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FINITELY ADDITIVE MEASURES FROM POSITIVE DEFINITE FUNCTIONS

BY

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1. Introduction

In 1953 it was conjectured by E. Hewitt (see [5, pp. 310–311] and the footnote on page 379 in [4]) that any (not necessarily continuous) positive definite function on **R** admits an integral representation by means of a nonnegative finitely additive set function on an algebra of Borel sets in **R**. This conjecture, which was proved in [2, p. 274], is generalized here in Theorem 3.2 to the case where **R** is replaced by an arbitrary commutative semigroup. We then apply Theorem 3.2 to prove theorem 4.2 on the existence of a finitely additive set function invariant under translations and surjective group homomorphisms. For the case where the group is $\mathbf{R}^n$, this is related to a result of Mycielski [8, p. 317] on the existence of a nonnegative finitely additive similarity invariant set function on the algebra of Lebesgue measurable sets in $\mathbf{R}^n$. In fact, our set function, though defined on a smaller yet important algebra of sets, is regular with respect to the zero and cozero sets and is invariant on a (larger) class of operators which includes the translations and the linear surjections on $\mathbf{R}^n$. Theorem 4.5 and Remark 4.7 provide means of evaluating this set function for the case where the group is $\mathbf{R}^n$ and **Z** respectively. Finally, we introduce the notion of the convolution of certain finitely additive set functions. In Theorem 5.7 we establish some properties of convolutions and in Theorem 5.8 we show the existence of (i) an algebra $\mathscr{A}$ of Borel sets in **R** or **Z**, which is sufficiently large to be of interest, and (ii) a finitely additive set function $\lambda: \mathscr{A} \to [0,1]$ such that the convolution, evaluated at any set $E \in \mathscr{A}$, of λ with certain finitely additive set functions $\mu: \mathscr{A} \to \mathbf{C}$ is equal to the product $\mu(\mathbf{R})\lambda(E)$ or $\mu(\mathbf{Z})\lambda(E)$.

Parts of Lemma 2.2 and Theorems 3.2, 4.2 and 4.6 were communicated without proof in [1, pp. 106–108].

2. Preliminaries

2.1. In this paper, X will denote a compact Hausdorff topological space and $\mathscr{B}(X)$ will designate the class of Borel sets in X. Given an algebra $\mathscr{A}$

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of subsets of a set S dense in X, we shall write $C(S, \mathscr{A})$ for the class of all complex-valued functions on S which can be uniformly approximated by $\mathscr{A}$-measurable step functions, and write I_E for the characteristic function of a set E in X. Recall that a function f lies in $C(S, \mathscr{A})$ only if $\int_S f\, d\lambda$ exists for every bounded finitely additive complex-valued set function λ on $\mathscr{A}$ (see [3, p. 293]), where the integral is defined by the usual Moore-Smith convergence method (see [9, pp. 183–191] and [12, pp. 401–404]) or equivalently [6] by the Dunford-Schwartz method [4, pp. 101–125]. Let $C(X)$ denote the space with supremum norm $\|\cdot\|_\infty$ consisting of the continuous complex-valued functions on X. A finitely additive set function $\lambda\colon \mathscr{A} \to \mathbf{C}$ will be said to be *regular* on $\mathscr{A}$ if, given $\varepsilon > 0$ and $E \in \mathscr{A}$, there exist constants a, $b > 0$ and real-valued functions f, $g \in C(X)$ such that the sets

$$K = \{x \in S\colon f(x) \leqslant a\} \quad \text{and} \quad V = \{x \in S\colon g(x) < b\}$$

have the properties (i) $K, V \in \mathscr{A}$, (ii) $K \subset E \subset V$ and (iii) $|\lambda|(V \setminus K) < \varepsilon$ where $|\lambda|$ is the total variation of λ. A linear functional L on $C(X)$ is said to be *nonnegative* if $L(f) \geqslant 0$ for all continuous functions $f\colon X \to [0, \infty)$. The norm $\|L\|$ of such a functional is equal to $L(1)$. Since a compact space is normal and so completely regular, the following useful analogue of the Riesz representation theorem in the context of finitely additive set functions can be easily deduced from [2, Corollary 2.9 and Remark 2.8, pp. 271].

2.2. LEMMA. *Let S be a dense subset of a compact Hausdorff space X and Λ be a bounded linear functional on $C(X)$. The class $\mathscr{A}$ of sets $E \cap S$ such that $E \in \mathscr{B}(X)$ and*

$$\inf_{f_1, f_2} \sup_{|g|=1} \{|\Lambda((f_2 - f_1)g)| : f_1 \leqslant I_E \leqslant f_2; f_1, f_2, g \in C(X)\} = 0$$

is an algebra of sets for which $f|_S \in C(S, \mathscr{A})$ for all $f \in C(X)$, and there exists a unique regular bounded finitely additive set function $\lambda : \mathscr{A} \to \mathbf{C}$ such that $\int_S f|_S d\lambda$ exists in the sense of Moore-Smith convergence and is equal to $\Lambda(f)$ for all $f \in C(X)$. Moreover, $\|\Lambda\| = |\lambda|(X)$. Also, when Λ is nonnegative, λ is nonnegative and $\mathscr{A}$ consists of those sets $E \cap S$ where $E \in \mathscr{B}(X)$ and

$$\inf\{\Lambda(f_2 - f_1) : f_1 \leqslant I_E \leqslant f_2; f_1, f_2 \in C(X)\} = 0. \qquad (2.2.1)$$

2.3. *Remark.* In the context of Lemma 2.2, it is not difficult to show that there exists a regular countably additive measure $\mu : \mathscr{B}(X) \to \mathbf{C}$ such that

$$\mathscr{A} = \{E \cap S : E \in \mathscr{B}(X), |\mu|(\overline{E} \setminus \mathring{E}) = 0\}$$

and $\lambda(E \cap S) = \mu(E)$ for all $E \cap S \in \mathscr{A}$: This fact, which is proved in [2, pp. 268–270] will be used later. For the familiar case where $X = [0, 1]$ and S is the rationals, we have the following:

(a) If μ is point mass at a rational r, then $\{r\} \notin \mathscr{A}$ and λ is again point mass. Moreover λ is countably additive.

(b) If μ is point mass at an irrational i, then $\mathscr{A}$ contains all closed subsets of S which do not contain i as a limit point. In this event λ is not countably additive.

(c) If μ is Lebesgue measure then $\mathscr{A}$ contains all singletons and λ is not countably additive.

(d) If μ has positive mass at each rational then $\mathscr{A}$ does not contain any singletons and λ is countably additive.

3. The moment problem

3.1. Given a semigroup G with commutative operation (denoted by $\cdot$) we consider a family $\pi = \{\pi_z : z \in G\}$ of continuous functions $\pi_z : X \to \mathbf{C}$ ($z \in G$) for which $\pi_{u\cdot v}(x) = \pi_u(x)\,\pi_v(x)$ for all $x \in X$ and all $u, v \in G$. We write $F_\pi(X)$ for the space, with supremum norm, consisting of all finite linear combinations of elements in π, and write $\overline{F_\pi(X)}$ for its supremum norm closure in $C(X)$. So given $f \in F_\pi(X)$, we shall write $\Sigma_{z\in G}\hat{f}(z)\pi_z(x)$ for $f(x)$, where the complex coefficients $\hat{f}(z)$ vanish for all but finitely many $z \in G$. We will say that a (not necessarily continuous) complex-valued function p on G is *positive definite with respect to* π if $\Sigma_{z\in G}\hat{f}(z)\,p(z) \geq 0$ for all nonnegative $f \in F_\pi(X)$. The analogue of Bochner's representation theorem, in the context of finitely additive set functions, is now given. It is on this result that much of this paper will depend.

