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CURVATURE AND EULER CHARACTERISTIC FOR SIX-DIMENSIONAL KÄHLER MANIFOLDS

BY

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0. Introduction

Perhaps the most basic problem in Riemannian geometry is the determination of which Riemannian metrics a given manifold can support, in particular which curvature properties can be realized on the manifold. A classical conjecture, due to H. Hopf, is that the Euler characteristic is a basic obstruction to the existence of a metric of nonnegative (or nonpositive) curvature; specifically, if M is a compact, $2n$-dimensional Riemannian manifold, with sectional curvature r,

$$(*)\qquad \begin{aligned} r \geq 0 \quad &\text{implies} \quad \chi(M) \geq 0, \\ r \leq 0 \quad &\text{implies} \quad (-1)^n\chi(M) \geq 0. \end{aligned}$$

This conjecture can be verified in dimensions 2 and 4 by the Gauss–Bonnet–Chern theorem (GBC) [4] (the 4-dimensional result is due to J. Milnor). The purpose of this article is to prove (*) for 6 real-dimensional Kähler manifolds.

The approach taken here is similar to that outlined in [4]. By the GBC, the Euler characteristic $\chi(M)$ is given by the integral over M of a homogeneous polynomial of degree n in the components R_{ijkl} of the Riemann curvature tensor R of M, which we denote by $\chi(R)$. We prove (*) in the case at hand by showing that $\chi(R) \geq 0$ at each point of M.

This theorem is actually a result in what B. O'Neill has dubbed "pointwise geometry", as only algebraic properties of R at a single point of M are used. This pointwise result does not hold in greater generality; that is, there are algebraic curvature tensors R with nonnegative sectional curvature but $\chi(R) < 0$. In [5], R. Geroch has found such an example in dimension six, which is of course non-Kählerian. More recently, Bourguignon and Karcher [3] have found a one-dimensional family of such tensors that are quite nearly Kählerian, the only non-Kähler component being a multiple of the identity operator. These results do not provide a counterexample to (*), however, since to our knowledge no compact manifold has been constructed realizing one of these operators as its curvature tensor at every point.

It should be pointed out that under the stronger hypothesis that the sectional curvature is strictly positive much stronger results have recently been shown. Block and Gieseker [2] have shown that any algebraic vector bundle

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over an algebraic manifold has positive Chern classes if it is ample, i.e., has positive holomorphic bisectional curvature. Furthermore, Mori [13] has verified a conjecture due to Frankel that a compact Kähler manifold M can have strictly positive holomorphic bisectional curvature only if $M = \mathbf{C}P^n$ (this has also been shown by Siu and Yau). Although these results are certainly stronger than those of this article, they are valid only in the much stronger hypothesis of strictly positive curvature.

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1. Algebraic preliminaries

Let V be an n-dimensional real inner product space with inner product $\langle\ ,\ \rangle$, and let $\Lambda^k(V)$ be the *kth* exterior power of V. There is a natural induced inner product structure on $\Lambda^k(V)$ defined as follows: for $\{v_i\}$ an orthonormal basis of V, $\{v_{i_1} \wedge \cdots \wedge v_{i_k}\}_{i_1 < \cdots < i_k}$ is an orthonormal basis of $\Lambda^k(V)$.

An *algebraic curvature operator* R on V is a symmetric linear operator on $\Lambda^2(V)$. The space of all algebraic curvature operators is denoted $\mathscr{R}(V)$. $\mathscr{R}(V)$ is naturally an inner product space with

$$\langle R, S\rangle = \text{trace}\,(RS).$$

$R \in \mathscr{R}(V)$ is *proper* if $b(R) = 0$, where $b\colon \mathscr{R}(V) \mapsto (V)$ is defined by

$$\langle b(R)x \wedge y, z \wedge w\rangle = \langle Rx \wedge y, z \wedge w\rangle + \langle Ry \wedge z, x \wedge w\rangle + \langle Rz \wedge x, y \wedge w\rangle.$$

This is the first Bianchi identity. Let $\mathscr{P}\mathscr{R}(V)$ be the linear subspace of proper operators.

There is a standard embedding of the oriented Grassmann manifold $G(2, V)$ of 2-planes in V into $\Lambda^2(V)$, defined by $P \mapsto v \wedge w$, where $\{v, w\}$ is an oriented orthonormal basis of P. Clearly this embeds $G(2, V)$ into the unit sphere of $\Lambda^2(V)$ [14]. Given $R \in \mathscr{R}(V)$, the *sectional curvature function* $r_R\colon G(2, V) \to \mathbf{R}$ of R is defined by $r_R(P) = \langle RP, P\rangle$.

Remark. If $V = T_*(M, m)$, the tangent space at m of a Riemannian manifold M, clearly all of these algebraic notions correspond to their geometric antecedents.

If $R \in \mathscr{R}(V)$ with dim $(V) = 2n$, define the *Gauss–Bonnet integrand* $\chi(R)$ *of* R to be the polynomial given by, for $\{v_i\}$ an orthonormal basis of V, $\Omega_{ij} = R(v_i \wedge v_j)$, and $*\colon \Lambda^k(V) \to \Lambda^{n-k}(V)$ the Hodge star operator [14],

$$\chi(R) = * \frac{1}{(2\pi)^n} \sum_{i_1, \ldots, i_{2n}} \varepsilon(i_1, \ldots, i_{2n}) \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2n-1} i_{2n}},$$

where the sum ranges over all shuffle permutations of $(1, \ldots, 2n)$ and ε is the sign of the permutation. The geometric significance of this polynomial is, of course, the following theorem, originally due to Allendoerfer and Weil; cf. [4].

THEOREM (1.1). *If* $V = T*(M, m)$, *where* M *is a compact, oriented,* $2n$-*dimensional Riemannian manifold with curvature tensor* R, *then* $\chi(M) = \int_M \chi(R)\, dV$.

Now let V be a $2n$-dimensional inner product space with a fixed complex structure automorphism $J\colon V \to V$, and assume that the inner product is hermitian. Define an element of $\mathscr{R}(V)$, also denoted J, by $J(v \wedge w) = Jv \wedge Jw$. $R \in \mathscr{R}(V)$ will be called *Kähler* if $RJ = JR = R$. As the name implies, the Riemannian curvature tensor of a Kähler manifold satisfies this property. The space of all Kähler operators is denoted by $\mathscr{K}(V)$, the proper ones by $\mathscr{PK}(V)$. If $\{v_1, v_{1*}, \ldots, v_n, v_{n*}\}$ $(v_{j*} = Jv_j)$ is an oriented, hermitian orthonormal basis of V (such a basis will be called *unitary*), let $R_{ijkl*} = \langle Rv_i \wedge v_j, v_k \wedge v_{l*}\rangle$ etc. R is Kähler if and only if $R_{ijk*l*} = R_{ijkl}$, $R_{ij*kl} = R_{ji*kl}$, etc. In the following, Latin indices will run from 1 to n, Greek indices will run over the set $1, 1*, \ldots, n, n*$.

$P \in G(2, V)$ is called *holomorphic* if $JP = P$, *nonholomorphic* otherwise. The space of holomorphic planes is naturally identified with two copies (+ and −) of $\mathbf{C}P(V)$; $P \in +\mathbf{C}P(V)$ (resp. $-\mathbf{C}P(V)$) if $P = v \wedge Jv$ (resp. $-v \wedge Jv$) for v a unit vector in P. The *holomorphic sectional curvature of* $R \in \mathscr{K}(V)$ is defined to be $r_R|_{\mathbf{C}P(V)}$.

It will be useful to note that $\{\xi \in \Lambda^2(V) \mid J\xi = \xi\} \subset \Lambda^2(V)$ is naturally identified with $u(V) \subseteq o(V)$, the skew-hermitian operators embedded into the skew-symmetric operators on V, using the identification of $o(V)$ with $\Lambda^2(V)$ by

$$v \wedge w \mapsto A_{v\wedge w}, \quad \langle A_{v\wedge w}\, x, y\rangle = \langle v \wedge w, x \wedge y\rangle.$$

(There are other identifications also used in the literature, usually differing only in sign.) Then $R \in \mathscr{R}(V)$ is Kähler if and only if $R(\Lambda^2(V)) \subseteq u(V)$, or $R|_{u(V)^\perp} = 0$.

2. Outline of the theorem

The proof of (*) for 4-manifolds in [2] was based on a simple argument. By a clever choice of basis, many of the components R_{ijkl} of the curvature tensor may be assumed to be zero. $\chi(R)$ reduces to a sum of squares and products of sectional curvatures, (*) then follows. Our approach will actually be more simpleminded; we show $\chi(R) \geq 0$ for $r_R \geq 0$ by locating the minimum point K of χ for all $R \in \mathscr{PK}(V)$ with $r_R \geq 0$ (and $|R| = 1$), then showing that $\chi(K)$ cannot be negative. Unfortunately, this last contradiction argument is very complicated; nonetheless, the main result is the following.

