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FOLIATIONS WITH LOCALLY REDUCTIVE NORMAL BUNDLE

BY

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1. Introduction

Let M be a connected smooth manifold and let $\mathscr{F}$ be a smooth codimension q foliation of M. Let $T(M)$ be the tangent bundle of M and let $E \subset T(M)$ be the subbundle consisting of vectors tangent to the leaves of $\mathscr{F}$. Let $Q = T(M)/E$ be the normal bundle of $\mathscr{F}$ and let $\pi: T(M) \to Q$ be the natural projection. We shall denote by $\chi(M)$, $\Gamma(E)$, and $\Gamma(Q)$ the spaces of smooth sections of the vector bundles $T(M)$, E, and Q respectively. Let

$$\nabla: \chi(M) \times \Gamma(Q) \to \Gamma(Q)$$

be a connection on Q. Following [10] we say that ∇ is an adapted connection if $\nabla_X Y = \pi([X, \tilde{Y}])$ for all $X \in \Gamma(E)$ and all $Y \in \Gamma(Q)$ where $\tilde{Y} \in \chi(M)$ is any vector field satisfying $\pi(\tilde{Y}) = Y$. Such a connection is called basic in [3] and is characterized by the condition that the parallel translation which it induces along a curve lying in a leaf of $\mathscr{F}$ coincides with the natural parallel translation along the leaves. Let $T: \chi(M) \times \chi(M) \to \Gamma(Q)$ be the torsion of ∇, that is, $T(X, Y) = \nabla_X(\pi Y) - \nabla_Y(\pi X) - \pi([X, Y])$. Then ∇ is adapted if and only if $i(X)T = 0$ for all $X \in \Gamma(E)$ where $i(X)T$ denotes the one-form on M with values in Q given by $(i(X)T)(Y) = T(X, Y)$ for $Y \in \chi(M)$. Let

$$R: \chi(M) \times \chi(M) \to \mathrm{Hom}_{\mathbf{R}}(\Gamma(Q), \Gamma(Q))$$

be the curvature of ∇, that is, $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ for $X, Y \in \chi(M)$, $Z \in \Gamma(Q)$. Following [10] we say that the adapted connection ∇ is basic if $i(X)R = 0$ for all $X \in \Gamma(E)$ where $i(X)R$ denotes the one-form on M with values in the bundle $\mathrm{End}(Q)$ given by $(i(X)R)(Y) = R(X, Y)$ for $Y \in \chi(M)$.

In Section 2 we study complete basic connections and prove:

THEOREM 1. *Let M and N be connected manifolds and let $f: M \to N$ be a submersion. Let ∇ be a connection on $Q = T(M)/\ker(f_*)$ and $\bar{\nabla}$ a linear connection on N such that $f^{-1}(\bar{\nabla}) = \nabla$. If ∇ is complete, then $f: M \to N$ is a locally trivial fiber bundle and $\bar{\nabla}$ is also complete.*

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We thank the referee for pointing out that Theorem 1 could also be proved along lines similar to the proof of Theorem 1 in [8]. The context there is Riemannian manifolds instead of affinely connected manifolds, but the geometric content is similar.

We say that $\mathscr{F}$ has a locally reductive normal bundle (a transverse locally reductive structure in the sense of [14]) if its normal bundle admits a basic connection ∇ satisfying $\nabla T = 0$, $\nabla R = 0$. In Section 3 we prove:

THEOREM 2. *Let $(M, \mathscr{F}, \nabla)$ be a foliated manifold with a complete locally reductive normaa bundle. Let $p\colon \tilde{M} \to M$ be the universal cover of M. Then there is a simply connected reductive homogeneous space G/H and a locally trivial fiber bundle $F\colon \tilde{M} \to G/H$ whose fibers are the leaves of $p^{-1}(\mathscr{F})$. Moreover, the lift of ∇ to $\tilde{M}$ agrees with the basic connection obtained by pulling back via F the canonical connection of the second kind on G/H.*

When $\mathscr{F}$ is zero-dimensional we obtain from Theorem 2 the theorem of Kobayashi [11] which states that a simply connected manifold with a complete linear connection with parallel torsion and curvature is isomorphic to a reductive homogeneous space with the canonical connection of the second kind.

In Section 4 we apply Theorem 2 to the case where $\mathscr{F}$ is a Riemannian foliation of M, that is, the normal bundle Q of $\mathscr{F}$ admits a smooth metric g such that the natural parallel transport along a curve lying in a leaf of $\mathscr{F}$ is an isometry. There is a unique torsion-free metric-preserving basic connection ∇ on Q (e.g., see [16], [13]). We say that g is complete if ∇ is complete and we say that $\mathscr{F}$ is Riemannian locally symmetric if $\nabla R = 0$. For each $x \in M$ and each two-dimensional subspace $\not{p}$ of Q_x, the (transverse) sectional curvature of $\not{p}$ is defined by

$$K(\not{p}) = -g(R(\tilde{X}_1, \tilde{X}_2)X_1, X_2)$$

where $\{X_1, X_2\}$ is an orthonormal basis of $\not{p}$ and $\tilde{X}_1, \tilde{X}_2 \in T_x(M)$ satisfy $\pi(\tilde{X}_1) = X_1$, $\pi(\tilde{X}_2) = X_2$.