3.2. THEOREM. *Suppose that* (i) *G is a commutative semigroup,* (ii) *S is a dense subset of a compact Hausdorff space X,* (iii) *$\pi = \{\pi_z : z \in G\}$ is a family of functions $\pi_z : X \to \mathbf{C}$ in $C(X)$ for which $\pi_{u\cdot v}(x) = \pi_u(x)\,\pi_v(x)$ for all $x \in X$ and all $u, v \in G$, and* (iv) *p is a complex-valued function on G which is positive definite with respect to π. Then there is an algebra $\mathscr{A}$ of sets in $\{E \cap S : E \in \mathscr{B}(X)\}$ for which* (1) *the functions $\{f|_S : f \in C(X)\}$ lie in $C(S, \mathscr{A})$, and* (2) *there exists a regular bounded finitely additive set function $\lambda : \mathscr{A} \to \mathbf{C}$ such that, in the sense of Moore-Smith convergence,*

$$p(z) = \int_S \pi_{z|S}\, d\lambda \quad \textit{for all } z \in G.$$

Moreover, if $\overline{F_\pi(X)} = C(X)$, then we can take for $\mathscr{A}$ the class of sets $E \cap S$ such that $E \in \mathscr{B}(X)$ and

$$\inf\left\{\left|\sum_{z\in G}(\hat{h}(z) - \hat{g}(z))\,p(z)\right| : g \leq I_E \leq h;\ g, h \in F_\pi(X)\right\} = 0; \qquad (3.2.1)$$

in this case λ is nonnegative and is uniquely determined on $\mathscr{A}$ by p.

Proof. On $F_\pi(X)$, the nonnegative linear functional L given by

$$L(f) = \sum_{z\in G} \hat{f}(z)\, p(z)$$

is well defined. For if $\Sigma_{z\in G} \tilde{f}_1(z)\, \pi_z = \Sigma_{z\in G} \tilde{f}_2(z)\, \pi_z$ then

$$\sum_{z\in G}[\hat{f}_1(z) - \hat{f}_2(z)]\, \pi(z) \geq 0$$

so that

$$\sum_{z\in G}(\hat{f}_1(z) - \hat{f}_2(z)) p(z) \geq 0,$$

since p is positive definite. Therefore $L(\Sigma_{z\in G}\hat{f}_1(z)\, \pi_z) \geq L(\Sigma_{z\in G}\hat{f}_2(z)\, \pi_z)$. Interchanging f_1 and f_2 shows L to be well defined. Now, for $f \in F_\pi(X)$ with $\|f\|_\infty = 1$, let θ_f be that complex number for which $|L(f)| = \theta_f\, L(f)$. Then,

$$|L(f)| = L(\theta_f f) = L(\mathrm{Re}(\theta_f f)) \leq L(1)$$

and so $|L(f)| \leq L(1)\|f\|_\infty$ for all $f \in F_\pi(X)$. Hence L is a bounded nonnegative linear functional on $F_\pi(X)$ and so, by the Hahn-Banach theorem, can be extended to a bounded linear functional Λ on $C(X)$ which is nonnegative and unique if $\overline{F_\pi(X)} = C(X)$. Taking for $\mathscr{A}$ the algebra given by Lemma 2.2, we have part (1). Also, there exists a unique regular bounded finitely additive set function λ on $\mathscr{A}$ such that $\int_S f|_S\, d\lambda$ exists in the sense of Moore-Smith convergence and is equal to $\Lambda(f)$ for all $f \in C(X)$. In particular, if $f = \pi_z$ for any given $z \in G$, we get part (2). Moreover, if $\overline{F_\pi(X)} = C(X)$, then the extension Λ is nonnegative and unique and so λ is nonnegative and uniquely determined by L. This completes the proof.

3.3. *Remarks.* (1) For $\mathscr{A}$ and λ as in Theorem 3.2, we have

$$\lambda(F) = \int_S f|_S\, d\lambda - \int_S (f|_S - I_F)\, d\lambda$$

for all $F \in \mathscr{A}$ and all $f \in F_\pi(X)$. Now, for the $\overline{F_\pi(X)} = C(X)$, (3.2.1) yields

$$\inf\left\{\int_S (h|_S - I_F)\, d\lambda : I_F \leq h|_S,\ h \in F_\pi(X)\right\}$$

$$\leq \inf\left\{\int_S (h|_S - g|_S)\, d\lambda : g|_S \leq I_F \leq h|_S;\ g, h \in F_\pi(X)\right\} = 0$$

and so

$$\lambda(F) = \inf\left\{\int_S h|_S\, d\lambda : I_F \leq h|_S;\ h \in F_\pi(X)\right\}. \tag{3.3.1}$$

Similarly,

$$\lambda(F) = \sup\left\{\int_S g|_S \, d\lambda : g|_S \leq I_F;\, g \in F_\pi(X)\right\}.$$

It now follows from the definition of $C(S, \mathscr{A})$ that if $\overline{F_\pi(X)} = C(X)$ then

$$\int_S f\, d\lambda = \inf\left\{\int_S h|_S \, d\lambda : f \leq h|_S;\, h \in F_\pi(X)\right\}$$
$$= \sup\left\{\int_S g|_S \, d\lambda : g|_S \leq f;\, g \in F_\pi(X)\right\} \tag{3.3.2}$$

for all $f \in C(S, \mathscr{A})$.

(2) For not necessarily continuous positive definite functions on the real line **R**, Hewitt tried to obtain a concrete representation theorem similar to that of Bochner in the continuous case, by means of finitely additive set functions on a subalgebra of Borel sets in **R** (see [5, pp. 310, 311] and the footnote on page 379 in [4]). The corrected version of Hewitt's result obtained in [2, p. 274] in the more general setting of an arbitrary locally compact abelian group G, can also be obtained from Theorem 3.2 by (a) taking S to be the dual group $G^\wedge$ of G, (b) taking X to be the Bohr compactification of S, (c) taking for π the characters $x \to \pi_z(x)$ on X ($z \in G$), and (d) showing, as we do in Example 3.4, that a function $p: G \to \mathbf{C}$ which is positive definite in the classical sense (i.e., $\Sigma_{i,j}\alpha_i\bar{\alpha}_j p(z_i \cdot z_j^{-1}) \geq 0$ for all finite sequences $(\alpha_i) \subset \mathbf{C}$ and $(z_i) \subset G$) is necessarily positive definite with respect to π.

3.4. *Example.* Let $\overline{m}$ designate the normalized Haar measure on the Bohr compactification X of the dual group S of a locally compact abelian group G with identity e. First let us show that if $p: G \to \mathbf{C}$ is positive definite in the classical sense then it is positive definite with respect to the class π of characters on X. Then we will establish the existence of a finitely additive set function on a subalgebra of the Borel sets in S, with properties that probably make it the most important such function.