THEOREM (2.1). *Let $V \simeq \mathbf{C}^3$. If $R \in \mathscr{P}\mathscr{K}(V)$ satisfies $r_R \geq 0$, then $\chi(R) \geq 0$. If $r_R \leq 0$, $\chi(R) \leq 0$.*

The proof of this theorem consists of several propositions, which are dealt with in the succeeding sections. Only the nonnegative case is proven, the nonpositive case follows from reversing inequalities.

Let $\mathscr{A} = \{R \in \mathscr{P}\mathscr{K}(V) \mid r_R \geq 0 \text{ and } |R| = 1\}$. Consider $\chi|_{\mathscr{A}}$. Let $K \in \mathscr{A}$ be a minimum point of $\chi|_{\mathscr{A}}$. *Assume* that $\chi(K) < 0$.

As a subset of the unit sphere in $\mathscr{P}\mathscr{K}(V)$, the interior

$$\text{Int}(\mathscr{A}) = \{R \in \mathscr{P}\mathscr{K}(V) \mid r_R > 0 \quad \text{and} \quad |R| = 1\}.$$

Assume first that $K \in \text{Int}(\mathscr{A})$. As a function on $\mathscr{P}\mathscr{K}(V)$, the gradient of χ at K, $\nabla\chi(K)$, which can and shall be identified with an element of $\mathscr{P}\mathscr{K}(V)$ in the usual manner (since it is an inner product space), satisfies $\nabla\chi(K) = \lambda K$ by Lagrange multipliers.

PROPOSITION (2.2). *For $V \simeq \mathbf{C}^3$, if $R \in \mathscr{P}\mathscr{K}(V)$, there is an $\bar{R} \in \mathscr{P}\mathscr{K}(V)$ so that $3\chi(R) = \langle R, \bar{R}\rangle$. Furthermore, considering χ as a function on all of $\mathscr{K}(V)$, $\nabla\chi(R) = \bar{R}$, and if $r_R \geq 0$, $r_{\bar{R}} > 0$.*

This proposition will be proven in Section 3. Using Proposition (2.2) it is now easy to show that $K \notin \text{Int}(\mathscr{A})$. If $K \in \text{Int}(\mathscr{A})$, $\bar{K} = \nabla\chi(K) = \lambda K$, $\lambda \geq 0$ necessarily since K and $\bar{K}$ both have nonnegative curvature. But, by assumption on K, $0 > 3\chi(K) = \langle \bar{K}, K\rangle = \lambda$. Thus, K must be on the boundary of $\mathscr{A}$.

Consider the function

$$r: \mathscr{P}\mathscr{K}(V) \times G(2, V) \to \mathbf{R}$$

defined by $r(R, P) = r_R(P)$. Let $W = r^{-1}(0)$. If R_0 has constant positive holomorphic sectional curvature [10],

$$\left.\frac{d}{dt}\right|_0 (r(R + tR_0, P)) = r(R_0, P) > 0,$$

thus ∇r is never zero, so W is smooth. Let

$$\Pi: \mathscr{P}\mathscr{K}(V) \times G(2, V) \to \mathscr{P}\mathscr{K}(V)$$

be the projection onto the first factor. $\Pi^{-1}(K) \cap W = \{(K, P_i) \mid r_K(P_i) = 0\}$ is essentially the minimal set of r_K. Note that the gradient of r at (K, P_i), $\nabla r(K, P_i)$, is in $T^*(\mathscr{P}\mathscr{K}(V), K)$ since $\nabla r_K(P_i) = 0$. Identify $\nabla r(K, P_i)$, and $\nabla\chi(K)$, with their corresponding elements of $\mathscr{K}(V)$ as usual.

PROPOSITION (2.3). *$\nabla\chi(K)$ is in the span of $\{\nabla r(K, P) \mid r_K(P) = 0\} \cup \{K\}$. Furthermore, a finite set $P_1, \ldots, P_k$ of zeroes of r_K may be chosen so that*

$$\bar{K} = \nabla\chi(K) = \sum_{i=1}^{k} \alpha_i \nabla r(K, P_i) + \lambda K,$$

where necessarily $\lambda < 0$.

Proof. The second assertion follows from the first by Proposition (2.2) and the fact that $\langle \nabla r(K, P_i), K \rangle = 0$, which is true since

$$\frac{d}{dt}\bigg|_0 r((1+t)K, P_i) = r(K, P_i) = 0.$$

Thus $\lambda = \langle \nabla \chi(K), K \rangle = \langle \bar{K}, K \rangle = 3\chi(K) < 0$. It is clear that, as a subset of the unit sphere in $\mathscr{P}\mathscr{K}(V)$, the boundary $\partial\mathscr{A}$ is a stratified set, that is, a disjoint union of smooth submanifolds of $\mathscr{P}\mathscr{K}(V)$, since $\partial\mathscr{A}$ is a component of $\partial\Pi(W_0)$, where $W_0 = \{(R, P) \in W \mid |R| = 1\}$, which is a real algebraic variety. Lagrange multipliers may then be applied to the stratum of $\partial\mathscr{A}$ containing K, showing that $\nabla\chi(K)$ is orthogonal to that stratum. Let $R \in \partial\mathscr{A}$, and $M \subseteq \mathscr{A}$ be the stratum of $\partial\mathscr{A}$ containing R, which is a smooth submanifold of $\mathscr{P}\mathscr{K}(V)$. Let R_t be a curve in M with $R_0 = R$, so that

$$\frac{d}{dt}\bigg|_0 R_t \in T*(M, R).$$

Choose $P \in G(2, V)$ so that $r(R, P) = 0$; that is, $(R, P) \in W_0$. Since $R_t \in \partial\mathscr{A} \subset \mathscr{A}$,

$$0 = \frac{d}{dt}\bigg|_0 r(R_t, P) = \left\langle \nabla r(R, P), \left(\frac{d}{dt}\bigg|_0 R_t, 0\right)\right\rangle,$$

so that $\nabla r(R, P) \perp M \subset \mathscr{P}\mathscr{K}(V)$. Thus $\{\nabla r(R, P) \mid (R, P) \in W_0\}$ is contained in the normal space at $R \in \partial\mathscr{A}$ to its stratum M. Now let S be in the normal space to the stratum M of $\partial\mathscr{A}$ at R, considered as a subspace of $\mathscr{P}\mathscr{K}(V)$. Since $\partial\mathscr{A}$ is stratified, S is in the span of $\{\lim_{B\to R} \nu(\partial\mathscr{A}, B) \mid B$ is in a maximal stratum of $\partial\mathscr{A}\}$, where $\nu(\partial\mathscr{A}, B)$ is the normal space at B. The maximal strata of $\partial\mathscr{A}$ are those of codimension one in $\{R \in \mathscr{P}\mathscr{K}(V) \mid |R| = 1\}$. If B is in $\partial\mathscr{A}$, $r_B \geq 0$, but $r_B \not> 0$. Since $\nabla r(B, P) = \Pi*(\nabla_r(B, P))$ is orthogonal to $\partial\mathscr{A}$ at B by the above, it along with B, which spans the normal space to $\{R \mid |R| = 1\}$ at B, must span $\nu(\partial\mathscr{A}, B)$ for any one such $P = P_B$ (which may be chosen continuously). Thus,

$$S \in \operatorname{span}\left\{R, \lim_{B\to R} \nabla r(B, P_B) \mid r_B(P_B) = 0\right\}.$$

If $\bar{P}_B = \lim_{B\to R} P_B$, clearly $r_R(\bar{P}_B) = 0$ by continuity, so that

$$S \in \operatorname{span}\{R, \nabla r(B, P) \mid r(R, P) = 0\}.$$

In particular, the normal space to $\partial\mathscr{A}$ at K is spanned by $\{K, \nabla r(K, P) \mid r_K(P) = 0\}$. The required finite subset $P_1, \ldots, P_k$ of zeroes may be chosen by the finite-dimensionality of $\mathscr{P}\mathscr{K}(V)$. ∎

PROPOSITION (2.4). *Each zero P_i of r_K is necessarily nonholomorphic.*

This proposition will be proven in Section 4, along with some technical relationships among the critical zeroes $\{P_i\}$. Section 5 will use these relationships to provide the various contradiction arguments which will complete the proof of the main theorem.