THEOREM 3. *Let $\mathscr{F}$ be a complete Riemannian locally symmetric foliation of a manifold M. If $K > 0$, then $M/\mathscr{F}$ is compact. If in addition $\mathscr{F}$ has a compact leaf with finite fundamental group, then M is compact with finite fundamental group.*

In Section 5 we give examples of foliations with locally reductive normal bundle. We will observe that a codimension one foliation of a compact manifold defined by a nonsingular closed one-form has a complete locally reductive normal bundle and so we will obtain from Theorem 2, Reeb's structure theorem [19] for such codimension one foliations. More generally, any Lie foliation of a compact manifold has a complete locally reductive normal bundle and we will obtain the structure theorem of Fedida [6]. Also see Molino's structure theory for Riemannian foliations [17].

2. Complete basic connections

Let M be a smooth manifold and let $\mathscr{F}$ be a smooth codimension q foliation of M.

DEFINITION [4]. We say that $Y \in \Gamma(Q)$ is parallel along the leaves of $\mathscr{F}$ if for each pair (U, f) where U is an open set in M and $f\colon U \to \mathbf{R}^q$ is a smooth submersion constant along the leaves of $\mathscr{F} \mid U$, we have $f_{*_p}(Y_p) = f_{*_q}(Y_q)$ whenever $f(p) = f(q)$ where $f_*\colon Q \to T(\mathbf{R}^q)$ is the map induced by $f_*\colon T(M) \to T(\mathbf{R}^q)$. We say that $\mathscr{F}$ is transversely parallelizable if there exist $Y_1, \ldots, Y_q \in \Gamma(Q)$ which are parallel along the leaves of $\mathscr{F}$ and are linearly independent at each point. We call such $Y_1, \ldots, Y_q$ a transverse e-structure for $\mathscr{F}$.

Given $Y \in \Gamma(Q)$, one can always choose $\tilde{Y} \in \chi(M)$ such that $\pi(\tilde{Y}) = Y$. Then Y is parallel along the leaves of $\mathscr{F}$ if and only if for any open set $U \subset M$, $[X, \tilde{Y}] \in \Gamma(E \mid U)$ for all $X \in \Gamma(E \mid U)$ [4].

DEFINITION. Let $Y \in \Gamma(Q)$. We say Y is complete if there exists a complete vector field $\tilde{Y} \in \chi(M)$ such that $\pi(\tilde{Y}) = Y$.

DEFINITION. Let $\mathscr{F}$ be transversely parallelizable and let $Y_1, \ldots, Y_q \in \Gamma(Q)$ be a transverse e-structure. We say this transverse e-structure is complete if Y_i is complete for $i = 1, \ldots, q$.

Let ∇ be an adapted connection on Q. Let $\rho\colon F(Q) \to M$ be the frame bundle of Q, a principal $GL(q, \mathbf{R})$-bundle and let $H \subset T(F(Q))$ be the horizontal distribution corresponding to ∇. Let $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta \in A}$ be an $\mathbf{R}^q$-cocycle defining $\mathscr{F}$. Let $F(\mathbf{R}^q)$ be the frame bundle of $\mathbf{R}^q$. Then

$$\{(\rho^{-1}(U_\alpha), f_{\alpha*}, g_{\alpha\beta*})\}_{\alpha,\beta \in A}$$

is an $F(\mathbf{R}^q)$-cocycle on $F(Q)$ and hence defines a codimension $q(q+1)$ foliation $\mathscr{F}'$ of $F(Q)$. Let $E' \subset T(F(Q))$ be the subbundle tangent to $\mathscr{F}'$. Since ∇ is adapted, we have $E' \subset H$ [15]. We may regard each $u \in F(Q)$ as the vector space isomorphism $u\colon \mathbf{R}^q \to Q_{\rho(u)}$ which sends the standard basis $\{e_1, \ldots, e_q\}$ of $\mathbf{R}^q$ to the frame u of $Q_{\rho(u)}$. Let $Q' = H/E'$, a q-plane bundle over $F(Q)$. Note that $\rho\colon F(Q) \to M$ induces $\rho_*\colon Q' \to Q$, an isomorphism on fibers. Let $h \in \mathbf{R}^q$. For $u \in F(Q)$, let $B(h)_u \in Q'_u$ be the unique element such that $\rho_{*_u}(B(h)_u) = u(h)$. Then $B(h)$ is a section of Q'. Note that $Q' \subset T(F(Q))/E'$ = normal bundle of $\mathscr{F}'$.

