ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 3, Fall 1984

ON POLYFREE GROUPS

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1. Introduction

A group with a subnormal series R: $1 = N_0 \lhd N_1 \lhd \cdots \lhd N_{k-1} \lhd N_k = G$ whose factor groups $F_i = N_i/N_{i-1}$ are free groups of finite rank r_i is called a polyfree group and R a polyfree series of G.

We proved in [2] that the length k and $c = \prod_{i=1}^{k} (r_i - 1)$, the so called Euler characteristic of G, are independent of the chosen polyfree series of G, and we gave examples which show that the ranks r_i are not independent of the choice of the polyfree series of G.

The free abelian group G on two generators x and y is a polyfree group of length 2. Let N_i be the subgroup of G generated by xy^i, $i \in \mathbf{Z}$. Then N_i and G/N_i are infinite cyclic, i.e. free of rank 1. If $i \neq j$, then $N_i \neq N_j$. The group G has therefore infinitely many polyfree series R_i: $1 \lhd N_i \lhd G$. A non-abelian example of a polyfree group with infinitely many polyfree series is Example 26 in [2]. In both cases the polyfree series contain infinite cyclic factors, or equivalently, the groups involved have Euler characteristic $c = 0$. In this note we consider polyfree groups of Euler characteristic $c \neq 0$. We show that in this case the number N of distinct polyfree series of a fixed group G is finite, and we give an upper bound for N which depends only on the invariants c and k.

On the other hand we give an example of a polyfree group G_n ($n = 1, 2, 3, \ldots$) of length 2 and Euler characteristic $cG_n = 2n - 1$ with 2^n polyfree series and an example of a polyfree group G_k ($k = 1, 2, \ldots$) of length k and Euler characteristic 1 with $k!$ polyfree series.

I would like to thank the referee for many valuable suggestions and for pointing out corollary 3.

2. Statement and proof of the main theorem

MAIN THEOREM. *A polyfree group G of length k and Euler characteristic $c \neq 0$ has only a finite number N of distinct polyfree series and*

$$N \leq (c + 1)^{(k-1)c + k^2 - 1}.$$

Received February 5, 1982.
[1] Research supported by a grant from the Swiss National Science Foundation.

Recall that two polyfree series

$$R\colon 1 = N_0 \lhd \cdots \lhd N_k = G \quad \text{and} \quad R'\colon 1 = M_0 \lhd \cdots \lhd M_k = G$$

are distinct if there exists i $(1 \le i \le k)$ such that $M_i \neq N_i$.

We need two lemmas for the proof.

LEMMA 1. *Let Q be a finite group of order q and U a subgroup of $Q_1 \times \cdots \times Q_n$, the direct product of n copies of Q, and let $dU = d$ denote the minimum number of generators of U. Assume that U contains for all pairs $i \neq j$ an element $(q_1, \ldots, q_n)$, $q_i \in Q$ with $q_i \neq q_j$, then $n \le q^d$.*

Proof. Let

$$g_1 = (u_{11}, \ldots, u_{1i}, \ldots, u_{1n}), \ldots, g_d = (u_{d1}, \ldots, u_{di}, \ldots, u_{dn}), \quad u_{ij} \in Q,$$

be a generating set of U. Since there is an element $(q_1, \ldots, q_n)$ in U with $q_i \neq q_j$ for every pair $i \neq j$, the column vectors

$$\begin{pmatrix} u_{1i} \\ \vdots \\ u_{di} \end{pmatrix}$$

are distinct for $i = 1, \ldots, n$. There exist q^d distinct column vectors of length d, and hence $n \le q^d$.

The proof of lemma 1 is similar to an argument used in [3].

LEMMA 2. *Suppose G is a finitely generated group with minimum number of generators $dG = d$ and $S = \{N_i,\ i \in I\}$ a set of normal subgroups of G. Let Q be a finite group of order q such that*

(i) *there exists a surjective homomorphism $G/N_i \twoheadrightarrow Q$ for all $i \in I$, and*
(ii) *$|G\colon N_i N_j| < q$ for all pairs $i \neq j$.*

Then S is finite and $|S| \le q^d$.