3. Behavior of the function χ (Proof of Proposition (2.2))

For the moment consider $V \simeq \mathbf{R}^6$. Let $R \in \mathscr{R}(V)$, and let $\{v_i\}$ be an orthonormal basis of V. As in Section 1, the curvature form Ω of R can be defined by $\Omega_{ij} = R(v_i \wedge v_j)$. Following [6], define $\Omega_{ijkl} \in \Lambda^4(V)$ by

$$\Omega_{ijkl} = \Omega_{ij} \wedge \Omega_{kl} - \Omega_{ik} \wedge \Omega_{jl} + \Omega_{il} \wedge \Omega_{jk},$$

and $\Omega_{123456} \in \Lambda^6(V)$ by

$$\Omega_{123456} = \Omega_{12} \wedge \Omega_{3456} - \Omega_{13} \wedge \Omega_{2456} + \Omega_{14} \wedge \Omega_{2356} - \Omega_{15} \wedge \Omega_{2346} + \Omega_{16} \wedge \Omega_{2345}.$$

Then

$$\chi(R) = \frac{1}{8\pi^3} *\Omega_{123456}.$$

An easy calculation shows that the coefficient of R_{1212} in the expression for $\chi(R)$ is

$$\frac{1}{8\pi^3} \langle \Omega_{3456}, v_3 \wedge v_4 \wedge v_5 \wedge v_6 \rangle.$$

Similarly, define $\Omega_{v_i \wedge v_j \wedge v_k \wedge v_l}(v_m \wedge v_n \wedge v_o \wedge v_p)$ to be $\langle \Omega_{ijkl}, v_m \wedge v_n \wedge v_o \wedge v_p \rangle$ and extend linearly to a map $\Omega\colon \Lambda^4(V) \to (\Lambda^4(V))^*$. The coefficient of R_{ijkl} in $\chi(R)$ is then

$$+\frac{1}{8\pi^3} \Omega_{*v_i \wedge v_j}(*v_k \wedge v_l).$$

The sign is always positive since the Hodge star corrects for the negative signs in the expression. Denote by Δ_{ijkl} the operator with a 1 in the $ijkl$-entry (and $klij$-entry), with zeroes elsewhere, and by

$$X_{ijkl}(R) = \begin{cases} \langle \Delta_{ijkl}/\sqrt{2}, R \rangle, & ij \neq kl, \\ \langle \Delta_{ijkl}, R \rangle, & ij = kl, \end{cases}$$

the associated coordinate function. Then

$$\frac{\partial}{\partial X_{ijkl}}(\chi)(R) = \begin{cases} \dfrac{\sqrt{2}}{8\pi^3}\,\Omega_{*v_i\wedge v_j}(*v_k \wedge v_l), & ij \neq kl, \\[2ex] \dfrac{1}{8\pi^3}\,\Omega_{*v_i\wedge v_j}(*v_k \wedge v_l), & ij = kl. \end{cases}$$

Since these coordinates are orthonormal, it is easily seen that

$$\nabla\chi(R) = \frac{1}{8\pi^3}\sum_{ij,kl}\Omega_{*v_i\wedge v_j}(*v_k \wedge v_l)\Delta_{ijkl},$$

so that

$$(\nabla\chi(R))_{ijkl} = \frac{1}{8\pi^3}\,\Omega_{*v_i\wedge v_j}(*v_k \wedge v_l).$$

It now follows (noting that χ is a homogeneous cubic) that

$$\langle \nabla\chi(R), R\rangle = 3\chi(R),$$

so, if $\bar{R} = \nabla\chi(R)$, the proof of Proposition (2.2) will be finished once it is shown that, if $R \in \mathscr{P}\mathscr{K}(V)$, so is $\bar{R}$; and $r_{\bar{R}} \geq 0$ if $r_R \geq 0$. That $\bar{R}$ is Kähler, if R is, is shown in [6]; in that paper Gray also gives a further identity

$$\sum_{(ijklp)} \langle \Omega_{ijkl}, v_p \wedge v_q \wedge v_r \wedge v_s\rangle = 0,$$

where the sum is taken over all cyclic permutations of $(ijklp)$. This identity easily yields the standard first Bianchi identity for $\bar{R}$:

$$\bar{R}_{ijkl} + \bar{R}_{jkil} + \bar{R}_{kijl} = \langle \Omega_{(*v_i\wedge v_j)}, *v_k \wedge v_l\rangle \\ + \langle \Omega_{(*v_j\wedge v_k)}, *v_i \wedge v_l\rangle + \langle \Omega_{(*v_k\wedge v_i)}, *v_j \wedge v_l\rangle.$$

Without loss of generality, i, j, k, l may be assumed to be distinct, otherwise the identity reduces to the fact that $\bar{R} \in \mathscr{R}(V)$. For definiteness, take $(i, j, k, l) = (1, 2, 3, 4)$. Then

$$\bar{R}_{1234} + \bar{R}_{2314} + \bar{R}_{3124} = \langle \Omega_{(v_3\wedge v_4\wedge v_5\wedge v_6)}, v_1 \wedge v_2 \wedge v_5 \wedge v_6\rangle \\ + \langle \Omega_{(v_1\wedge v_4\wedge v_5\wedge v_6)}, v_2 \wedge v_3 \wedge v_5 \wedge v_6\rangle \\ + \langle \Omega_{(v_2\wedge v_4\wedge v_5\wedge v_6)}, v_3 \wedge v_1 \wedge v_5 \wedge v_6\rangle.$$

Gray's identity implies that

$$\begin{aligned}0 = &\langle \Omega_{(v_3\wedge v_4\wedge v_5\wedge v_6)}, v_1\wedge v_2\wedge v_5\wedge v_6\rangle \\ &+ \langle \Omega_{(v_4\wedge v_5\wedge v_6\wedge v_1)}, v_3\wedge v_2\wedge v_5\wedge v_6\rangle \\ &+ \langle \Omega_{(v_5\wedge v_6\wedge v_1\wedge v_3)}, v_4\wedge v_2\wedge v_5\wedge v_6\rangle \\ &+ \langle \Omega_{(v_6\wedge v_1\wedge v_3\wedge v_4)}, v_5\wedge v_2\wedge v_5\wedge v_6\rangle \\ &+ \langle \Omega_{(v_1\wedge v_3\wedge v_4\wedge v_5)}, v_6\wedge v_2\wedge v_5\wedge v_6\rangle.\end{aligned}$$

Since the last two terms are zero and since $\Omega\colon \Lambda^4(V)\to\Lambda^4(V)$ is symmetric, the Bianchi identity follows. Now assume that $r_R \geq 0$. Let P be any plane section; $r_{\bar{R}}(P) = \Omega_{*P}(*P)$, which is nonnegative by essentially Milnor's result, since (up to a positive factor) $\Omega_{*P}(*P)$ is the Gauss–Bonnet integrand of the restriction of R to $\Lambda^2(P^\perp)$. ∎

Remark. It is at this point that it becomes clear that these arguments cannot be readily extended to higher dimensions, due to Geroch's counterexample in the real case.

4. Relations among critical points of K

The following elementary result appears in [11].

LEMMA (4.1). *If $V \simeq \mathbf{C}^2$ and if $R \in \mathscr{P}\mathscr{K}(V)$ satisfies $r_R \geq 0$, then*

$$R\colon \Lambda^2(V)\to\Lambda^2(V)$$

is positive semi-definite.

Proof of Proposition (2.4). Assume that P is a holomorphic zero of the operator K defined in Section 2, a minimum point of $\chi|_{\mathscr{A}}$ with $\chi(K) < 0$. Choose a unitary basis of $V \simeq \mathbf{C}^3$ so that $P = v_1\wedge v_{1*}$. Since $r_K(P) = 0$, $K_{11^*11^*} = 0$. For any such choice of basis, the restriction of K to an operator on $\Lambda^2(v_1\wedge v_{1*}\wedge v_2\wedge v_{2*})$ is positive semi-definite by Lemma (4.1), whence

$$0 \geq K^2_{11^*22^*} - K_{11^*11^*}K_{22^*22^*} = K^2_{11^*22^*},$$

since $\langle K(av_1\wedge c_{1*} + bv_2\wedge v_{2*}), av_1\wedge v_{1*} + bv_2\wedge v_{2*}\rangle \geq 0$. Thus $K_{11^*22^*} = 0$; also, by changing bases,

$$K_{11^*\alpha\beta} = 0 \quad \text{for any } \alpha, \beta \in \{1, 1*, 2, 2*, 3, 3*\}.$$