If G_d denotes the group G with discrete topology, then (i) the function $p: G_d \to \mathbf{C}$ is a continuous positive definite function, and (ii) X is the set of all characters on G_d. Hence, by Bochner's theorem (see [10, p. 19–21] and [11, pp. 285, 286, 290]) there exists a nonnegative measure ν on the Borel subsets of X such that

$$p(z) = \int_X \pi_z(x)\, d\nu(x).$$

Thus, $\Sigma_{z\in G}\hat{f}(z)\,\pi_z \geq 0$ implies

$$\sum_{z\in G}\hat{f}(z)\,p(z) = \int_X \left(\sum_{z\in G}\hat{f}(z)\,\pi_z(x)\right) d\nu(x) \geq 0.$$

This shows that p is positive definite with respect to π.

It now follows from Theorem 3.2 that a function $p : G \to \mathbf{C}$ is positive definite in the classical sense if and only if there is an algebra $\mathscr{A}$ of Borel sets in S on which is defined a unique regular finitely additive set function $\lambda : \mathscr{A} \to [0, 1]$ for which $p(z) = \int_S \pi_z|_S \, d\lambda$ $(z \in G)$. In particular, to the discontinuous function

$$p(z) = \begin{cases} 0 & \text{if } z \neq e \\ 1 & \text{if } z = e \end{cases} \tag{3.4.1}$$

which is positive definite in the classical sense, can be associated (by (3.2.1)) an algebra $\mathscr{E}$ of subsets of S on which is defined a unique regular finitely additive set function $m : \mathscr{E} \to [0, 1]$ such that

$$\int_S \pi_z|_S(x) \, dm(x) = \begin{cases} 0 & \text{if } z \neq e \\ 1 & \text{if } z = e \end{cases}. \tag{3.4.2}$$

This algebra and set function each have a property that will be used in Section 4. First, $\mathscr{E}$ is translation invariant in the sense that if $F \in \mathscr{E}$, then $s \cdot F \in \mathscr{E}$ for all $s \in S$. Secondly, m is translation invariant in the sense that $m(F) = m(s \cdot F)$ for all $F \in \mathscr{E}$ and all $s \in S$. The first property follows directly from (3.2.1). The second property follows from the regularity of m and the fact that $f \in F_\pi(X)$ if and only if $f_s \in F_\pi(X)$ for all $s \in S$; where $f_s(x) = f(x \cdot s)$ for all $x \in X$ and $s \in S$. For, given $F \in \mathscr{E}$ and given $s \in S$, then $\hat{f}(e) = \hat{f}_s(e)$ and

$$\begin{aligned} m(F) &= \inf\left\{ \left| \sum_{z \in G} \hat{f}(z) \, p(z) \right| : I_F \leq f|_S; f \in F_\pi(X) \right\} \\ &= \inf\{\hat{f}(e) : I_F \leq f|_S; f \in F_\pi(X)\} \quad \text{(see (3.4.1))} \\ &= \inf\{\hat{f}_s(e) : I_{s \cdot F} \leq f_{s|S}; f_s \in F_\pi(X)\} \\ &= \inf\{\hat{g}(e) : I_{s \cdot F} \leq g|_S; g \in F_\pi(X)\} \\ &= \inf\left\{ \left| \sum_{z \in G} \hat{g}(z) \, p(z) \right| : I_{s \cdot F} \leq g|_S; g \in F_\pi(X) \right\} \\ &= m(s \cdot F). \end{aligned}$$

Furthermore, by remark 2.3, there exists a regular countably additive measure $\mu : \mathscr{B}(X) \to [0, 1]$ such that if $\overline{\mathscr{A}}$ is the algebra of sets $E \in \mathscr{B}(X)$ for which $|\mu|(\overline{E} \setminus \mathring{E}) = 0$, then $\mathscr{E} = \{E \cap S : E \in \overline{\mathscr{A}}\}$ and $m(E \cap S) = \mu(E)$ for all $E \in \overline{\mathscr{A}}$. Since

$$\int_X \pi_z \, d\mu = \int_S \pi_z|_S \, dm = \begin{cases} 0 & \text{if } z \neq e \\ 1 & \text{if } z = e \end{cases},$$

it follows that $\mu = \overline{m}$.

3.5. By a tuple $(\lambda, \mathscr{A})$ we mean a bounded finitely additive complex-valued set function λ defined on an algebra $\mathscr{A}$ of Borel sets in S. Let $M(S)$ denote the family of tuples $(\lambda, \mathscr{A})$ such that (i) for some regular complex-valued countably additive measure $\bar{\lambda}$ on an algebra $\bar{\mathscr{A}} \subset \mathscr{B}(X)$ we have $\bar{\lambda}(\bar{E}\setminus\mathring{E}) = 0$ for all $E \in \bar{\mathscr{A}}$, (ii) $f \in C(X, \bar{\mathscr{A}})$ for all $f \in C(X)$, and (iii) $\mathscr{A} = \{E \cap S : E \in \bar{\mathscr{A}}\}$ and $\lambda(E \cap S) = \bar{\lambda}(E)$ for all $E \in \mathscr{A}$. Note that the regularity of λ on $\mathscr{A}$ follows from Lemma 2.5 in [2, p. 268]. We shall say that $(\lambda, \mathscr{A}) \in M(S)$ lies in $M_\pi(S)$ if, given $E \in \mathscr{A}$, there exist uniformly bounded sequences (g_n), $(h_n) \subset F_\pi(X)$ for which

$$g_{n-1} \leq g_n \leq I_E \leq h_n \leq h_{n-1}$$

and $h_n - g_n \to 0$ (as $n \to \infty$) on $X\setminus(\bar{E}\setminus\mathring{E})$. In Sections 5.3 and 5.4 examples of such tuples are given.

3.6. *Remark.* If S is a locally compact abelian group with dual group $S^\wedge$, if $\langle s, \hat{s}\rangle$ denotes the character $\hat{s} \in S^\wedge$ evaluated at $s \in S$, and if $(\mu, \mathscr{A}) \in M(S)$ and $(\lambda, \mathscr{C}) \in M(S^\wedge)$, then it follows (see [2, pp. 276, 277]) that (1) the Fourier-Stieltjes transforms

$$\hat{\mu}(\hat{s}) = \int_S \langle s, \hat{s}\rangle \, d\mu(s) \quad \text{and} \quad \hat{\lambda}(s) = \int_{S^\wedge} \langle s, \hat{s}\rangle \, d\lambda(\hat{s})$$

lie in $C(S^\wedge, \mathscr{C})$ and $C(S, \mathscr{A})$ respectively and (2)

$$\int_S \int_{S^\wedge} \langle s, \hat{s}\rangle \, d\lambda(\hat{s}) \, d\mu(s) = \int_{S^\wedge} \int_S \langle s, \hat{s}\rangle \, d\mu(s) \, d\lambda(\hat{s}).$$

In particular, for any given $\hat{t} \in S^\wedge$, the function $p_{\hat{t}} : S^\wedge \to [0, 1]$ given by

$$p_{\hat{t}}(\hat{s}) = \int_S \langle s, \hat{s} \cdot \hat{t}^{-1}\rangle \, dm(s),$$

where m is the set function of Section 3.4, lies in $C(S^\wedge, \mathscr{C})$. Since

$$p_{\hat{t}}(\hat{s}) = \begin{cases} 0 & \text{if } \hat{s} \neq \hat{t} \\ 1 & \text{if } \hat{s} = \hat{t}, \end{cases}$$

it follows that the singleton $\{\hat{t}\}$ lies in $\mathscr{C}$. So $\mathscr{C}$ contains the set $\{\{\hat{t}\}, \hat{t} \in S^\wedge\}$ of singletons.