DEFINITION. We say ∇ is complete if $B(h)$ is complete for all $h \in \mathbf{R}^q$.

Let E_h^k be the $q \times q$ matrix with a 1 in the h^{th} column and k^{th} row and 0 elsewhere and let $\sigma(E_h^k)$ be the corresponding fundamental vector field on

$F(Q)$. Since the vector fields $\sigma(E_h^k)$ project via the maps $f_{\alpha *}$ to the fundamental vector fields on $F(\mathbf{R}^q)$, it follows that

$$\pi(\sigma(E_h^k)) \in \Gamma(T(F(Q))/E')$$

is parallel along the leaves of $\mathscr{F}'$. Note that it is also complete.

Suppose now that ∇ is basic. Let θ be the $\mathbf{R}^q$-valued one-form on $F(Q)$ defined by $\theta_u(Y) = u^{-1}(\pi\rho_{*u}(Y))$ for $u \in F(Q)$, $Y \in T_u(F(Q))$. The torsion form of ∇ is the $\mathbf{R}^q$-valued two-form Θ on $F(Q)$ defined by

$$\Theta_u(X, Y) = (d\theta)_u(X_H, Y_H) \quad \text{for } u \in F(Q) \quad \text{and} \quad X, Y \in T_u(F(Q)).$$

Since $i(X)T = 0$ for all $X \in \Gamma(E)$, it follows that $i(X)\Theta = 0$ for all $X \in \Gamma(E')$. Let ω be the connection form of ∇ and let Ω be the curvature form. Since $i(X)R = 0$ for all $X \in \Gamma(E)$, it follows that $i(X)\Omega = 0$ for all $X \in \Gamma(E')$. For $i = 1, \ldots, q$ let $E_i = B(e_i)$ and let Y_i be a horizontal vector field on $F(Q)$ satisfying $\pi(Y_i) = E_i$. If $X \in \Gamma(E')$, then

$$\begin{aligned} 0 &= (i(X)\Omega)(Y_i) = \Omega(X, Y_i) = d\omega(X_H, Y_{iH}) = d\omega(X, Y_i) \\ &= X\omega(Y_i) - Y_i\omega(X) - \omega([X, Y_i]) = -\omega([X, Y_i]) \end{aligned}$$

and so $[X, Y_i]$ is horizontal. Now

$$\begin{aligned} 0 &= (i(X)\Theta)(Y_i) = \Theta(X, Y_i) = d\theta(X_H, Y_{iH}) = d\theta(X, Y_i) \\ &= X\theta(Y_i) - Y_i\theta(X) - \theta([X, Y_i]) = -\theta([X, Y_i]) \end{aligned}$$

and so $[X, Y_i] \in \Gamma(E' \oplus V)$ where $V \subset T(F(Q))$ denotes the subbundle consisting of vertical vectors. Hence $[X, Y_i] \in \Gamma(E')$ and so E_i is parallel along the leaves of $\mathscr{F}'$. Thus $\{E_i, \pi(\sigma(E_h^k)): i, h, k = 1, \ldots, q\}$ is a transverse e-structure for $\mathscr{F}'$. If ∇ is complete, then this transverse e-structure is complete.

We now prove Theorem 1. Let $\mathscr{F}$ be the foliation of M whose leaves are the connected components of the level sets of the submersion $f: M \to N$. Since ∇ is the pull-back via f of a connection on N, it follows that ∇ is a basic connection for $\mathscr{F}$. Since ∇ is complete, we have from the above discussion that $\{E_i, \pi(\sigma(E_h^k)): i, h, k = 1, \ldots, q\}$ is a complete transverse e-structure for $\mathscr{F}'$. For each $i = 1, \ldots q$ let Y_i be a complete horizontal vector field satisfying $\pi(Y_i) = E_i$ and let $\phi^i: \mathbf{R} \times F(Q) \to F(Q)$ be the action of $\mathbf{R}$ on $F(Q)$ generated by Y_i. Let $X_1, \ldots, X_r$ $(r = q^2)$ be the vertical vector fields $\sigma(E_h^k)$ and for each $j = 1, \ldots, r$ let $\psi^j: \mathbf{R} \times F(Q) \to F(Q)$ be the action of $\mathbf{R}$ on $F(Q)$ generated by X_j.