Applying the first Bianchi identity and the Kähler identities to the equation $K_{11^*22^*} = 0$, $K_{1212} + K_{12^*12^*} = 0$, so that $K_{1212} = K_{12^*12^*} = 0$. Again changing bases, $K_{1\alpha 1\beta} = 0$ for all α, β. Thus, whenever any 2 of α, β, γ, δ are 1 or $1*$, $K_{\alpha\beta\gamma\delta} = 0$. This would imply that $\chi(K) = 0$ since each term of $\chi(K)$ has a factor with two or more indices in $\{1, 1*\}$. ∎

Define $\eta\colon \Lambda^2(V) \to u(V) \subset \Lambda^2(V)$ by $\eta(\xi) = \xi + J\xi$. Up to normalization, η is the orthogonal projection onto $u(V)$. Note that $R(\eta(\xi)) = 2R(\xi)$ for $R \in \mathscr{K}(V)$. Also, if $\eta(P) = \eta(Q)$ (there is a circle of planes P_θ satisfying this for any nonholomorphic plane P), and if P is a critical point of r_R, then so is Q, and moreover $\nabla r(R, Q) = \nabla r(R, P)$. For any choice of unitary basis $\{v_\alpha\}$, define $I \in u(V)$ by

$$I = v_1 \wedge v_{1*} + v_2 \wedge v_{2*} + v_3 \wedge v_{3*}.$$

I is independent of unitary basis chosen, corresponding to $i \cdot (\mathrm{Id}) \in u(V)$ as a skew-hermitian operator on V. It is a simple calculation to show that, for $\xi \in u(V)$, $\langle \xi, I \rangle = -i \operatorname{trace}(\xi)$.

Lemma (4.2). *$\eta(G)/|\eta(G)|$, defined to be $\{\eta(P)/|\eta(P)| \mid P \in G(2, V)\}$, is given by*

$$\eta(G)/|\eta(G)| = \{\xi \in u(V) : |\xi| = 1,\ \xi^3 = 0, \quad \text{and} \quad |\langle \xi, I \rangle| \le 1\}.$$

$|\langle \eta(P)/|\eta(P)|, I \rangle| = 1$ if and only if P is holomorphic.

Proof. Let $P \in G(2, V)$. There is a unitary basis of V so that

$$P = av_1 \wedge v_{1*} + bv_1 \wedge v_2 \quad [10];$$

thus

$$\eta(P)/|\eta(P)| = \frac{(2av_1 \wedge v_{1*} + b(v_1 \wedge v_2 + v_{1*} \wedge v_{2*}))}{(4a^2 + b^2)^{1/2}}.$$

Clearly, then $(\eta(P)/|\eta(P)|)^3 = 0$, and

$$|\langle \eta(P)/|\eta(P)|, I \rangle| = \frac{2|a|}{(4a^2 + b^2)^{1/2}} \le 1,$$

equalling 1 only when $|a| = 1$, $b = 0$. Conversely, let $\xi \in u(V)$. Diagonalizing the operator ξ, there is a unitary basis so that

$$\xi = av_1 \wedge v_{1*} + bv_2 \wedge v_{2*} + cv_3 \wedge v_{3*}.$$

If $\xi^3 = 0$, $abc = 0$; say $c = 0$ for definiteness. If $|\xi| = 1$ but $|\langle \xi, I \rangle| \le 1$, then $|a + b| \le 1$ and $a^2 + b^2 = 1$, so that $ab \le 0$. $ab = 0$ if and only if ξ itself is a holomorphic plane, in which case $\xi = \eta(\xi)/|\eta(\xi)|$. If $ab < 0$, take $a > 0$, $b < 0$. Then for the plane we have

$$P = (\sqrt{a}\, v_1 + \sqrt{-b}\, v_{2*}) \wedge (\sqrt{a}\, v_{1*} + \sqrt{-b}\, v_2), \quad \eta(P)/|\eta(P)| = \xi. \qquad \blacksquare$$

Let P be a (necessarily nonholomorphic) zero of r_K defined above. Choose a unitary basis so that $P = av_1 \wedge v_{1*} + bv_1 \wedge v_2$, $b \neq 0$.

LEMMA (4.3). *With respect to this unitary basis of V,*

$$\nabla r(K, P) = a^2\Delta_{11^*11^*} + (3b^2/16)\Delta_{1212} - (b^2/16)\Delta_{12^*12^*} + (b/2)\Delta_{11^*12} + (b^2/8)\Delta_{11^*22^*},$$

where $\Delta_{\alpha\beta\gamma\delta} \in \mathscr{K}(V)$ *is defined to have a* 1 *in the* $\alpha\beta\gamma\delta$-*component* (*also the* $\gamma\delta\alpha\beta$-, $a*\beta*\gamma\delta$-, *etc.*), *zeroes elsewhere.*

Remark. It can be seen that $\nabla r(K, P)$ has 2 positive, 2 negative eigenvalues, thus an 11-dimensional kernel. From this it will follow that there necessarily are at least 4 distinct zeroes of r_K (where here P_i, P_j are *distinct* only if $\pm\eta(P_i) \neq \eta(P_j)$), although this fact will not be needed.

Proof. Choose coordinate functions $X_{\alpha\beta\gamma\delta}$ on $\mathscr{PK}(V)$ as follows:

$$X_{ii^*ii^*}(R) = \langle \Delta_{ii^*ii^*}, R\rangle$$
$$X_{ijij}(R) = \langle \Delta_{ijij} + \Delta_{ij^*ij^*} + 2\Delta_{ii^*jj^*}, R\rangle/4$$
$$X_{ij^*ij^*}(R) = \langle \Delta_{ijij} - \Delta_{ij^*ij^*}, R\rangle/\sqrt{8},$$
$$X_{ii^*ij}(R) = \langle \Delta_{ii^*ij}, R\rangle/2, \text{ etc.},$$

extending $\Delta_{ii^*ii^*}$, $(\Delta_{ijij} + \Delta_{ij^*ij^*} + 2\Delta_{ii^*jj^*})/4$, etc., to an orthonormal basis of $\mathscr{PK}(V)$. One can easily compute that

$$\partial/\partial X_{11^*11^*}(r)(K, P) = a^2,$$
$$\partial/\partial X_{1212}(r)(K, P) = b^2/4,$$
$$\partial/\partial X_{12^*12^*}(r)(K, P) = b^2/\sqrt{8},$$
$$\partial/\partial X_{11^*12}(r)(K, P) = ab,$$

and all other derivatives vanish, in particular, all derivatives in the $G(2, V)$ factor vanish since P is a critical point of r_K. The claimed expression for $\Delta r(K, P)$ then follows. ∎

LEMMA (4.4). *Let* $\xi \in u(V)$, $V \simeq \mathbf{C}^3$, *satisfy* $\xi^3 = 0$, *and* $R \in \mathscr{PK}(V)$ *satisfy* $r_R \geq 0$. *Then* $\langle R\xi, \xi\rangle \geq 0$. *Furthermore, if* $L \subset u(V)$ *is a subspace of dimension* ≥ 2 *there is a* $\xi \in L$ *with* $\xi^3 = 0$.

Proof. Let $\xi \in u(V)$ satisfy $\xi^3 = 0$. Then there is a unitary basis so that $\xi = av_1 \wedge v_{1^*} + bv_2 \wedge v_{2^*}$. Lemma (4.1), applied to the restriction of R to $\Lambda^2(v_1 \wedge v_{1^*} \wedge v_2 \wedge v_{2^*})$, yields the first statement. For the second, let ξ_0, $\xi_1 \in u(V)$ be independent. Either $\xi_1^3 = 0$, in which case there is nothing to show, or the cubic $*(\xi_0 + t\xi_1)^3$ in t has a real root t_0; $\xi = \xi_0 + t_0\xi_1$ will do. ∎

Let $\{P_i\}_{i=1,\ldots,k}$ be distinct (in the sense that $\pm\eta(P_i)$ are distinct) zeroes of r_K so that, in the notation of Section 2,

$$\bar{K} = \sum_{i=1}^{k} \alpha_i \nabla r(K, P_i) + \lambda K.$$

Proposition (2.3) implies that $r_{\bar{K}} \geq 0$ and $\lambda < 0$, for K as defined earlier, a minimum point of $\chi|_{\mathscr{A}}$ with $r_K \geq 0$, $|K| = 1$, and $\chi(K) < 0$ assumed. In [10] the author proved the following useful result, here stated only in the special case that $r_K(P_i) = 0$, i.e., that P_i is a critical zero of r_K (since $r_K \geq 0$, P_i is critical).

LEMMA (4.5).

$$K(P_i) = B_i {*}\eta(P_i)^2 = 2B_i {*} P_i \wedge JP_i.$$

Moreover,

$$B_i = -\langle KP_i, I\rangle / |\eta(P_i)^2|.$$

The following proposition is the key technical result in this section, and will be used extensively.