4. A Set Function Invariant Under Translations and Homomorphisms

4.1. Mycielski [8, p. 317] proved the existence of a finitely additive set function μ over the algebra L_n of all Lebesgue measurable sets in $\mathbf{R}^n$ ($n \geq 1$) such that $\mu(\mathbf{R}^n) = 1$ and $\mu(TF) = \mu(F)$ for all $F \in L_n$ and all similarities T of $\mathbf{R}^n$. In this section we show the existence of a *regular* finitely additive set function m defined over an algebra $\mathscr{C}$ of Borel sets in $\mathbf{R}^n$ such that $m(\mathbf{R}^n) = 1$ and $m(TF) = m(F)$ for all $F \in \mathscr{C}$ and all T in a class of operators on $\mathbf{R}^n$ which includes the similarities. Though this generalization is obtained at the expense of the size of $\mathscr{E}$, we show that it is still large enough to be of interest. This result of ours follows from the following general theorem.

4.2 THEOREM. *Given a locally compact abelian group S, there exist a translation invariant algebra $\mathscr{E}$ of Borel sets in S and a translation invariant finitely additive set function $m : \mathscr{E} \to [0, 1]$ with the following properties:*

(1) *Any continuous almost periodic function on S is the uniform limit of $\mathscr{E}$-measurable step functions.*

(2) *m is inner and outer regular with respect to the zero and cozero sets (in $\mathscr{E}$), respectively, and $m(S) = 1$.*

(3) *For any surjective homomorphism $T : S_1 \to S_2$ between locally compact abelian groups, T^{-1} maps $\mathscr{E}_2$ into $\mathscr{E}_1$ and $m_1(T^{-1}F) = m_2(F)$ for all $F \in \mathscr{E}_2$; where $(m_i, \mathscr{E}_i)$ is the tuple associated, as above, with S_i ($i = 1, 2$).*

Proof. Let X be the Bohr compactification of S and let $(m, \mathscr{E})$ be the translation invariant tuple given in Example 3.4. Since $\{f|_S : f \in C(X)\}$ consists of the family of all continuous almost periodic functions of S (see [7, p. 168]), and since, by Theorem 3.2, $\{f|_S : f \in C(X)\} \subset C(S, \mathscr{E})$, we get part (1) of the theorem. Part (2) follows from Theorem 3.2 and equation (3.4.2).

There remains to prove part (3). Given $\hat{s}_2 \in S_2\hat{}$, the complex-valued function on S_1 given by $s_1 \to \langle Ts_1, \hat{s}_2\rangle$ is positive definite. So, by Theorem 3.2, there exists a bounded finitely additive set function $\lambda_{\hat{s}_2}$ defined on an algebra $\mathscr{A}_{\hat{s}_2}$ of Borel subsets of $S_1\hat{}$, for which

$$\lambda_{\hat{s}_2}(S_1\hat{}) = 1, \quad (\lambda_{\hat{s}_2}, \mathscr{A}_{\hat{s}_2}) \in M(S_1\hat{}),$$

and

$$\langle Ts_1, \hat{s}_2\rangle = \int_{S_1\hat{}} \langle s_1, \hat{w}\rangle \, d\lambda_{\hat{s}_2}(\hat{w}) \quad (s_1 \in S_1). \tag{4.2.1}$$

Since $|\langle Ts_1, \hat{s}_2\rangle| = |\langle s_1, \hat{w}\rangle| = 1 (s_1 \in S_1, \hat{w} \in \hat{S_1}, \hat{s}_2 \in \hat{S_2})$ and since $\lambda_{\hat{s}_2}$ takes its values in $[0, 1]$, it follows that if $\lambda_{\hat{s}_2}(\{\hat{e}_1\}) \neq 0$ ($\hat{e}_1$ the identity in $\hat{S_1}$), then $\lambda_{\hat{s}_2}$ would have its support concentrated at $\hat{e}_1$. So, we would have $\langle Ts_1, \hat{s}_2\rangle = 1$ for all $s_2 \in S_1$, and since T is onto, this would imply that $\hat{s}_2 = \hat{e}_2$ (the identity in $\hat{S_2}$). So, $\lambda_{\hat{s}_2}(\{\hat{e}_1\}) = 0$ whenever $\hat{s}_2 \neq \hat{e}_2$. Hence, by equation (3.4.2) and Section 3.6, we have

$$\begin{aligned}\int_{S_1} \langle Ts_1, \hat{s}_2\rangle \, dm_1(s_1) &= \int_{S_1}\int_{S_1\hat{}} \langle s_1, \hat{w}\rangle \, d\lambda_{\hat{s}_2}(\hat{w}) \, dm_1(s_1) \\ &= \int_{S_1\hat{}}\int_{S_1} \langle s_1, \hat{w}\rangle \, dm_1(s_1) \, d\lambda_{\hat{s}_2}(\hat{w}) \\ &= \lambda_{\hat{s}_2}(\{\hat{e}_1\}) \\ &= 0\end{aligned}$$

for all $\hat{s}_2 \in S_2\hat{}$, $\hat{s}_2 \neq \hat{e}_2$. It now follows from Section 3.6 that $\phi \circ T \in C(S_1, \mathscr{E}_1)$ and

$$\int_{S_1} \phi \circ T\, dm_1 = \int_{S_2} \phi\, dm_2 \tag{4.2.2}$$

for all $\phi \in F_{\pi_2}(X_2)$, where $\pi_j = \hat{S_j}$ $(j = 1, 2)$

Choose an arbitrary $E \in \mathscr{E}_2$. We now show that $T^{-1}E \in \mathscr{E}_1$ and that

$$m_1(T^{-1}E) = m_2(E).$$

By (3.2.1), given $\varepsilon > 0$, there exist $g', h' \in F_{\pi_2}(X_2)$ with

$$g'|_{S_2} \leq I_E \leq h'|_{S_2} \quad \text{and} \quad \int_{S_2} (h'|_{S_2} - g'|_{S_2})\, dm_2 < \varepsilon.$$

So, by (4.2.2) we have

$$\int_{S_1} (h'|_{S_2} - g'|_{S_2}) \circ T\, dm_1 < \varepsilon$$

where

$$g'|_{S_1} \circ T \leq I_{T^{-1}E} \leq h'|_{S_2} \circ T$$

and, by (4.2.1) and Section 3.6, the functions $g'|_{S_2} \circ T$ and $h'|_{S_2} \circ T$ lie in $C(S_1, \mathscr{E}_1)$. Thus, by (3.3.2), we get

$$\inf\left\{ \int_{S_1} (h|_{S_1} - g|_{S_1})\, dm_1 : g|_{S_1} \leq I_{T^{-1}E} \leq h|_{S_1};\, g, h \in F_{\pi_1}(X_1) \right\} = 0.$$