Let $u_0 \in F(Q)$ and let L' be the leaf of $\mathscr{F}'$ passing through u_0. Define

$$\Phi: \mathbf{R}^r \times \mathbf{R}^q \times L' \to F(Q)$$

by

$$\Phi(s_1, \ldots, s_r, t_1, \ldots, t_q, u) = \psi_{s_1}^1 \circ \cdots \circ \phi_{t_1}^1 \circ \cdots \circ \psi_{s_r}^r \circ \phi_{t_q}^q(u).$$

Note that the leaves of $\mathscr{F}'$ are closed since $\mathscr{F}'$ is defined by the submersion

$$f_*: F(Q) \to F(N).$$

Hence by the proof of Proposition 4 in [16] (or the proof of Lemma 1 in Section II. 2 of [17]), there is a neighborhood Ω of 0 in $\mathbf{R}^r \times \mathbf{R}^q = \mathbf{R}^{r+q}$ such that $\Phi: \Omega \times L' \to U$ is a diffeomorphism where U is an open saturated set in $F(Q)$. We remark that this fact is closely related to a classical result of Ehresmann [5]. Note that Φ maps the foliation of $\Omega \times L'$ whose leaves are the sets $\{\text{point}\} \times L'$ to $\mathscr{F}'$ and induces on each leaf of $\Omega \times L'$ a diffeomorphism onto a leaf of $\mathscr{F}'$. We may assume that Ω is of the form $\Omega_1 \times \Omega_2$ where Ω_1 is a neighborhood of 0 in $\mathbf{R}^r$ and Ω_2 is a neighborhood of 0 in $\mathbf{R}^q$. Note that $\rho: F(Q) \to M$ maps each leaf of $\mathscr{F}'$ diffeomorphically onto a leaf of $\mathscr{F}$. Let $L = \rho(L') \in \mathscr{F}$. Since $X_1, \ldots, X_r$ are vertical, Φ induces a smooth map $\Psi: \Omega_2 \times L \to M$ such that the diagram

$$\begin{array}{ccc} \Omega_1 \times \Omega_2 \times L' & \xrightarrow{\Phi} & U \\ \Big\downarrow{\scriptstyle p_2 \times \rho} & & \Big\downarrow{\scriptstyle \rho} \\ \Omega_2 \times L & \xrightarrow{\Psi} & \rho(U) \end{array}$$

commutes where $p_2: \Omega_1 \times \Omega_2 \to \Omega_2$ is the projection onto the second factor. Then $\rho(U)$ is an open saturated set in M and Ψ is a local diffeomorphism which maps the foliation of $\Omega_2 \times L$ whose leaves are the sets $\{\text{point}\} \times L$ to $\mathscr{F}$ and induces on each leaf of $\Omega_2 \times L$ a diffeomorphism onto a leaf of $\mathscr{F}$. Let $x_0 = \rho(u_0)$ and consider the composition

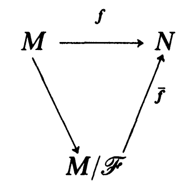
$$\Omega_2 \xrightarrow{i_{x_0}} \Omega_2 \times L \xrightarrow{\Psi} \rho(U) \xrightarrow{f} N$$

where $i_{x_0}(y) = (y, x_0)$. Since this composition is a local diffeomorphism we may assume, by shrinking Ω_2 if necessary, that it is a diffeomorphism. Thus

$$\Omega_2 \times L \xrightarrow{\Psi} \rho(U)$$

is one-one and hence is a diffeomorphism. Let K be a compact neighborhood of 0 in $\mathbf{R}^q$ contained in Ω_2. Then $\Psi(K \times L)$ is a closed saturated neighborhood of L in M. Hence each point of $M/\mathscr{F}$ has a neighborhood base consisting of closed sets. Since the points of $M/\mathscr{F}$ are closed sets, it follows that $M/\mathscr{F}$ is Hausdorff. Thus $M/\mathscr{F}$ is a smooth Hausdorff manifold and the natural projection $M \to M/\mathscr{F}$ is a locally trivial fiber bundle.

Now f induces a local diffeomorphism $\bar{f}$: $M/\mathscr{F} \to N$ such that the diagram



commutes. Also, ∇ induces a complete linear connection $\tilde{\nabla}$ on $M/\mathscr{F}$ such that $\bar{f}^{-1}(\bar{\nabla}) = \tilde{\nabla}$. Since $\tilde{\nabla}$ is complete and $\bar{f}$ is a connection-preserving local diffeomorphism, it follows that $\bar{f}$ is a covering and $\bar{\nabla}$ is complete [9]. Hence f is a locally trivial fiber bundle.

3. Locally reductive normal bundle

(3.1) PROPOSITION. *Let $(M, \mathscr{F}, \nabla)$ be a foliated manifold with a locally reductive normal bundle. Let p: $\tilde{M} \to M$ be the universal cover of M. Then there is a simply connected reductive homogeneous space G/H and a smooth submersion F: $\tilde{M} \to G/H$ such that the leaves of $p^{-1}(\mathscr{F})$ are the connected components of the sets $F^{-1}\{x\}$, $x \in G/H$. Moreover, the lift of ∇ to $\tilde{M}$ agrees with the basic connection obtained by pulling back via F the canonical connection of the second kind on G/H.*