PROPOSITION (4.6). *For some $i, j \leq k$, assume that $\langle KP_i, P_j\rangle \neq 0$. Then, there is a P_i' with $\eta(P_i') = \eta(P_i)$ so that there are orthonormal bases $\{x_1, x_2\}$ of P_i', $\{z_1, z_2\}$ of P_j so that:*

(a) $z_1 + Jz_2 = -(x_1 + Jx_2)$;
(b) x_1 *is orthogonal to* Jz_1 *and* z_2;
(c) x_2 *is orthogonal to* Jz_2 *and* z_1.

Note. As remarked earlier, the change from P_i to P_i' does not alter $\nabla r(K, P_i)$, $K(P_i)$, etc.

Proof. By Lemma (4.5), since $\langle KP_i, P_j\rangle \neq 0$, $P_i \wedge JP_i \wedge P_j \neq 0$. Thus

$$P_i \oplus JP_i \oplus P_j \simeq V,$$

so there are $x_1, x_2 \in P_i$, $y_1, y_2 \in JP_i$, and $z_1, z_2 \in P_j$ so that

$$JP_j = (x_1 + y_1 + z_1) \wedge (x_2 + y_2 + z_2)$$

with $\{(x_1 + y_1 + z_1), (x_2 + y_2 + z_2)\}$ orthonormal. From either [10] or [14] it is easy to show:

LEMMA (4.7). *If $z \in P_j$ or JP_j, then $\langle K(JP_j), v \wedge z\rangle = 0$ for any v.*

Proof. In [10] and [14] it is shown that, if v is orthogonal to JP_j, $z \in JP_j$, then $\langle K(JP_j), v \wedge z\rangle = 0$ since JP_j is critical. This extends to v not orthogonal to JP_j since the critical value is zero, and to $z \in P_j$ by the Kähler identities. ∎

By this lemma, $0 = \langle K(JP_j), (x_n + y_n + z_n) \wedge v\rangle$, where $v \in \{x_1, x_2, y_1, y_2\}$ and $n = 1, 2$, and also $0 = \langle K(JP_j), (x_n + y_n) \wedge v\rangle$ since $z_n \in P_j$. In particular,

$$0 = \langle K(JP_j), \xi\rangle$$

if

$$\xi \in \{x_1 \wedge y_1, x_1 \wedge x_2 + y_1 \wedge x_2, x_1 \wedge x_2 + x_1 \wedge y_2, \\ x_2 \wedge y_2, x_1 \wedge y_2 + y_1 \wedge y_2, x_2 \wedge y_1 + y_2 \wedge y_1\}.$$

Since

$$\begin{aligned} 0 &= \langle K(JP_j), (x_1 \wedge x_2 + x_1 \wedge y_2) - (x_1 \wedge y_2 + y_1 \wedge y_2)\rangle \\ &= \langle K(JP_j), x_1 \wedge x_2 + y_1 \wedge y_2\rangle, \end{aligned}$$

and

$$x_1 \wedge x_2 = \lambda_x P_i, \quad y_1 \wedge y_2 = \lambda_y JP_i,$$

necessarily $\lambda_x = \lambda_y$ because $\langle K(JP_j), P_i\rangle \neq 0$. First assume that $\lambda_x = \lambda_y = 0$. Then $x_2 = \alpha x_1$, $y_2 = \beta y_1$. Let $\{v, w\}$ be an orthonormal basis of P_i. Then

$$P_i(\theta) = (\cos\theta v + \sin\theta Lv) \wedge (\cos\theta w + \sin\theta Jw)$$

satisfies $\eta(P_i(\theta)) = \eta(P_i)$ ($P_i(\theta)$ parametrizes $\eta^{-1}(\eta(P_i))$ [10]), $P_i(0) = P_i$, and $P_i(\pi/2) = JP_i$. By varying θ from 0 to $\pi/2$, the roles of P_i and JP_i are reversed. However, $0 \neq P_i(\theta) \wedge JP_i(\theta) \wedge P_j$ still, so the above construction may be repeated, continuously in θ for $P_i(\theta)$, $JP_i(\theta)$. Passing from $\theta = 0$ to $\theta = \pi/2$ then reverses the roles of α and β; thus, for some θ, $\alpha(\theta) = \beta(\theta)$. Dropping the θ's, it may be assumed that $\alpha = \beta$, so

$$JP_j = (x_1 + y_1 + z_1) \wedge (\alpha(x_1 + y_1) + z_2).$$

Rotating the orthonormal basis $\{(x_1 + y_1 + z_1), (\alpha(x_1 + y_1) + z_2)\}$ in JP_j, this basis can be replaced by $\{(x_1' + y_1' + z_1'), (\alpha'(x_1' + y_1') + z_2')\}$, where

$$\begin{aligned} x_1' + y_1' + z_1' &= \cos t(x_1 + y_1 + z_1) + \sin t(\alpha(x_1 + y_1) + z_2) \\ &= (\cos t + \alpha \sin t)(x_1 + y_1) + ((\cos t)z_1 + (\sin t)z_2) \end{aligned}$$

and

$$\begin{aligned} \alpha'(x_1' + y_1') + z_2' &= -\sin t(x_1 + y_1 + z_1) + \cos t(\alpha(x_1 + y_1) + z_2) \\ &= (-\sin t + \alpha\cos t)(x_1 + y_1) + (-\sin t)z_1 + (\cos t)z_2. \end{aligned}$$

α_2' is then given by $\alpha' = (-\sin t + \alpha\cos t)/(\cos t + \alpha\sin t)$, so that for an appropriate choice of t, $\alpha' = 0$. But this then implies $JP_j \wedge P_j = 0$, contradicting the fact that P_j is necessarily nonholomorphic.

Hence $\lambda_x = \lambda_y \neq 0$, implying that $\{x_1, x_2\}$ is a basis of P_i, and $\{y_1, y_2\}$ is a basis of JP_i. Thus, there are numbers a, b, c, and d so that

$$y_1 = aJx_1 + bJx_2, \quad y_2 = cJx_1 + dJx_2.$$

$J(x_1 \wedge x_2) = y_1 \wedge y_2$ implies that $ad - bc = 1$. Returning to the equations $0 = \langle K(JP_j), \xi\rangle$ above, and substituting for y_1, y_2 in terms of x_1 and x_2, $K(JP_j) = K(P_j)$ is orthogonal to

$$\{ax_1 \wedge Jx_1 + bx_1 \wedge Jx_2,\ x_1 \wedge x_2 + aJx_1 \wedge x_2 + bJx_2 \wedge x_2,\ x_1 \wedge x_2 + cx_1 \wedge Jx_1 + dx_1 \wedge Jx_2,\ cx_2 \wedge Jx_1 + dx_2 \wedge Jx_2\}.$$

Using the Kähler identities, $K(JP_j)$ would be orthogonal to all of $\Lambda^2(P_i \wedge JP_i)$, implying in particular that $\langle K(P_j), P_i\rangle = 0$ in contradiction to the hypotheses of this proposition, unless

$$0 = \begin{vmatrix} a & 0 & b & 0 \\ 0 & -1 & a & b \\ c & 1 & d & 0 \\ 0 & 0 & c & d \end{vmatrix} = -(a + d)(ad - bc).$$

Thus $d = -a$. A rotation of the basis $\{(x_1 + y_1 - z_1), (x_2 + y_2 + z_2)\}$ corresponds to the adjoint action of $SO(2)$ on

$$\begin{bmatrix} ab \\ cd \end{bmatrix} \in Sl(2, \mathbf{R}),$$

so by an appropriate rotation it may be assumed that $a = d = 0$, and $bc = -1$.