This proves that $T^{-1}E \in \mathscr{E}_1$. Furthermore,

$$\begin{aligned}
m_1(T^{-1}E) &= \inf\left\{ \int_{S_1} h|_{S_1}\, dm_1 : I_{T^{-1}E} \leq h|_{S_1};\, h \in F_{\pi_1}(X_1) \right\} \\
&\leq \inf\left\{ \int_{S_1} h'|_{S_2} \circ T\, dm_1 : I_{T^{-1}E} \leq h'|_{S_2} \circ T;\, h' \in F_{\pi_2}(X_2) \right\} \\
&= \inf\left\{ \int_{S_1} h'|_{S_2} \circ T\, dm_1 : I_E \leq h'|_{S_2};\, h' \in F_{\pi_2}(X_2) \right\} \\
&= \inf\left\{ \int_{S_2} h'\, |_{S_2}\, dm_2 : I_E \leq h'|_{S_2};\, h' \in F_{\pi_2}(X_2) \right\} \quad \text{(see (4.2.2))} \\
&= m_2(E) \quad \text{(see (3.3.1))}.
\end{aligned}$$

Similarly, $m_1(T^{-1}(S_2 \setminus E)) \leq m_2(S_2 \setminus E)$. Also,

$$\begin{aligned}
1 = m_2(S_2) &= m_2(E) + m_2(S_2 \setminus E) \geq m_1(T^{-1}E) + m_1(T^{-1}(S_2 \setminus E)) \\
&= m_1(T^{-1}S_2) = m_1(S_1) = 1.
\end{aligned}$$

So, $m_1(T^{-1}E) = m_2(E)$. This completes the proof.

4.3. By an isometry of a subset E of $\mathbf{R}^n$ onto another, we mean an (invertible surjective) operator T on E which, like its inverse, preserves distances. It is an elementary fact that such an operator can be extended to an isometry of $\mathbf{R}^n$ (onto itself). An isometry of $\mathbf{R}^n$ is a rigid motion of $\mathbf{R}^n$ and so is necessarily an affine transformation of the form $z \to Uz + z_0$ ($z \in \mathbf{R}^n$) for fixed $z_0 \in \mathbf{R}^n$ and a bijective homomorphism U of $\mathbf{R}^n$. Now, a similarity is, by definition, the product of a magnification from the origin (which, like its inverse, is surjective and affine) and an isometry of $\mathbf{R}^n$. So Theorem 4.2 for the case $S = \mathbf{R}^n$ yields the analogue on $\mathscr{E}$ of Mycielski's result on the Lebesgue subsets of $\mathbf{R}^n$.

4.4. If, for a given Borel measurable function $f : \mathbf{R}^n \to \mathbf{C}$, the sequence of Lebesgue integrals

$$\frac{1}{|B_k|} \int_{B_k} f$$

converges (as $k \to \infty$) to a common limit regardless of the sequence (B_k) of spheres in $\mathbf{R}^n$ for which their volumes $|B_k| \to \infty$, then we write

$$\lim_{|B| \to \infty} \frac{1}{|B|} \int_B f$$

for the common limit and we say that

$$\lim_{|B| \to \infty} \frac{1}{|B|} \int_B f$$

exists. This notion provides us with a means of evaluating integrals with respect to m for the case $S = \mathbf{R}^n$.

4.5. Theorem. *Let $(m, \mathscr{E})$ be the tuple of Theorem 4.2 for the case $S = \mathbf{R}^n$. Then*

$$\lim_{|B| \to \infty} \frac{1}{|B|} \int_B f$$

exists and

$$\int_{\mathbf{R}^n} f\, dm = \lim_{|B| \to \infty} \frac{1}{|B|} \int_B f \tag{4.5.1}$$

for all $f \in C(\mathbf{R}^n, \mathscr{E})$; where the integral on the left hand side exists in the sense of Moore-Smith convergence, while those on the right exist in the sense of Lebesgue.

Proof. Since, by definition, any given $f \in C(\mathbf{R}^n, \mathscr{E})$ can be uniformly approximated by $\mathscr{E}$-measurable (Borel) step functions on $\mathbf{R}^n$, it follows that the function $s \to f(s)$ on an arbitrary sphere $B \subset \mathbf{R}^n$ is the uniform limit of Borel measurable step functions. Hence, the integrals $\int_B f$ exist in the sense Lebesgue.

Since f can be uniformly approximated by $\mathscr{E}$-measurable step functions, it is sufficient to take $f = I_E$ for a set $E \in \mathscr{E}$. Given $\varepsilon > 0$, there exist two real-valued functions g, $h \in F_\pi(\mathbf{R}^n)$ (where $\pi = (\mathbf{R}^n)\hat{}$ and $\mathbf{R}^n$ is the Bohr compactification of $\mathbf{R}^n$) such that $g|_{\mathbf{R}^n} \leqslant I_E \leqslant h|_{\mathbf{R}^n}$ and

$$\int_{\mathbf{R}^n} g|_{\mathbf{R}^n}\, dm - \varepsilon \leqslant \int_{\mathbf{R}^n} I_E\, dm \leqslant \int_{\mathbf{R}^n} h|_{\mathbf{R}n}\, dm + \varepsilon. \tag{4.5.2}$$

Clearly

$$\lim_{k\to\infty} \frac{1}{|B_k|} \int_{B_k} \langle z, \hat{z}\rangle = \begin{cases} 1 & \text{if } \hat{z} = \text{identity} \\ 0 & \text{otherwise} \end{cases}$$

for all sequences (B_k) of spheres such that $|B_k| \to \infty$. Hence,

$$\lim_{|B|\to\infty} \frac{1}{|B|} \int_B \phi$$

exists and is equal to $\int_{\mathbf{R}^n} \phi\, dm$ for all $\phi \in F_\pi(\mathbf{R}^n)$. So, for any sequence (B_k) of spheres such that $|B_k| \to \infty$, we have

$$\begin{aligned}
\int_{\mathbf{R}^n} g|_{\mathbf{R}^n}\, dm &= \lim_{|B|\to\infty} \frac{1}{|B|} \int_B g|_{\mathbf{R}^n} \\
&= \lim \frac{1}{|B_k|} \int_{B_k} g|_{\mathbf{R}^n} \\
&\leqslant \liminf \frac{1}{|B_k|} \int_{B_k} I_E \\
&\leqslant \limsup \frac{1}{|B_k|} \int_{B_k} I_E \\
&\leqslant \limsup \frac{1}{|B_k|} \int_{B_k} h|_{\mathbf{R}^n} \\
&= \lim_{|B|\to\infty} \frac{1}{|B|} \int_B h|_{\mathbf{R}^n} \\
&= \int_{\mathbf{R}^n} h|_{\mathbf{R}^n}\, dm.
\end{aligned}$$

Thus

$$\begin{aligned}
\limsup \frac{1}{|B_k|} \int_{B_k} I_E - \liminf \frac{1}{|B_k|} \int_{B_k} I_E &\leqslant \lim_{|B|\to\infty} \frac{1}{|B|} \int_B (h - g)|_{\mathbf{R}^n} \\
&= \int_{\mathbf{R}^n} (h - g)|_{\mathbf{R}^n}\, dm \\
&< 2\,\varepsilon.
\end{aligned}$$

So

$$\lim_{|B|\to\infty} \frac{1}{|B|} \int_B I_E$$

exists and, by (4.5.2), is equal to $\int_{\mathbf{R}^n} I_E\, dm$. The theorem is proved.