Proof. Let U be an open set in M such that the leaves of $\mathscr{F} \mid U$ are the level sets of a smooth submersion f: $U \to V$ where V is an open subset of $\mathbf{R}^q$. Let $\bar{X}$, $\bar{Y} \in \chi(V)$. Let $Y \in \Gamma(Q \mid U)$ be the unique section of $Q \mid U$ which is f-related to $\bar{Y}$ and let $X \in \chi(U)$ be any vector field which is f-related to $\bar{X}$. Let $Z \in \Gamma(E \mid U)$. Then

$$\begin{aligned} \nabla_Z \nabla_X Y &= R(Z, X)Y + \nabla_X \nabla_Z Y + \nabla_{[Z,X]} Y \\ &= (i(Z)R)(X)(Y) + \nabla_X \nabla_Z Y + \nabla_{[Z,X]} Y \\ &= 0 \end{aligned}$$

since Z, $[Z, X] \in \Gamma(E \mid U)$. Thus $\nabla_X Y$ is parallel along the leaves of $\mathscr{F} \mid U$ and hence is f-related to a vector field $\bar{\nabla}_{\bar{X}} \bar{Y}$ on V. If $X_1 \in \chi(U)$ is also f-related to $\bar{X}$, then $\nabla_X Y - \nabla_{X_1} Y = \nabla_{X - X_1} Y = 0$ since $X - X_1 \in \Gamma(E \mid U)$ and so $\bar{\nabla}_{\bar{X}} \bar{Y}$ depends only on $\bar{X}$ and $\bar{Y}$. Clearly $\bar{\nabla}$: $\chi(V) \times \chi(V) \to \chi(V)$ defines a linear connection on V such that $f^{-1}(\bar{\nabla}) = \nabla$. Moreover, the torsion $\bar{T}$ and curvature $\bar{R}$ of $\bar{\nabla}$ satisfy $\bar{\nabla}\bar{T} = 0$, $\bar{\nabla}\bar{R} = 0$. Hence V is locally representable as a reductive homogeneous spacse with the canonical connection of the second kind [18]. Thus, by shrinking U if necessary, we may assume that V is an open subset of a simply connected reductive homogeneous space G/H and that $\bar{\nabla}$ is the restriction of the canonical connection of the second kind. Hence we can find an open cover $\{U_\alpha\}_{\alpha \in A}$ of M such that for each $\alpha \in A$ the leaves of $\mathscr{F} \mid U_\alpha$ are the level sets of a smooth submersion f_α: $U_\alpha \to V_\alpha$ where

V_α is an open subset of a simply connected reductive homogeneous space $(G/H)_\alpha$ and $f_\alpha^{-1}(\bar{\nabla}_\alpha) = \nabla \mid U_\alpha$ where $\bar{\nabla}_\alpha$ is the canonical connection of the second kind on $(G/H)_\alpha$. For each α, $\beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$ we have a diffeomorphism

$$g_{\alpha\beta}: f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$$

satisfying $f_\alpha = g_{\alpha\beta} \circ f_\beta$ on $U_\alpha \cap U_\beta$. Since

$$f_\beta^{-1}(g_{\alpha\beta}^{-1}(\bar{\nabla}_\alpha)) = (g_{\alpha\beta} \circ f_\beta)^{-1}(\bar{\nabla}_\alpha) = f_\alpha^{-1}(\bar{\nabla}_\alpha) = \nabla = f_\beta^{-1}(\bar{\nabla}_\beta)$$

on $U_\alpha \cap U_\beta$ it follows that $g_{\alpha\beta}^{-1}(\bar{\nabla}_\alpha) = \bar{\nabla}_\beta$ on $f_\beta(U_\alpha \cap U_\beta)$ and so $g_{\alpha\beta}$ is an affine transformation.

Let $\alpha \in A$. Since $(G/H)_\alpha$ is a reductive homogeneous space and $\bar{\nabla}_\alpha$ is the canonical connection of the second kind, we have that $(G/H)_\alpha$ is an analytic manifold and $\bar{\nabla}_\alpha$ is a complete analytic linear connection [12]. Without loss of generality we may assume that $U_\alpha \cap U_\beta$ is connected whenever it is non-empty. Hence $g_{\alpha\beta}$ can be uniquely extended to an affine isomorphism from $(G/H)_\beta$ to $(G/H)_\alpha$ [12]. If α, $\alpha' \in A$ are arbitrary, let σ: $[0, 1] \to M$ be a continuous curve with $\sigma(0) \in U_\alpha$, $\sigma(1) \in U_{\alpha'}$, and choose a covering of σ by a finite sequence $U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_n}$ with $U_{\alpha_0} = U_\alpha$, $U_{\alpha_n} = U_{\alpha'}$ such that $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$ for $i = 0, 1, \ldots, n-1$. Since $(G/H)_{\alpha_i}$ and $(G/H)_{\alpha_{i+1}}$ are affinely isomorphic for $i = 0, 1, \ldots, n-1$ it follows that $(G/H)_\alpha$ and $(G/H)_{\alpha'}$ are afffinely isomorphic. Hence there is a simply connected reductive homogeneous space G/H such that $\mathscr{F}$ is defined by a G/H-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta \in A}$ such that $f_\alpha^{-1}(\bar{\nabla}) = \nabla \mid U_\alpha$ where $\bar{\nabla}$ is the canonical connection of the second kind on G/H and each $g_{\alpha\beta}$ is the restriction of an affine isomorphism of G/H. Thus $\mathscr{F}$ is transversely homogeneous and so there is a smooth submersion F: $\tilde{M} \to G/H$ constant along the leaves of $p^{-1}(\mathscr{F})$ [1]. Clearly $F^{-1}(\bar{\nabla}) = p^{-1}(\nabla)$. This completes the proof of the proposition.