Thus far,

$$JP_j = (x_1 + bJx_2 + z_1) \wedge (x_2 + cJx_1 + z_2).$$

Since $P_i \wedge JP_i \wedge P_j \neq 0$, $P_i \wedge JP_i \wedge JP_j \neq 0$ by hypothesis, $\{z_1, z_2\}$ are linearly independent. Then there are numbers α, β, γ, and δ so that

$$\alpha Jz_1 + \beta Jz_2 = x_1 + bJx_2 + z_1, \quad \gamma Jz_1 + \delta Jz_2 = x_2 + cJx_1 + z_2$$

Applying bJ to the second equation, and subtracting that from the first, yields $\alpha = \delta = 0$, $\beta = -b$, $\gamma = -c$. However, the right-hand sides are chosen to be orthonormal, thus $\{cz_1, bz_2\}$ are orthonormal; that is,

$$\langle z_1, z_2\rangle = 0, \quad \langle z_2, z_2\rangle = 1/b^2, \quad \langle z_1, z_1\rangle = 1/c^2.$$

The situation is now completely symmetric with respect to P_i and P_j; so, repeating the above with the roles of P_i and P_j reversed,

$$\langle x_1, x_2\rangle = 0, \quad \langle x_1, x_1\rangle = 1/c^2, \quad \langle x_2, x_2\rangle = 1/b^2.$$

However, $P_i = x_1 \wedge x_2$ may be replaced with any $P_i' \in \eta^{-1}\eta(P_i)$,

$$P_i' \in \{P_i(\theta) = (\cos\theta x_1 + \sin\theta Jx_1) \wedge (\cos\theta x_2 + \sin\theta Jx_2)\}$$

as before. If

$$x_1' = x_1 \cos\theta + Jx_1 \sin\theta,\ x_2' = x_x \cos\theta + Jx_2 \sin\theta,$$

then

$$x_1'' = \cos\theta x_1' + b\sin\theta x_2',\ x_2'' = \cos\theta x_2' + c\sin\theta x_1'$$

also satisfies

$$-bJz_2 = x_1'' + bJx_2'' + z_1.$$

The symmetry between P_i and P_j also holds between $P_i(\theta)$ and P_j, so $\langle x_1'', x_2''\rangle = 0$. Since this holds for all θ, an easy calculation shows $1/c + 1/b = 0$, thus $b = \pm 1$, $c = \mp 1$. Choose signs (switch to $-P_i$ if need be) so that $b = 1$, $c = -1$. This now yields equation (a) of the proposition, along with the fact that $\{x_1, x_2\}$ and $\{z_1, z_2\}$ are orthonormal.

For equation (b), $0 = \langle -(z_1 + Jz_2), Jz_1\rangle = \langle x_1, Jz_1\rangle + \langle x_2, z_1\rangle$. However, this again holds for $P_i(\theta)$:

$$\langle x_1'', Jz_1\rangle = (\cos^2\theta - \sin^2\theta)\langle x_1, Jz_1\rangle + \cos\theta\sin\theta(\langle x_1, z_1\rangle + \langle x_2, Jz_1\rangle).$$

By an appropriate choice of θ, $\langle x_1'', Jz_1\rangle$ may be assumed to be zero. Dropping the primes, we may choose P_i so that $\langle x_1, Jz_1\rangle = 0 = \langle x_2, z_1\rangle$. Equation (c) follows after a short calculation. ∎

COROLLARY (4.8). *In addition to the conditions of the proposition, if* $\langle KP_i, P_j\rangle \neq 0$ *the orthonormal bases* $\{x_1, x_2\}$ *of* P_i' *and* $\{z_1, z_2\}$ *of* P_j *chosen also satisfy*

$$\langle x_1, z_1\rangle = \langle x_2, z_2\rangle,\ \langle x_2, Jz_1\rangle = \langle z_2, Jx_1\rangle,$$

and

$$\langle z_2, Jz_1\rangle = \langle x_2, Jx_1\rangle.$$

Remark. The last equation is independent of the choice of $P_i(\theta)$ above. Note that, by [10], $\langle x_2, Jx_1\rangle^2 = \langle P_i, JP_i\rangle = \langle P_i, I\rangle^2$, which in a sense measures the degree to which P_i is nonholomorphic. The corollary implies in particular that this degree is the same for each critical plane P_j such that $\langle KP_i, P_j\rangle \neq 0$.

Proof.

$$1 = \langle Jz_1, Jz_1\rangle = \langle x_1 + Jx_2 + Jz_2, x_1 + Jx_2 + Jz_2\rangle$$
$$= 3 + 2\langle x_1, Jx_2\rangle + 2\langle x_1, Jz_2\rangle + 2\langle x_2, z_2\rangle,$$

so

$$1 = \langle x_2, Jx_1\rangle - \langle x_2, z_2\rangle + \langle z_2, Jx_1\rangle.$$

However,

$$1 = \langle Jx_1, Jx_1\rangle = \langle Jx_1, x_2 - Jz_1 + z_2\rangle = \langle Jx_1, x_2\rangle - \langle x_1, z_1\rangle + \langle Jx_1, z_2\rangle,$$

thus, $\langle x_1, z_1\rangle = \langle x_2, z_2\rangle$.

$$\begin{aligned}\langle x_2, Jz_1\rangle &= \langle x_2, x_2 - Jx_1 + z_2\rangle\\ &= 1 - \langle x_2, Jx_1\rangle + \langle x_2, z_2\rangle\\ &= 1 - (1 - \langle z_2, Jx_1\rangle) = \langle z_2, Jx_1\rangle.\end{aligned}$$

Finally, $\langle x_2, Jx_1\rangle = 1 + \langle x_2, z_2\rangle - \langle z_2, Jx_1\rangle$, and

$$\begin{aligned}\langle z_2, Jz_1\rangle &= \langle z_2, z_2 - Jx_1 + x_2\rangle\\ &= 1 - \langle z_2, Jx_1\rangle + \langle z_2, x_2\rangle\\ &= \langle x_2, Jx_1\rangle. \blacksquare\end{aligned}$$

COROLLARY (4.9). *If* $\langle KP_j, P_i\rangle \neq 0$, *then*

$$\Pi_{\Lambda^2(P_i\wedge JP_i)}(KP_j) = \alpha(P_i + JP_i) + \beta I_{\Lambda^2(P_i\wedge JP_i)},$$

where $\Pi_{\Lambda^2(P_i\wedge JP_i)}$ *is the orthogonal projection onto* $\Lambda^2(P_i\wedge JP_i)$, *and*

$$I_{\Lambda^2(P_i\wedge JP_i)} \in u(P_i\wedge JP_i)$$

is given by

$$I_{\Lambda^2(P_i\wedge JP_i)} = v_1\wedge v_{1*} + v_2\wedge v_{2*}$$

for any unitary basis of $P_i\wedge JP_i$. *Furthermore, there are only two possible choices of* α/β, *which depend only on* P_i.

Proof. Within the proof of Proposition (4.6) (set $a = d = 0$, $b = 1$, $c = -1$) it has been shown that $K(P_j) = K(JP_j)$, since not orthogonal to all of $\Lambda^2(P_i\wedge JP_i)$, is orthogonal to

$$\{x_1\wedge Jx_2, x_1\wedge x_2 - x_2\wedge Jx_2, x_1\wedge x_2 - x_1\wedge Jx_1\}.$$

Since $K(P_j) \in u(V)$, and, recalling that $P_i = x_1\wedge x_2$, $K(P_j)$ is orthogonal to

$$\{x_1\wedge Jx_2 + x_2\wedge Jx_1, P_i + JP_i - 2x_2\wedge Jx_2, P_i + JP_i - 2x_1\wedge Jx_1\}.$$

However, it is clear that $u(P_i\wedge JP_i) = u(V) \cap \Lambda^2(P_i\wedge JP_i)$ is spanned by these three vectors and $I_{\Lambda^2(P_i\wedge JP_i)}$. Thus, consider the orthonormal basis

$$\Big\{(x_1\wedge Jx_2 + x_2\wedge Jx_1)/\sqrt{2}, (x_1\wedge Jx_1 - x_2\wedge Jx_2)/\sqrt{2},$$
$$(P_i + JP_i)/\sqrt{2 - 2\langle P_i, JP_i\rangle},$$
$$\left(I_{\Lambda^2(P_i\wedge JP_i)} - \frac{\langle I, P_i\rangle}{2 - 2\langle P_i, JP_i\rangle}(P_i + JP_i)\right)\Big/|\cdot|\Big\};$$

$K(P_j)$ is orthogonal to the first two vectors, so

$$K(P_j) = \alpha(P_i + JP_i) + \beta I_{\Lambda^2(P_i\wedge JP_i)}.$$

Also, up to scale, $K(P_j)$ is a holomorphic plane by Lemma (4.5); thus $K(P_j)^2 = 0$. If $\langle P_i, I\rangle = a$, then $0 = (2a\alpha + \beta)\beta - (1 - a^2)\alpha^2$, so

$$\alpha/\beta = \frac{1}{1-a} \quad \text{or} \quad \frac{-1}{1+a}. \quad \blacksquare$$

PROPOSITION (4.10). *If $K(P_i) \neq 0$, and $\langle KP_i, P_j\rangle = 0$, then*

$$P_j \in \Lambda^2(P_i \wedge JP_i).$$

Proof. First assume that $KP_j \neq 0$ as well. Then necessarily

$$P_i \wedge JP_i \wedge P_j = 0 = P_j \wedge JP_j \wedge P_i$$

by Lemma (4.5). Thus, for any a, b, $0 = (a\eta(P_i) + b\eta(P_j))^3$. Moreover,

$$0 = \langle K(a\eta(P_i) + b\eta(P_j)), a\eta(P_i) + b\eta(P_j)\rangle$$

so that, since there are no holomorphic zeroes of r_K, each $a\eta(P_i) + b\eta(P_j)$ is, up to scale, $\eta(Q)$ for a non-holomorphic zero of r_K. Pick a, b so that

$$\langle K(a\eta(P_i) + b\eta(P_j)), I\rangle = 0,$$

which is clearly possible by linearity (with not both a and b zero). Since $K(P_i) \neq 0$, $K(P_j) \neq 0$, $ab \neq 0$. But, Lemma (4.5) then implies that (since $a\eta(P_i) + b\eta(P_j) = \eta(Q)$ for Q a zero of r_K)

$$K(a\eta(P_i) + b\eta(P_j)) = 0;$$

thus $*P_i \wedge JP_i = \mu K(P_i) = \lambda K(P_j) = \tau * P_j \wedge JP_j$, with none of the scale factors zero. Thus $P_j \in \Lambda^2(P_j \wedge JP_j) = \Lambda^2(P_i \wedge JP_i)$, completing the proposition in this case.