4.6. *Remark.* When $S = \mathbf{R}$, we can use Theorem 4.5 to show that sets of the form

$$F = \cup \{[a, b] + nc : n = 0, \pm 1 \pm 2, \dots\}$$

lie in $\mathscr{E}$ for all $a \leqslant b$ and all $c \geqslant b - a$. In fact, since

$$AP(\mathbf{R}) = \{f|_{\mathbf{R}} : f\, C(\overline{\mathbf{R}})\},$$

then, by (2.2.1), $\mathscr{E}$ consists of all sets $E \cap \mathbf{R}$ such that $E \in \mathscr{B}(\overline{\mathbf{R}})$ and

$$\inf\left\{\int_S (f_2 - f_1)\, dm : f_1 \leqslant I_{E\cap\mathbf{R}} \leqslant f_2; f_1, f_2 \in AP(\mathbf{R})\right\} = 0.$$

If we write $\hat{z}$ for the character given by $z \to \langle z, \hat{z}\rangle = \exp\,(i2\pi z/c)$ $(z \in \mathbf{R})$, and if we let I denote the interval on the unit circle in $\mathbf{C}$ consisting of all points $\langle z, \hat{z}\rangle$ such that $z \in F$, then the set $E = \{z \in \overline{\mathbf{R}} : \langle z, \hat{z}\rangle \in I\}$ lies in $\mathscr{B}(\overline{\mathbf{R}})$ and $F = E \cap \mathbf{R}$. Also, given $\varepsilon > 0$, there clearly exist functions $g, h \in AP(\mathbf{R})$ such that $g \leqslant I_F \leqslant h$ and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (h - g) < \varepsilon.$$

By Theorem 4.5, this implies the inequality $\int_{\mathbf{R}}(h - g)\, dm < \varepsilon$ and so we have $F \in \mathscr{E}$. Moreover, $m(F) = (b - a)/c$.

4.7 *Remark.* Let $(m, \mathscr{E})$ be the tuple of Theorem 4.2 for the case $S = \mathbf{Z}$. Then we can repeat the ideas in the proof of Theorem 4.5 to show that the following limit exists and is given by

$$\int_Z f\, dm = \lim_{N\to\infty} \frac{1}{2N + 1} \sum_{j=-N}^{N} f(j) \qquad (4.7.1)$$

for all $f \in C(\mathbf{Z}, \mathscr{E})$ where the integral on the left exists in the sense of Moore-Smith convergence. By an argument identical to that used in the previous remark, it can be shown that sets of the form

$$F = \cup \{[a, b] + nc : n = 0, \pm 1, \pm 2, \dots\} \cap \mathbf{Z}$$

lie in $\mathscr{E}$ for all $a \leqslant b$ and all $c > 1$.

5. Convolutions of Finitely Additive Set Functions

5.1. In the following sections we shall introduce the notion of convolution for finitely additive set functions. To do this, we shall work on the space

$M_{S^\wedge}(S)$ introduced in Section 3.5 for the case where S is a locally compact abelian group with character group $S^\wedge$.

5.2. THEOREM. *Given* (i) *a locally compact abelian group* S *with character group* $S^\wedge$, (ii) $(\lambda, \mathscr{A}) \in M_{S^\wedge}(S)$ *where* $\mathscr{A}$ *is translation invariant, and* (iii) $E \in \mathscr{A}$, *then the function* $z \to \lambda(E \cdot z)$ *lies in* $C(S, \mathscr{A})$.

Proof. Taking the Jordan decomposition of the real and imaginary parts of λ (see [12, pp. 374, 401]) we see that λ may be assumed to take its values in $[0, \infty)$. Now, if $\hat{z} \in S^\wedge$, then

$$\int_S \langle w \cdot z, \hat{z}\rangle \, d\lambda(w) = \hat{\lambda}(\hat{z})\langle z, \hat{z}\rangle,$$

and since $\hat{z} \in C(S, \mathscr{A})$ it follows that the function $z \to \int_S \langle w \cdot z, \hat{z}\rangle \, d\lambda(w)$ also lies in $C(S, \mathscr{A})$. So, for any $f \in F_{S^\wedge}(\overline{S})$ where $\overline{S}$ is the Bohr compactification of S, the function

$$z \to \int_S f|_S(w \cdot z) \, d\lambda(w)$$

lies in $C(S, \mathscr{A})$. Hence it is sufficient to show that, given $\varepsilon > 0$, there exist functions f, $g \in F_{S^\wedge}(\overline{S})$ such that $g|_S \leqslant I_E \leqslant h|_S$ and

$$\int_S (h|_S - g|_S)(w \cdot z) \, d\lambda(w) < \varepsilon$$

for all $z \in S$.

Let $F \in \mathscr{B}(\overline{S})$ be such that $F \cap S = E$ and let (g_n), (h_n) be uniformly bounded sequences in $F_{S^\wedge}(\overline{S})$ for which

$$g_{n-1} \leqslant g_n \leqslant I_F \leqslant h_n \leqslant h_{n-1} \quad \text{and} \quad h_n - g_n \to 0 \text{ on } \overline{S} \setminus (\overline{F} \setminus \mathring{F}).$$

Now, by definition of $M(S)$, there exists an algebra $\overline{\mathscr{A}} \subset \mathscr{B}(\overline{S})$ and there exists a bounded regular countably additive set function $\overline{\lambda} : \mathscr{B}(\overline{S}) \to [0, \infty)$ such that (i) $\mathscr{A} = \overline{\mathscr{A}} \cap S$, (ii) $\overline{\lambda}(\overline{G} \setminus \mathring{G}) = 0$ $(G \in \overline{\mathscr{A}})$ and (iii) $\lambda(G \cap S) = \overline{\lambda}(G)$ $(G \in \overline{\mathscr{A}})$. So, it is sufficient to show that there exists n' such that

$$\int_{\overline{S}} (h_{n'} - g_{n'})(w \cdot z) \, d\overline{\lambda}(w) < \varepsilon \tag{5.2.1}$$

for all $z \in \overline{S}$.

Let

$$E_n = \{z \in \overline{S} : \int_{\overline{S}} (h_n - g_n)(w \cdot z) \, d\overline{\lambda}(w) \geqslant \varepsilon\}$$

Since $z \to \langle z, \hat{z}\rangle$ is continuous on $\overline{S}$ $(\hat{z} \in S^\wedge)$, so is

$$z \to \int_{\overline{S}} (h_n - g_n)(w \cdot z) \, d\overline{\lambda}(w).$$

(see first part of proof and [7, p. 168]). Hence, E_n is closed in $\overline{S}$. Furthermore, since $h_{n+1} - g_{n+1} \leq h_n - g_n$, it follows that $E_{n+1} \subset E_n$ for all n. Let $E_\infty = \cap E_n$. If $E_n \neq \emptyset$ for all n (i.e., (5.2.1) does not hold), then by the compactness of $\overline{S}$, $E_\infty \neq \emptyset$.That is, there exists $z_\infty \in E_\infty \subset \overline{S}$ for which

$$\int_{\overline{S}} (h_n - g_n)(w \cdot z_\infty)\, d\overline{\lambda}(w) \geq \varepsilon$$

for all n. But this cannot hold since

$$\lim_{n \to \infty} \int_{\overline{S}} (h_n - g_n)(w \cdot z)\, d\overline{\lambda}(w) = 0$$

for all $z \in \overline{S}$. To see this, note that $(F \cdot z)^- \setminus (F \cdot z)^\circ = (\overline{F} \setminus \mathring{F}) \cdot z$ for all $z \in \overline{S}$ and so the function $w \to (h_n - g_n)(w \cdot z)$ convergences to 0 (as $n \to \infty$) almost everywhere with respect to $\overline{\lambda}$. Hence there must exist n' such that $E_{n'} = \emptyset$ (i.e. (5.2.1) holds for some n'). The theorem is proved.