Proof. Suppose that $N_1, \ldots, N_n$ are n normal subgroups contained in S and

$$f\colon G \to G/N_1 \times \cdots \times G/N_n \twoheadrightarrow \underbrace{Q \times \cdots \times Q}_{n}$$

is the composition of the canonical map $G \to G/N_1 \times \cdots \times G/N_n$ with the direct product of the maps $G/N_i \twoheadrightarrow Q$ of (i). We verify that

$$U = f(G) \le Q \times \cdots \times Q$$

satisfies the condition of lemma 1: Let $i \neq j$, then since $|G\colon N_i N_j| < q$, the order of Q, the image of N_j under $f1\colon G \twoheadrightarrow G/N_i \twoheadrightarrow Q$ is non-trivial. Let

$n_i \in N_j$ be such that $f_i(n_i) \neq 1$, then $f(n_i) \in U$ has components $(q_1, \ldots, q_n)$ where $q_i = f_i(n_i) \neq 1$ and $q_j = 1$. Application of lemma 1 gives now $n \leq q^d$. Any finite subset of S is therefore of cardinality $\leq q^d$, and hence S is finite and $|S| \leq q^d$. This completes the proof.

Now let $1 \lhd N_1 \lhd \cdots \lhd N_{k-1} \lhd N_k = G$ be a polyfree series with finitely generated free factors N_i/N_{i-1} of rank r_i, then we have for dG, the minimal number of generators of G, the equation $dG \leq r_1 + \cdots + r_k$. If we now use the equation $(r_1 - 1) \cdots (r_k - 1) = c$, where c is the Euler characteristic of G, it is easy to see that $r_1 + \cdots + r_k \leq c + 2k - 1$, and therefore

$$dG = d \leq c + 2k - 1. \tag{1}$$

For the Euler characteristic of N_{k-1}, cN_{k-1}, we have

$$0 \neq cN_{k-1} \leq c, \tag{2}$$

since it divides c.

Proof of the main theorem. We use induction on k.

$k = 1$. G is then a free group, and since non-trivial finitely generated normal subgroups of a free group are of finite index, R: $1 \lhd G$ is the only polyfree series of G.

$k > 1$. Let $S = \{N_{k-1,i} \mid i \in I\}$ be the set of distinct $(k-1)$th terms of polyfree series of G. Using property (2) above and induction hypotheses for $N_{k-1,i}$ we get that the number of polyfree series is finite and $\leq (c+1)^{(k-2)c+(k-1)2-1}$ for any $N_{k-1,i}$ and therefore

$$N \leq |S|(c+1)^{(k-2)c+(k-1)2-1}. \tag{3}$$

Now suppose $1 = N_0 \lhd \cdots \lhd N_{k-1} \lhd N_k = G$ and $1 = M_0 \lhd \cdots \lhd M_{k-1} \lhd M_k = G$ are two polyfree series with $N_{k-1} \neq M_{k-1}$, and let i be such that $N_{i-1} \subseteq M_{k-1}$ but $N_i \not\subseteq M_{k-1}$. Then the map g: $N_i \twoheadrightarrow G/M_{k-1}$ is non-trivial and has a factorisation

$$g: N_i \twoheadrightarrow N_i/N_{i-1} \rightarrow G/M_{k-1}. \tag{4}$$

Since N_i sn G, $g(N_i)$ sn G/M_{k-1}. A non-trivial finitely generated subnormal subgroup of a free group is of finite index and the index formula applies for the ranks:

$$\text{rank}\,(g(N_i)) - 1 = |G/M_{k-1}: g(N_i)|(s_k - 1) \quad \text{where } s_k = \text{rank } G/M_{k-1},$$

i.e.

$$|G/M_{k-1}: g(N_i)| = (\text{rank}\,(g(N_i)) - 1)/(s_k - 1).$$

Now $(s_k - 1) \mid c \neq 0$, therefore $s_k - 1 \geq 1$, and with (4) above

$$\text{rank}\,(g(N_i)) - 1 \leq r_i - 1 \quad \text{where } r_i = \text{rank } N_i/N_{i-1} \text{ and } r_i - 1 \leq c.$$

Hence

(5) $$c \geq |G/M_{k-1}: g(N_i)| = |G: M_{k-1}N_i| \geq |G: M_{k-1}N_{k-1}|.$$

Let now Q be the finite cyclic group of order $c + 1$. Then

(i) there exists $G/N_{k-1,i} \twoheadrightarrow Q$ for all i since $G/N_{k-1,i}$ is free, and
(ii) $|G: N_{k-1,i}N_{k-1,j}| < c + 1$ for $i \neq j$ by (5).