Theorem 2 now follows from Proposition (3.1) and Theorem 1.

(3.2) COROLLARY. *Let $(M, \mathscr{F})$ be a foliated manifold and let ∇ be a complete basic connection on the normal bundle of $\mathscr{F}$. Let p: $\tilde{M} \to M$ be the universal cover of M.*

(a) *If $T = 0$, $\nabla R = 0$, there is a simply connected symmetric space G/H and a locally trivial fiber bundle F: $\tilde{M} \to G/H$ whose fibers are the leaves of $p^{-1}(\mathscr{F})$. Moreover, $p^{-1}(\nabla) = F^{-1}(\bar{\nabla})$ where $\bar{\nabla}$ is the canonical connection on G/H.*

(b) *If $R = 0$, $\nabla T = 0$, there is a simply connected Lie group K and a locally trivial fiber bundle F: $\tilde{M} \to K$ whose fibers are the leaves of $p^{-1}(\mathscr{F})$. Moreover, $p^{-1}(\nabla) = F^{-1}(\bar{\nabla})$ where $\bar{\nabla}$ is the linear connection on K whose parallel transport is defined by the left translations of K.*

(c) *If $T = 0$, $R = 0$, then $\tilde{M}$ is diffeomorphic to a product $\tilde{L} \times \mathbf{R}^q$ where $\tilde{L}$ is the universal cover of the leaves of $\mathscr{F}$ and $p^{-1}(\mathscr{F})$ is the product foliation.*

Moreover, $p^{-1}(\nabla)$ *is the basic connection on* $\tilde{L} \times \mathbf{R}^q$ *determined by the canonical linear connection on* $\mathbf{R}^q$.

4. Riemannian locally symmetric foliations

We prove Theorem 3. Let $(M, \mathscr{F}, g)$ be a complete Riemannian locally symmetric foliation. That is, $\mathscr{F}$ is a foliation of the manifold M and g is a holonomy-invariant metric on the normal bundle Q of $\mathscr{F}$. Moreover, the unique basic connection ∇ on Q with $T = 0$, $\nabla g = 0$ is complete and satisfies $\nabla R = 0$. We assume that the (transverse) sectional curvature K of $(M, \mathscr{F}, g)$ is positive. Let $p\colon \tilde{M} \to M$ be the universal cover of M and let $\tilde{\mathscr{F}} = p^{-1}(\mathscr{F})$. By Theorem 1, the space of leaves $\tilde{M}/\tilde{\mathscr{F}}$ is a smooth Hausdorff manifold and the natural projection $\tilde{M} \to \tilde{M}/\tilde{\mathscr{F}}$ is a locally trivial fiber bundle. The lift of g to the normal bundle of $\tilde{\mathscr{F}}$ projects to a complete Riemannian metric on $\tilde{M}/\tilde{\mathscr{F}}$ with parallel curvature. Thus $\tilde{M}/\tilde{\mathscr{F}}$ is a complete Riemannian locally symmetric space and hence, since it is simply connected, is Riemannian symmetric [12]. Since $K > 0$, it follows that $\tilde{M}/\tilde{\mathscr{F}}$ has positive sectional curvature. Thus $\tilde{M}/\tilde{\mathscr{F}}$ is compact [21]. Now $p\colon \tilde{M} \to M$ induces a continuous surjection $\tilde{M}/\tilde{\mathscr{F}} \to M/\mathscr{F}$ and so $M/\mathscr{F}$ is compact. If $\mathscr{F}$ has a compact leaf with finite fundamental group, then the fibers of the bundle $\tilde{M} \to \tilde{M}/\tilde{\mathscr{F}}$ are compact. Hence $\tilde{M}$ is compact and so M is compact with finite fundamental group.