5.3. *Example*. Here, we provide an example of a tuple $(\lambda, \mathscr{A}) \in M_{\mathbf{R}^\wedge}(\mathbf{R})$ with λ and $\mathscr{A}$ translation invariant. Let $\mathscr{A}$ be the algebra of sets in $\mathbf{R}$ generated by the singletons and by sets of the form

$$F = \cup \{[a, b] + nc : n = 0, \pm 1, \pm 2, \ldots\}$$

where $a \leq b$ and $-\infty < c < \infty$. By Remark 4.6, $\mathscr{A}$ is a (translation invariant) subalgebra of the algebra $\mathscr{E}$ in Example 3.4. Let $\lambda : \mathscr{A} \to [0, 1]$ be the restriction to $\mathscr{A}$ of the set function m in Example 3.4. Now, for any set such as F above, there clearly exist uniformly bounded sequences

$$(g_n), (h_n) \subset AP(\mathbf{R})$$

such that

$$g_{n-1} \leq g_n \leq I_F \leq h_n \leq h_{n-1} \quad \text{for all } n$$

and $h_n - g_n \to 0$ (as $n \to \infty$) on the set

$$\mathbf{R} \setminus \{a + kc, b + nc : k, n = 0, \pm 1 \pm 2, \ldots\}.$$

Hence, there exist uniformly bounded sequences

$$(\overline{g}_n), (\overline{h}_n) \subset C(\overline{\mathbf{R}})$$

such that

$$\overline{g}_{n-1} \leq \overline{g}_n \leq I_E \leq \overline{h}_n \leq \overline{h}_{n-1} \quad \text{for all } n,$$

and $\overline{h}_n - \overline{g}_n \to 0$ (as $n \to \infty$) on $\overline{\mathbf{R}} \setminus (\overline{E} \setminus \mathring{E})$ where E is a set in $\mathscr{B}(\overline{\mathbf{R}})$ such that $E \cap \mathbf{R} = F$ (see the final paragraph in Example 3.4). Since $\overline{F_{\mathbf{R}^\wedge}(\overline{\mathbf{R}})} = C(\overline{\mathbf{R}})$, we can take $\overline{g}_n$, $\overline{h}_n$ in $F_{\mathbf{R}^\wedge}(\overline{\mathbf{R}})$.

Since $M_{\mathbf{R}^\wedge}(\mathbf{R}) \subset M(\mathbf{R})$, we must also show that any function in $AP(\mathbf{R})$ lies in $C(\mathbf{R}, \mathscr{A})$. Since $\overline{F_{\mathbf{R}^\wedge}(\overline{\mathbf{R}})} = C(\overline{\mathbf{R}})$ and since $AP(\mathbf{R}) = \{f|_{\mathbf{R}} : f \in C(\overline{\mathbf{R}})\}$ it is sufficient to note that any given $\hat{z} \in \mathbf{R}^\wedge$ lies in $C(\mathbf{R}, \mathscr{A})$; which is evident from the fact that to any given $\hat{z} \in \mathbf{R}^\wedge$ one can associate a point $t \in \mathbf{R}$ such that $\langle z, \hat{z}\rangle = \exp(itz)$ for all $z \in \mathbf{R}$ (see Remark 4.6 and [7, pp. 139, 140]).

5.4. *Example.* For the case $S = \mathbf{Z}$, let $\mathscr{A}$ be the algebra of sets in $\mathbf{Z}$ generated by the singletons and by sets of the form

$$F = \cup\{[a, b] + nc : n = 0, \pm 1, \pm 2, \dots\} \cap \mathbf{Z}$$

for all $a \leq b$ and all $c > 1$. Then by Remark 4.7, $\mathscr{A}$ is a subalgebra of the algebra $\mathscr{E}$ in Example 3.4. If we let $\lambda : \mathscr{A} \to [0, 1]$ be the restriction to $\mathscr{A}$ of the set function m in Example 3.4, then we can repeat the reasoning in the previous example to show that $(\lambda, \mathscr{A}) \in M_{\mathbf{Z}^\wedge}(\mathbf{Z})$ with λ and $\mathscr{A}$ translation invariant.

5.5. *Remark.* Given a countably additive regular measure $\bar{\mu} : \mathscr{B}(\bar{S}) \to \mathbf{C}$ which is absolutely continuous with respect to the Haar measure on the Bohr compactification $\bar{S}$ of $S = \mathbf{R}$ or $S = \mathbf{Z}$, then clearly $(\mu, \mathscr{A}) \in M_{S^\wedge}(S)$ when $\mathscr{A}$ is the algebra of example 5.3 or 5.4 respectively, and μ is given by $\mu(E \cap S) = \bar{\mu}(E)$ for all $E \in \mathscr{B}(\bar{S})$ such that $E \cap S \in \mathscr{A}$.

5.6. Let $(\mu, \mathscr{A})$ and $(\nu, \mathscr{A})$ be two tuples in $M_{S^\wedge}(S)$ with $\mathscr{A}$ translation invariant. By Theorem 5.2, $\int_S \mu(E \cdot z^{-1})\, d\nu(z)$ exists, in the sense of Moore-Smith convergence, for all $E \in \mathscr{A}$. We shall write $\mu * \nu$ for the bounded finitely additive complex-valued set function on $\mathscr{A}$ given by the convolution

$$(\mu * \nu)(E) = \int_S \mu(E \cdot z^{-1})\, d\nu(z) \quad (E \in \mathscr{A}).$$

If $\bar{S}$ denotes the Bohr compactification of S, then (by definition of $M_{S^\wedge}(S)$) there exists an algebra $\bar{\mathscr{A}} \subset \mathscr{B}(\bar{S})$ (which is necessarily translation invariant) for which $\mathscr{A} = \{E \cap S : E \in \bar{\mathscr{A}}\}$, and there exist regular (countably additive) measures $\bar{\mu}, \bar{\nu} : \mathscr{B}(\bar{S}) \to \mathbf{C}$ such that

$$|\mu|(\bar{E} \setminus \mathring{E}) = |\nu|(\bar{E} \setminus \mathring{E}) = 0, \quad \mu(E \cap S) = \bar{\mu}(E) \quad \text{and} \quad \nu(E \cap S) = \bar{\nu}(E)$$

for all $E \in \bar{\mathscr{A}}$. Now, if L is the bounded linear functional on $C(\bar{S})$ given by

$$L(f) = \int_{\bar{S}} f\, d(\bar{\mu} * \bar{\nu}) \quad (f \in C(\bar{S})),$$

where $\bar{\mu} * \bar{\nu}$ denotes the convolution given by

$$\bar{\mu} * \bar{\nu}(E) = \int_{\bar{S}} \bar{\mu}(E \cdot z^{-1})\, d\bar{\nu}(z) \quad (E \in \bar{\mathscr{A}})$$

