coprod_{(\varrho)\in R_{(H)}} G \times_{N_\rho(H)} B^\rho = \coprod_{(\varrho)\in R_{(H)}} \overline{G} \times_{N_\rho(H)} B^\rho,$$

with projection induced by the quotient map $E\Gamma^\rho \to B^\rho = E\Gamma^\rho/A^\rho$. If X is a

G-space of orbit type (H) only, equivalence classes of G-A bundles over X are in bijective correspondence with $[X,B(H)]_G$.

COROLLARY 1.11. *Let $\theta : B(H) \to BA$ classify $E(H)$ as an A bundle. If X is a G-space of orbit type* (H) *only, an A bundle over X admits a G-A structure if and only if its classifying map $f : X \to BA$ factors up to homotopy through an equivariant map $\phi : X \to B(H)$; i.e., $f \sim \theta\phi$.*

The inclusion of $\overline{N}_\rho(H)$ in $\overline{N}(H)$ allows us to consider $B\overline{N}_\rho(H)$ and hence $B\Gamma^\rho$ as bundles over $B\overline{N}(H)$. $B\Gamma^\rho$ has fibre $BA^\rho \times_{N_\rho(H)} \overline{N}(H)$. (Since $\overline{N}(H)/\overline{N}_\rho(H)$ is finite, this is just a finite number of copies of BA^ρ when we forget the action.)

COROLLARY 1.12. *Let X be a G space of orbit type* (H) *only, and let*

$$\overline{f} : \overline{X} \to B\overline{N}(H)$$

be a classifying map for X as an $\overline{N}(H)$ bundle over $\overline{X}$. The equivalence classes of G-A bundles over X are in bijective correspondence with homotopy classes of lifts of $\overline{f}$ to $\coprod_{(\varrho) \in R_{(H)}} B\Gamma^\rho$.

Examples. 1. X a free G-space, $H = (1)$, ϱ trivial. Then $\Lambda^\rho = G \times A$, H^ρ trivial, $\Gamma^\rho = \Lambda^\rho = G \times A$, $N_\rho(H) = G$, $A^\rho = A$. Thus

$$B^\rho = E(G \times A)/A = EG \times (EA/A) = EG \times BA$$

and G-A bundles over a free G-space X are classified by equivariant homotopy classes of maps of $X \to EG \times BA$. But $[\overline{X}, EG \times BA]_G = [\overline{X}, BA]$. So G-A bundles over X are in bijective correspondence with A bundles over $\overline{X} = X/G$ (as is well known).

2. X a trivial G-space, $H = G$, $\varrho : G \to A$ any homomorphism. Then Λ^ρ is isomorphic to $G \times A^\rho$ by $\phi : G \times A^\rho \to \Lambda^\rho$, $\phi(g,a) = (g, \varrho(g)a)$ and $\Gamma^\rho = A^\rho$. Of course, $N^\rho(H) = G$ and $\overline{G} = \Gamma^\rho/A^\rho$ is trivial. Thus $B\varrho = EA^\rho/A^\rho = BA^\rho$ with trivial action. So G-A bundles over a connected trivial G-space X are classified by homotopy classes of maps of X into BA^ρ, some $\varrho : G \to A$.

3. G abelian, X has orbit type (H) only. Suppose $\varrho : H \to A$ extends to a homomorphism $\hat{\varrho} : G \to A$ with $A^\rho = \hat{A}^\rho$. (This is always true if A is the unitary group $U(n)$.) Then $N(H) = G$, $\Lambda^\rho = G \times A^\rho$, $N_\rho(H) = G$. Further $\Gamma^\rho = G \times A^\rho/H^\rho$ is isomorphic to $G/H \times A^\rho$. In fact, let $\hat{\phi} : G \times A^\rho \to G \times A^\rho$ be the isomorphism

$$\phi(g,a) = (g, \varrho(g)^{-1}a).$$

Then $\phi(h, \varrho(h)) = (h, 1)$. So ϕ induces

$$\phi : \Gamma^\rho = G \times A^\rho/H^\rho \cong G/H \times A^\rho.$$

Thus

$$B^\rho = E(G/H) \times EA^\rho/A^\rho = E(G/H) \times BA^\rho$$

and

$$[X, E(G/H) \times BA^\rho]_G = [\overline{X}, BA^\rho].$$

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