Now assume that $K(P_j) = 0$. If $P_j \wedge JP_j \wedge P_i \neq 0$, using the proof of Proposition (4.6), which depends only on the condition $P_j \wedge JP_j \wedge P_i \neq 0$ to a certain point, it can be shown that $K(P_i) = K(JP_i)$ is orthogonal to

$$\{x_1 \wedge Jx_2, x_1 \wedge x_2 - bx_2 \wedge Jx_2, x_1 \wedge x_2 + cx_1 \wedge Jx_1\}$$

with $x_1 \wedge x_2 = P_1$, $\langle x_1, x_2\rangle = 0$ and $bc = -1$. Beyond that point (where it is determined that $b = \pm 1$) symmetry between P_i and P_j is assumed; thus it cannot be concluded here that $b = 1$ and $c = -1$ as was the case when $\langle K(P_i), P_j > \neq 0$. Nontheless, similarly to Corollary (4.9),

$$K(P_i) = \alpha(P_j + JP_j) + \beta I_{\Lambda^2(P_j \wedge JP_j)},$$

as can be seen by precisely the same argument, with α, β both necessarily nonzero (since $K(P_i)$ is, up to scale, a holomorphic plane). In fact,

$$\alpha/\beta = \frac{1}{1-a} \quad \text{or} \quad -\frac{1}{1+a},$$

where $a = \langle P_j, I\rangle$, as in Corollary (4.9). On the other hand,

$$\begin{aligned} 0 &= \langle K(P_i), P_j\rangle \\ &= \langle \alpha(P_j + JP_j) + \beta I_{\Lambda^2(P_j \wedge JP_j)}, P_j\rangle \\ &= \alpha(1 + a^2) + \beta a, \end{aligned}$$

since $\langle P_j, JP_j\rangle = \langle P_j, I\rangle^2$. Thus

$$0 = \beta\left(\frac{1}{1-a}(1 + a^2) + a\right) = \beta(1 + a),$$

or

$$0 = \beta\left(-\frac{1}{1+a}(1 + a^2) + a\right) = \beta(\alpha - 1),$$

which is contradicted by the facts that $\beta \neq 0$ and P_j is nonholomorphic.

Therefore, $P_j \wedge JP_j \wedge P_i = 0$ necessarily. Again, for any a, b,

$$(a\eta(P_i) + b\eta(P_j))^3 = 0,$$

and

$$\langle K(a\eta(P_i) + b\eta(P_j)), a\eta(P_i) + b\eta(P_j)\rangle = 0.$$

Arguing as above, up to a scale factor, $a\eta(P_i) + b\eta(P_j) = \eta(Q)$ for a nonholomorphic zero Q of r_K. Lemma (4.5) implies that

$$K(a\eta(P_i) + b\eta(P_j)) = \mu_{\alpha\beta}{*}(a\eta(P_i) + b\eta(P_j))^2.$$

The current assumptions imply that $K(a\eta(P_i) + b\eta(P_j)) = a\alpha{*}\eta(P_i)^2$; thus

$$a\alpha{*}\eta(P_i)^2 = \mu_{ab}(a^2{*}\eta(P_i)^2 + 2ab{*}\eta(P_i) \wedge \eta(P_j) + b^2{*}\eta(P_j)^2).$$

Hold a fixed (α is constant, but μ_{ab} depends on a and b); then,

$$\left(\frac{a\alpha}{\mu_{ab}} - a^2\right){*}\eta(P_i)^2 = 2ab{*}\eta(P_i) \wedge \eta(P_j) + b^2{*}\eta(P_j)^2.$$

Since ${*}\eta(P_i)^2 \neq 0$ and ${*}\eta(P_j)^2 \neq 0$, as $b \to \infty$, $a\alpha/\mu_{ab} - a^2$ is asymptotic to b^2, and ${*}\eta(P_i)^2 = {*}\eta(P_j)^2$, completing the proof of the Proposition. ▮

5. Final arguments

We are now prepared to complete the proof of the main theorem of this paper. The method will be to analyze the various possibilities for $\langle KP_i, P_j\rangle$, using the results of the previous section to derive a contradiction in each case. Recall that K is a minimum of χ, the Gauss–Bonnet integrand, on $\mathscr{A} =$

$\{R \in \mathscr{PK}(V) \mid r_R \geq 0, |R| = 1\}$, with $V \simeq \mathbf{C}^3$. $\{P_i\}_{i=1,\ldots,k}$ is a chosen set of distinct zeroes of r_K, so that

$$\bar{K} = \sum_{i=1}^{k} \alpha_i \nabla r(K, P_i) + \lambda K \quad \text{with } \lambda < 0.$$

The assumption to be contradicted at each point is that $\chi(K) < 0$. Note that, without loss of generality, it may be assumed that

$$\pm(P_i + JP_i) \neq (P_j + JP_j) \quad \text{for } i \neq j,$$

since if that condition is not satisfied one of the points could be eliminated from the expression for $\bar{K}$.

LEMMA (5.1). *If Q is a holomorphic plane orthogonal to P_i, then*

$$\langle \nabla r(K, P_i)Q, Q\rangle = 0.$$

Proof. Choose a unitary basis so that

$$P_i = av_1 \wedge v_{1*} + bv_1 \wedge v_2, \quad b \neq 0,$$

$$Q = \sum_{\alpha < \beta} q_{\alpha\beta} v_\alpha \wedge v_\beta, \quad JQ = Q.$$

By hypothesis, $aq_{11*} + bq_{12} = 0$. Using the expression for $\nabla r(K, P_i)$ given in Lemma (4.3), an easy calculation shows

$$\langle \nabla r(K, P_i)Q, Q\rangle = a^2 q_{11*}^2 + \frac{3b^2}{4} q_{12}^2 - \frac{b^2}{4} q_{12*}^2 + 2abq_{11*}q_{12} + \frac{b^2}{4} q_{11*}q_{22*}.$$

Since $Q \wedge Q = 0$, $q_{11*}q_{22*} = q_{12}^2 + q_{12*}^2$; thus

$$\langle \nabla r(K, P_i)Q, Q\rangle = a^2 q_{11*}^2 + 2abq_{11*}q_{12} + b^2 q_{12}^2 = 0. \quad ∎$$

PROPOSITION (5.2) *$\langle KP_i, P_j\rangle$ cannot be zero for all i, j.*

Proof. Assume $\langle KP_i, P_j\rangle = 0$ for all i, j. By re-ordering the indices, let l be so that $K(P_i) = 0$ for $i \leq l$, $K(P_i) \neq 0$ for $i > l$.

Case 1. $l \neq k$. Then $K(P_k) = B_k P_k \wedge JP_k$ where $B_k \neq 0$. $\langle KP_k, P_i\rangle = 0$ implies that $\langle *P_k \wedge JP_k, P_i\rangle = 0$ for all i. Let $Q = *P_k \wedge JP_k / |P_k \wedge JP_k|$. Using the expression for $\bar{K}$ given in Proposition (2.3), $r_{\bar{K}}(Q) = \lambda r_K(Q) < 0$ since $\lambda < 0$, contradicting the nonnegativity of $r_{\bar{K}}$.

Case 2. $l = k$. Assume there are i, j with $P_i \wedge P_j \wedge JP_j \neq 0$. Let

$$\xi_0 = a\eta(P_i) + b\eta(P_j), \quad ab \neq 0.$$

Then $\xi_0^3 \neq 0$ for almost all such a, b, and $\xi_0 \in \ker(K)$. By an umpublished result of B. Kostant (cf. [12]), since $\chi(K) < 0$ there must be a negative eigenvalue of K: $\Lambda^2(V) \to \Lambda^2(V)$. Let $\xi_1 \in u(V)$ with $K(\xi_1) = \mu\xi_1$ where $\mu < 0$.