5. Applications and examples

(5.1) *Application to Fedida's structure theorem* [6] *for Lie foliations.* Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Let M be a compact manifold and suppose ω is a smooth $\mathfrak{g}$-valued one-form of rank q on M satisfying $d\omega + \frac{1}{2}[\omega, \omega] = 0$. Then ω defines a smooth codimension q foliation $\mathscr{F}$ on M which is a Lie foliation modeled on $\mathfrak{g}$ [6]. Let $X_1, \ldots, X_q$ be a basis of $\mathfrak{g}$. Then $\omega = \sum_{i=1}^q \omega_i X_i$ where $\omega_1, \ldots, \omega_q$ are smooth linearly independent one-forms on M satisfying

$$d\omega_i = \sum_{1 \le j < k \le q} c^i_{jk}\, \omega_j \wedge \omega_k \quad \text{where } c^i_{jk} \in \mathbf{R}.$$

Let $\tilde{Y}_1, \ldots, \tilde{Y}_q \in \chi(M)$ be such that $\omega_i(\tilde{Y}_j) = \delta_{ij}$. For each $i = 1, \ldots, q$ let

$$Y_i = \pi(\tilde{Y}_i) \in \Gamma(Q).$$

Define a connection ∇ on Q by requiring $\nabla_X Y_i = 0$, $i = 1, \ldots, q$ for all $X \in \chi(M)$.

LEMMA. ∇ *is adapted.*

Proof. Let $X \in \Gamma(E)$, $Y \in \Gamma(Q)$. Write $Y = \sum_{i=1}^{q} f_i Y_i$ where the f_i are smooth functions on M. Then

$$\begin{aligned} \nabla_X Y &= \nabla_X(\sum f_i Y_i) \\ &= \sum \nabla_X f_i Y_i \\ &= \sum (f_i \nabla_X Y_i + (Xf_i) Y_i) \\ &= \sum (Xf_i) Y_i \\ &= \sum \pi((Xf_i)\tilde{Y}_i) \\ &= \sum \pi([X, f_i \tilde{Y}_i]) - f_i[X, \tilde{Y}_i]) \\ &= \pi(\sum [X, f_i \tilde{Y}_i]) - \sum f_i \pi([X, \tilde{Y}_i]). \end{aligned}$$

But for $i, l = 1, \ldots, q$ we have

$$\begin{aligned} 0 &= \sum_{1 \le j < k \le q} c^i_{jk} \omega_j \wedge \omega_k(X, \tilde{Y}_l) \\ &= d\omega_i(X, \tilde{Y}_l) \\ &= X\omega_i(\tilde{Y}_l) - \tilde{Y}_l \omega_i(X) - \omega_i([X, \tilde{Y}_l]) \\ &= -\omega_i([X, \tilde{Y}_l]) \end{aligned}$$

and so $[X, \tilde{Y}_l] \in \Gamma(E)$. Thus $\nabla_X Y = \pi(\sum [X, f_i \tilde{Y}_i]) = \pi([X, \sum f_i \tilde{Y}_i])$.

LEMMA. $\nabla T = 0$.

Proof. For $i, l, r = 1, \ldots, q$ we have

$$\begin{aligned} -\omega_i([\tilde{Y}_l, \tilde{Y}_r]) &= \tilde{Y}_l \omega_i(\tilde{Y}_r) - \tilde{Y}_r \omega_i(\tilde{Y}_l) - \omega_i([\tilde{Y}_l, \tilde{Y}_r]) \\ &= d\omega_i(\tilde{Y}_l, \tilde{Y}_r) \\ &= \sum_{1 \le j < k \le q} c^i_{jk} \omega_j \wedge \omega_k(\tilde{Y}_l, \tilde{Y}_r) \\ &= \sum_{1 \le j < k \le q} c^i_{jk}(\delta_{jl}\delta_{kr} - \delta_{jr}\delta_{kl}) \\ &= -b^i_{lr} \in \mathbf{R}. \end{aligned}$$

Thus $[\tilde{Y}_l, \tilde{Y}_r] = X + \sum_{i=1}^{q} b^i_{lr} \tilde{Y}_i$ where $X \in \Gamma(E)$ and so

$$\pi([\tilde{Y}_l, \tilde{Y}_r]) = \sum_{i=1}^{q} b^i_{lr} Y_i.$$

Hence

$$T(\tilde{Y}_l, \tilde{Y}_r) = \nabla_{\tilde{Y}_l} Y_r - \nabla_{\tilde{Y}_r} Y_l - \pi([\tilde{Y}_l, \tilde{Y}_r]) = -\sum_{i=1}^{q} b^i_{lr} Y_i$$

which shows that T is parallel.

Clearly, $R = 0$. In particular ∇ is basic. Since M is compact, we have that $\tilde{Y}_1, \ldots, \tilde{Y}_q$ are complete and so ∇ is complete. Hence, by Corollary 3.2 (b), the leaves of the lift of $\mathscr{F}$ to the universal cover $\tilde{M}$ of M are the fibers of a locally trivial fiber bundle $\tilde{M} \to K$ where K is a simply connected Lie group which is Fedida's result. Of course, K is the simply connected Lie group whose Lie algebra is $\mathscr{g}$.