then by the Riesz representation theorem, there exists a regular (countably) additive) measure $\bar{\lambda} : \mathscr{B}(\bar{S}) \to \mathbf{C}$ such that

$$L(f) = \int_{\bar{S}} f\, d\bar{\lambda} \quad (f \in C(\bar{S})).$$

If we write λ for the finitely additive set function regular on $\mathscr{A}$ (see Lemma 2.2 and remark 2.3) given by $\lambda(E \cap S) = \overline{\lambda}(E)$ for all $E \in \overline{\mathscr{A}}$, then

$$L(f) = \int_S f|_S \, d\lambda \quad (f \in C(\overline{S}))$$

and so

$$\int_S f|_S \, d(\mu * \nu) = \int_{\overline{S}} f \, d(\overline{\mu} * \overline{\nu}) = \int_S f \, d\overline{\lambda} = \int_S f|_S \, d\lambda \quad (f \in C(\overline{S}));$$

the first equality being a consequence of the definition of the Moore-Smith type integral. So,

$$\lambda(E) = \mu * \nu(E) \quad (E \in \mathscr{A})$$

and so $\mu * \nu$ is regular on $\mathscr{A}$. Similarly, $\overline{\lambda}(E) = \overline{\mu} * \overline{\nu}(E)$ for all $E \in \mathscr{B}(\overline{S})$.

5.7. THEOREM. *Given a locally compact abelian group* S *with dual group* S^, *and given* $(\mu, \mathscr{A}), (\nu, \mathscr{A}) \in M_{S^\wedge}(S)$ *with* $\mathscr{A}$ *translation invariant, then*

(1) $(\mu * \nu, \mathscr{A}) \in M_{S^\wedge}(S)$,
(2) $\mu * \nu = \mu * \nu$, *and*
(3) $(\mu * \nu)^\wedge = \hat{\mu}\hat{\nu}$

Proof. Since $(\mu, \mathscr{A})$ and $(\nu, \mathscr{A})$ lie in $M(S)$ there exists, by definition, a (translation invariant) algebra $\overline{\mathscr{A}} \subset \mathscr{B}(\overline{S})$ for which

$$\mathscr{A} = \{E \cap S : E \in \overline{\mathscr{A}}\},$$

and there exist regular (countably additive) measures $\overline{\mu}, \overline{\nu} : \mathscr{B}(\overline{S}) \to \mathbf{C}$ such that

$$|\overline{\mu}|(\overline{E} \setminus \mathring{E}) = |\overline{\nu}|(\overline{E} \setminus \mathring{E}) = 0, \quad \mu(E \cap S) = \overline{\mu}(E) \quad \text{and} \quad \nu(E \cap S) = \overline{\nu}(E)$$

for all $E \in \overline{\mathscr{A}}$. Now, the measure $\overline{\lambda} : \mathscr{B}(\overline{S}) \to \mathbf{C}$ given by the convolution

$$\overline{\lambda}(E) = \int_{\overline{S}} \overline{\mu}(E \cdot z^{-1}) \, d\overline{\nu}(z) \quad (E \in \mathscr{B}(\overline{S}))$$

is regular (see Section 5.6) and

$$\overline{\lambda}(\overline{E} \setminus \mathring{E}) = 0$$

for all $E \in \overline{\mathscr{A}}$ (since $\overline{\mu}((\overline{E} \setminus \mathring{E}) \cdot z) = 0$ for all $z \in \overline{S}$). So, the tuple $(\lambda, \mathscr{A})$ where $\lambda(E \cap S) = \overline{\lambda}(E)$ for all $E \in \overline{\mathscr{A}}$, lies in $M(S)$. Furthermore, since the function $z \to \mu(E \cdot z^{-1})$ lies in $C(S, \mathscr{A})$ (see Theorem 5.2), it follows from the definition of the Moore-Smith type integral (see, for example, [12, pp. 401–403]) that

$$\int_{\overline{S}} \overline{\mu}(E \cdot z^{-1}) \, d\overline{\nu}(z) = \int_S \mu(E \cap S) \cdot z^{-1}) \, d\nu(z) \quad (E \in \overline{\mathscr{A}}).$$

So $\lambda = \mu * \nu$ and so $(\mu * \nu, \mathcal{A}) \in M(S)$. In fact, it is easy to see that $(\mu * \nu, \mathcal{A}) \in M_{S^\wedge}(S)$. This proves part (1). Also, we have

$$\mu * \nu(E \cap S) = \overline{\mu} * \overline{\nu}(E) \quad \text{and} \quad \nu * \mu(E \cap S) = \overline{\nu} * \overline{\mu}(E)$$

for all $E \in \overline{\mathcal{A}}$. So, by Fubini's theorem, we have $\mu * \nu = \nu * \mu$ which is part (2). Now if $f \in C(\overline{S})$, then $\int_{\overline{S}} f\, d\overline{\lambda} = \int_S f\, d\lambda$. Since any given $\hat{z} \in S^\wedge$ admits a unique continuous extension (which we also denote by $\hat{z}$) to $\overline{S}$ then

$$\hat{\lambda}(\hat{z}) = \hat{\overline{\lambda}}(\hat{z}) = \hat{\overline{\mu}}(\hat{z})\hat{\overline{\nu}}(\hat{z}) = \hat{\mu}(\hat{z})\hat{\nu}(\hat{z})$$

and so we have part (3). The theorem is proved.

5.8. Theorem. *For $S = \mathbf{R}$ or $S = \mathbf{Z}$, there exists a tuple $(\lambda, \mathcal{A}) \in M_{S^\wedge}(S)$, with $\mathcal{A}$ and $\lambda : \mathcal{A} \to [0, 1]$ translation invariant, such that*

$$\mu * \lambda(E) = \mu(S)\,\lambda(E) \quad (E \in \mathcal{A})$$

for all finitely additive set functions $\mu : \mathcal{A} \to \mathbf{C}$ for which $(\mu, \mathcal{A}) \in M_{S^\wedge}(S)$. Furthermore, the following limits exist and are given by

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mu(E + z)\, dz = \mu(\mathbf{R})\,\lambda(E) \quad (E \in \mathcal{A}) \tag{5.8.1}$$

when $S = \mathbf{R}$, and

$$\lim_{N\to\infty} \frac{1}{2N + 1} \sum_{j=N}^{N} \mu(E + j) = \mu(\mathbf{Z})\,\lambda(E) \quad (E \in \mathcal{A}) \tag{5.8.2}$$

when $S = \mathbf{Z}$.

Proof. Let $(\lambda, \mathcal{A})$ be that tuple given in example 5.3 or 5.4 when $S = \mathbf{R}$ or $S = \mathbf{Z}$ respectively. Then, by part (2) of Theorem 5.7, we have

$$\mu * \lambda(E) = \lambda * \mu(E) = \int_S \lambda(E \cdot z^{-1})\, d\mu(z) = \int_S \lambda(E)\, d\mu(z) = \mu(S)\,\lambda(E)$$

for all $E \in \mathcal{A}$ and all finitely additive set functions $\mu : \mathcal{A} \to \mathbf{C}$ such that

$$(\mu, \mathcal{A}) \in M_{S^\wedge}(S).$$

Also, (5.8.1) and (5.8.2) follow from the definition of λ and equations (4.5.1) and (4.7.1) respectively. This completes the proof of the theorem.

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