By Lemma (4.4), $(\xi_1)^3 \neq 0$. Let t_0 be a zero of the cubic $*(\xi_0 + t\xi_1)^3$, and let $\xi = \xi_0 + t_0\xi_1$. $\xi^3 = 0$ and $\langle K\xi, \xi\rangle < 0$, which is contradicted by Lemma (4.4). Thus $P_i \wedge P_j \wedge JP_j = 0$ for all i, j. But then $\langle *P_1 \wedge JP_1, P_i\rangle = 0$ for all i, and the argument of case 1 may be applied again to derive the final contradiction needed for this proposition. ∎

PROPOSITION (5.3). *For any i, $\langle KP_i, P_j\rangle$ cannot be nonzero for all j.*

Proof. Assume that $\langle KP_i, P_j\rangle \neq 0$ for all j. Using Proposition (4.6) and Corollary (4.8), let x_n^j, $n = 1, 2$, $j = 1, \ldots, k$, be so that $\{x_1^j, x_2^j\}$ is an orthonormal basis of P_j, and $\pm(x_1^i + Jx_2^i) = x_1^j + Jx_2^j$. The 4-dimensional subspaces $P_j \wedge JP_j$ all intersect in the holomorphic plane

$$Q = (x_1^j + Jx_2^j) \wedge (Jx_1^j - x_2^j)/(2 + 2\langle x_1^j, Jx_2^j\rangle)$$

with the scale factor independent of j by Corollary (4.8). Choose a unitary basis so that $x_1^j = v_1$, $x_2^j = av_{1*} + bv_2$, as usual. Then

$$Q = [(1-a)^2 v_1 \wedge v_{1*} - b(1-a)(v_1 \wedge v_2 + v_{1*} \wedge v_{2*}) + b^2 v_2 \wedge v_{2*}]/(2-2a).$$

By Lemma (4.3),

$$\langle \nabla r(K, P_j)Q, Q\rangle = (1-a)^2/4,$$

which may be readily computed. Thus $0 < \langle \nabla r(K, P_i)Q, Q\rangle = \langle \nabla r(K, P_j)Q, Q\rangle$ for all j, since a is independent of j. By Proposition (2.3),

$$\begin{aligned} 0 \leq r_{\bar{K}}(Q) &= \lambda r_K(Q) + \sum_{j=1}^{k} \alpha_j \langle \nabla r(K, P_j)Q, Q\rangle \\ &= \lambda r_K(Q) + \left(\sum_{j=1}^{k} \alpha_j\right) \frac{(1-a)^2}{4}. \end{aligned}$$

Since $\lambda < 0$ by hypothesis, $r_k(Q) > 0$ by Proposition (2.4), and $1 - a^2 > 0$ since P_j is nonholomorphic, $\sum_{j=1}^k \alpha_j > 0$.

Restrict $\nabla r(K, P_j)$ to an operator on the tangent space

$$T*(G(2, V), Q) = \{P \in G(2, V) \mid P \wedge Q = 0, \langle P, Q\rangle = 0\};$$

denote this restriction by $\nabla r(K, P_j)|_{T*(G,Q)}$.

LEMMA (5.4).

$$\text{Trace } (\nabla r(K, P_j)|_{T*(G,Q)}) = \frac{a^2 - 1}{4} < 0.$$

Proof. Recall that $a = \langle P_j, I\rangle$, and is independent of j. Let

$$x_1^j = v, \quad \text{and} \quad x_2^j = av_{1*} + bv_2$$

as before. $T*(G, Q)$ is spanned by

$$(x_1^j + Jx_2^j) \wedge v_3, \quad (x_1^j + Jx_2^j) \wedge v_{3*},$$
$$(x_1^j + Jx_2^j) \wedge (x_2^j + Jx_1^j), \quad (x_1^j + Jx_2^j) \wedge (-x_1^j + Jx_2^j),$$

and their images under J; so an orthonormal basis of $T*(G, Q)$ consists of

$$v_1 \wedge v_{2*}, \quad v_2 \wedge v_{1*}, \quad [b(v_1 \wedge v_{1*} - v_2 \wedge v_{2*}) - a(v_1 \wedge v_2 + v_{1*} \wedge v_{2*})]/\sqrt{2},$$
$$(v_1 \wedge v_2 - v_{1*} \wedge v_{2*})/\sqrt{2},$$

and 4 more vectors involving v_3 or v_{3*}, which will not enter into the calculation. The lemma follows by computing the trace using this orthonormal basis and Lemma (4.3). ∎

For all $T \in T*(G, Q)$, $(T + JT)^3 = 0$ and $\langle T, I \rangle = 0$, since Q is a holomorphic plane. Lemma (4.2) then implies that, up to scale, $T + JT = \eta(P)$ for some nonholomorphic plane P. Lemma (4.4) then implies that

$$K|_{T*(G,Q)} \quad \text{and} \quad \bar{K}|_{T*(G,Q)}$$

must be positive semi-definite. However, then

$$\begin{aligned} 0 &\leq \text{trace } (\bar{K}|_{T*(G,Q)}) \\ &= \lambda(\text{trace } K|_{T*(G,Q)}) + \sum_{j=1}^{k} \alpha_j \text{ trace } (\nabla r(K, P_j)|_{T*(G,Q)}) \\ &= \lambda(\text{trace } K|_{T*(G,Q)}) + \left(\sum_{j=1}^{k} \alpha_j\right)\left(\frac{a^2 - 1}{4}\right), \end{aligned}$$

by Lemma (5.4). The first term is nonpositive since $\lambda < 0$ and $K|_{T*(G,Q)}$ is positive semidefinite. Also, $(a^2 - 1)/4 < 0$; thus $\sum_{j=1}^{k} \alpha_j \leq 0$ in contradiction to the above, completing the proof of the Proposition. ∎

PROPOSITION (5.5). *Assume that for some i, there is a j so that $\langle KP_i, P_j \rangle \neq 0$, and an l so that $\langle KP_i, P_l \rangle = 0$. This also contradicts the hypotheses on K.*

Remark. Since this exhausts all possible cases, the proof of this proposition will complete the proof of Theorem (2.1).

Proof. Since $\langle KP_i, P_j \rangle \neq 0$, in particular $KP_i \neq 0$. Proposition (4.10) then implies that $P_l \in \Lambda^2(P_i \wedge JP_i)$, for any P_l so that $\langle KP_i, P_l \rangle = 0$. If in addition $\langle KP_j, P_l \rangle = 0$, then $P_l \in \Lambda^2(P_j \wedge JP_j)$. But, since $P_i \wedge JP_i$ is a complex 2-dimensional subspace of V as is $P_j \wedge JP_j$,

$$JP_l \in \Lambda^2(P_i \wedge JP_i) \quad \text{and} \quad JP_l \in \Lambda^2(P_j \wedge JP_j).$$

Thus $P_l \wedge JP_l = \alpha P_i \wedge JP_i$, and $P_l \wedge JP_l = \beta P_j \wedge JP_j$, which contradicts the fact that $\langle KP_i, P_j \rangle \neq 0$. Thus, for all $h \neq j$, $\langle KP_j, P_h \rangle \neq 0$. However, Proposition (5.3) precludes this possibility, completing the proof of the Proposition and, finally, finishing the proof of Theorem (2.1). ▮

Judging from the extremely complicated arguments needed to establish this result, and the strong dependence on the algebra of $\Lambda^2(V)$ for $V \simeq \mathbf{C}^3$, it is highly unlikely that this proof can be modified to higher dimensions. The work of Geroch [5] also contraindicates this possibility, even though his example is non-Kählerian. The separate examples due to Bourguignon and Karcher [3] suggest even more strongly that these pointwise methods will make little more progress into this problem, since their examples are almost Kähler (in the technical as well as heuristic sense).

On the other hand, the work of Bloch and Gieseker [2], Mori [13], and Siu and Yau described earlier suggests that, at least for Kähler manifolds, nonnegative curvature is a rather strong restriction, since positive curvature is completely rigid. Thus it is still quite possible that (*) will be true, and provable, for Kähler manifolds, although much stronger analytic methods will certainly be needed. The full conjecture for all compact, even-dimensional manifolds will probably be considerably more difficult, if it is possible at all to resolve.

Remark. Recent work of A. Gray [8] seems to indicate that, using this result, it should be possible to show that similar inequalities will hold for other Chern numbers involving c_3 for higher-dimensional Kähler manifolds.

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