We may therefore apply Lemma 2 and get $|S| \leq (c + 1)^{dG}$. Now we use (1) and (3) to get the result:

$$N \leq (c + 1)^{c+2k+1}(c + 1)^{(k-2)c+(k-1)2-1} = (c + 1)^{(k-1)c+k2-1}.$$

COROLLARY 3. *The automorphism group of a polyfree group G of positive Euler characteristic is residually finite.*

Proof. The finite set of polyfree series of G is permuted by Aut G. Therefore Aut G has a normal subgroup P_1 of finite index such that P_1 leaves invariant each term of a polyfree series

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_k = G.$$

Each Aut (N_{i+1}/N_i) is residually finite, as the automorphism group of a finitely generated free group. Hence there is a $P_2 \lhd$ Aut G such that P_1/P_2 is residually finite and P_2 stabilizes the series. Hence $[G, {}_kP_2] = 1$. But the Three Subgroup Lemma shows that $[G, P_2]$ is nilpotent, and of course $[G, P_2] \lhd G$. Since G has positive characteristic, $[G, P_2] = 1$ and $P_2 = 1$. Hence Aut G is residually finite.

Remark. Consider the following:

(a) the residual finiteness of free groups;
(b) a subgroup of finite index in a finitely generated group G contains a characteristic subgroup of finite index of G;
(c) if $N \lhd G \twoheadrightarrow F$ is exact, N finite and F free, then G contains a free subgroup U of finite index such that $U \cap N = 1$;

Using (a)–(c) and induction on the length k, it is not difficult to prove that polyfree groups are residually finite. Since they are also finitely generated, their automorphism groups are residually finite by a result of G. Baumslag [1], independent of their Euler characteristic.

COROLLARY 4. *A polyfree group with positive Euler characteristic has a normal subgroup of finite index which has a normal polyfree series.*

Proof. The finite set of polyfree series of G is permuted by the inner automorphisms of G. Therefore there is a subgroup U of finite index leaving all of them fixed. The intersection of any polyfree series of G with U is a normal polyfree series of U.

The following example to Corollary 4 shows a polyfree group which has no normal polyfree series. It contains a subgroup of index 2 with a normal polyfree series.

Example 1. Let $X = \langle x_1, x_2 \rangle$, $Y = \langle y_1, y_2 \rangle$ and $U = \langle s, t \rangle$ be free groups of rank 2. Let $N = X \times Y$ and let U operate on N by $x_i^s = y_i$, $y_i^s = x_i$, $x_i^t = x_i$, $y_i^t = y_i$, $i = 1, 2$, then the semidirect product $G = N \rtimes U$ is a polyfree group of length 3 and Euler characteristic 1. Then

$$R_1\colon 1 \lhd X \lhd N \lhd G \quad \text{and} \quad R_2\colon 1 \lhd Y \lhd N \lhd G$$

are polyfree series, but not normal series of G. The following considerations show that R_1 and R_2 are the only polyfree series of G. Assume that

$$R\colon 1 \lhd M_1 \lhd M_2 \lhd G$$

is any polyfree series of G. Since G is of characteristic 1, G/M_2 is non-cyclic free.

We show first that $N \subseteq M_2$: Since $X^s = Y$, either both X and Y are in M_2 or none of them. In the first case, $N \subseteq M_2$. The second case leads to a contradiction: Consider $p\colon G \twoheadrightarrow G/M_2$, then $p(X)$ and $p(Y)$ are nontrivial finitely generated subnormal subgroups of G/M_2. They are therefore of finite index in G/M_2. On the other hand $[X, Y] = 1$; therefore $p(X) \cap p(Y)$ is abelian, a contradiction.

Now consider $N \lhd M_2 \lhd G$. We have G/N and G/M_2 non-trivial and free. M_2/N is a finitely generated normal subgroup of G/N. Since its index is not finite, M_2/N is trivial and hence $M_2 = N$.

A similar consideration shows that either $M_1 = X$ or $M_1 = Y$.