(5.2) *Application to Reeb's structure theorem* [19] *for codimension one foliations defined by a closed one-form.* Let M be a compact manifold and let $\mathscr{F}$ be a codimension one foliation of M defined by a nonsingular one-form ω on M satisfying $d\omega = 0$. Then $\mathscr{F}$ is Riemannian and hence the canonical torsion-free connection is curvature-free. Hence, by Corollary 3.2 (c), $\tilde{M}$ is a product $\tilde{L} \times \mathbf{R}$ and $p^{-1}(\mathscr{F})$ is the product foliation which is Reeb's result.

(5.3) *Example.* Let M be a manifold and let ω be a smooth nonsingular one-form on M satisfying $d\omega = \omega \wedge \omega_1$, $d\omega_1 = 0$. Then ω defines a smooth codimension one foliation $\mathscr{F}$ of M which is transversely affine and which can be defined by an $\mathbf{R}$-cocycle

$$\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta \in A}$$

where each $g_{\alpha\beta}$ is of the form $g_{\alpha\beta}(t) = a_{\alpha\beta} t + b_{\alpha\beta}$ [7], [20]. The canonical linear connection on $\mathbf{R}$ induces a basic connection ∇ on the normal bundle of $\mathscr{F}$ satisfying $T = 0$, $R = 0$. If M is compact and ∇ is complete, then $\mathscr{F}$ has no exceptional minimal sets [2].

(5.4) *Example.* This example is a special case of (5.3). Let $F\colon \mathbf{R}^2 \to \mathbf{R}$ be the smooth submersion given by $F(x, y) = e^y \sin 2\pi x$. Then F defines a codimension one foliation $\tilde{\mathscr{F}}$ of $\mathbf{R}^2$ which passes to a codimension one transversely affine foliation $\mathscr{F}$ of the two-dimensional torus T^2. The basic connection on the normal bundle of $\mathscr{F}$ induced by the canonical linear connection on $\mathbf{R}$ is not complete. Observe that $F\colon \mathbf{R}^2 \to \mathbf{R}$ is not a locally trivial fiber bundle.

(5.5) *Example.* Let

$$K = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b, \in \mathbf{R}, a > 0\right\},$$

a two-dimensional Lie group. Let $\tilde{M} = \{(x, y, z) \in \mathbf{R}^3 : z > 0\}$ and let $F\colon \tilde{M} \to K$ be the smooth submersion given by

$$F(x, y, z) = \begin{pmatrix} z & e^y \sin 2\pi x \\ 0 & 1 \end{pmatrix}.$$

Then F defines a codimension two foliation $\tilde{\mathscr{F}}$ of $\tilde{M}$. Define a left action of $Z \times Z$ on $\tilde{M}$ by $((n, m), (x, y, z)) \mapsto (x + n, y + m, z)$. Then $\tilde{\mathscr{F}}$ passes to a codimension two foliation $\mathscr{F}$ of $(Z \times Z)\backslash\tilde{M} = T^2 \times \mathbf{R}^+$ which can be defined by a K-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta \in A}$ where each $g_{\alpha\beta}$ is of the form $g_{\alpha\beta}(k) = a_{\alpha\beta} k a_{\alpha\beta}^{-1}$. The linear connection on K whose parallel transport is given by left translations of K induces a basic connection ∇ on the normal bundle of $\mathscr{F}$ satisfying $R = 0$, $\nabla T = 0$, $T \neq 0$.

(5.6) *Example.* Let G/H be a reductive homogeneous space. That is, the Lie algebra $\mathfrak{g}$ of G may be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of $\mathfrak{h}$ and $\mathfrak{m}$ is an ad (H)-invariant subspace of $\mathfrak{g}$. Let $\bar{\nabla}$ be the canonical connection of the second kind on G/H. Then $\bar{\nabla}$ is a complete G-invariant linear connection on G/H satisfying $\bar{\nabla}\bar{T} = 0$, $\bar{\nabla}\bar{R} = 0$. Let Γ be a discrete subgroup of G. The foliation of G whose leaves are the left cosets gH of H induces on $M = \Gamma\backslash G$ a foliation $\mathscr{F}$ with a complete locally reductive normal bundle.

(5.7) *Example.* Define a left action of $\pi_1(T^2) = Z \times Z$ on S^2 by

$$(1, 0) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(3),$$

$$(0, 1) \mapsto \begin{pmatrix} \cos 2\pi\alpha & \sin 2\pi\alpha & 0 \\ -\sin 2\pi\alpha & \cos 2\pi\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O(3)$$

where $0 < \alpha < 1$ is irrational. Let $M = \mathbf{R}^2 \times_{(Z \times Z)} S^2$ be the associated bundle over T^2 with fiber S^2. The foliation of $\mathbf{R}^2 \times S^2$ whose leaves are the sets $\mathbf{R}^2 \times \{x\}$, $x \in S^2$ induces on M a complete Riemannian locally symmetric foliation $\mathscr{F}$ with (transverse) sectional curvature $K \equiv 1$.

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