3. Two examples

We give an example of a polyfree group of length k and Euler characteristic 1 with $k!$ distinct polyfree series and an example of a polyfree group of length 2 and Euler characteristic $2n - 1$ which has at least 2^n distinct polyfree series.

Example 2. Let $X_i = \langle x_i, y_i \rangle$ be free groups of rank 2 for $i = 1, \ldots, k$, and let

$$G = X_1 \times \cdots \times X_k.$$

Then G is polyfree of length k and Euler characteristic 1. Let $(i_1, i_2, \ldots, i_k)$ be a permutation of $(1, 2, \ldots, k)$, then

$$1 \lhd X_{i_1} \lhd X_{i_1} \times X_{i_2} \lhd X_{i_1} \times X_{i_2} \times X_{i_3} \lhd \cdots \lhd G$$

is a polyfree series of G. There are $k!$ such polyfree series. In fact, these are the only polyfree series of the group G. This can be shown by a consideration similar to that in Example 1.

Example 3. Suppose that $f_1, \ldots, f_r$ are automorphisms of the free group $\langle y_1, \ldots, y_s \rangle$ of rank s. Then the group with the presentation

$$G = \langle x_1, \ldots, x_r, y_1, \ldots, y_s;\ y_j^{x_i} = f_i(y_j),\ i = 1, \ldots, r, j = 1, \ldots, s \rangle$$

is free by free. More precisely, $y_1, \ldots, y_s$ freely generate a free normal subgroup N and G/N is free on the images of $x_1, \ldots, x_r$ under $G \twoheadrightarrow G/N$.

PROPOSITION 5. *Let G_n be the group with the presentation*

$$(^*)G_n = \langle t, x_1, \ldots, x_n, s, y_1, \ldots, y_n; \quad x_i^t = x_i,\ x_i^s = x_i^{y_i},\ y_i^t = y_i^{x_i},\ y_i^s = y_i,\ 1 \le i \le r \rangle$$

Then for any partition $A \cup B$ of $\{1, \ldots, n\}$ the $2n$ elements x_i, y_i, $i \in A$ and $x_j t^{-1}$, $y_j s^{-1}$, $j \in B$ freely generate a free normal subgroup N of G whose quotient G/N is free on two generators.

Proof. We have

$$y_k^t = y_k^{x_k} \Leftrightarrow y_k = y_k^{x_k t^{-1}} \Leftrightarrow x_k t^{-1} = (x_k t^{-1})^{y_k} \Leftrightarrow (x_k t^{-1})^{s^{-1}} = (x_k t^{-1})^{y_k s^{-1}},$$

and similarly

$$x_k^s = x_k^{y_k} \Leftrightarrow (y_k s^{-1})^{t^{-1}} = (y_k s^{-1})^{x_k t^{-1}}.$$

Moreover, if we use $x_k t = t x_k$, we get

$$y_k^t = y_k^{x_k} \Leftrightarrow y_k^{t^{-1}} = y_k^{x_k^{-1}}.$$

The system (*) of relations is therefore equivalent to (**):

$$(^{**}) \begin{cases} x_i^{t^{-1}} = x_i, & x_i^{s^{-1}} = x_i^{y_i^{-1}} \quad (i \in A), \\ (x_j t^{-1})^{t^{-1}} = x_j t^{-1}, & (x_j t^{-1})^{s^{-1}} = (x_j t^{-1})^{y_j s^{-1}} \quad (j \in B), \\ y_i^{t^{-1}} = y_i^{x_i^{-1}}, & y_i^{s^{-1}} = y_i, \quad (i \in A), \\ (y_j s^{-1})^{t^{-1}} = (y_j s^{-1})^{x_j t^{-1}}, & (y_j s^{-1})^{s^{-1}} = y_j s^{-1} \quad (j \in B). \end{cases}$$

t^{-1} and s^{-1} operate as automorphisms of the subgroup generated by x_i, y_i, $i \in A$ and $x_j t^{-1}$, $y_j s^{-1}$, $j \in B$ and the result follows therefore by the remark at the begining of the section.

There are 2^n such partitions of $\{1, \ldots, n\}$. Therefore G_n has 2^n free normal subgroups with free quotients. Since any two of them generate G_n, they are distinct. Therefore G_n is a polyfree group of length 2 and Euler characteristic $2n - 1$ with at least 2^n distinct polyfree series.

References

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