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N-DIMENSIONAL SUBMANIFOLDS OF $\mathbf{R}^{N+1}$ AND S^{N+2}

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1. Introduction

(1.1) Let $x\colon M^n \to Q_c^{n+q}$ be an immersion of a differentiable manifold M^n into a $(n+q)$-dimensional Riemannian manifold of constant curvature c and let $\alpha\colon T_pM \times T_pM \to (T_pM)^\perp$ be the second fundamental form of x at $p \in M$; here T_pM is the tangent space of M at p and $(T_pM)^\perp$ is the orthogonal complement of $dx_p(T_pM)$ in $T_{x(p)}Q_c$. We say that $U_p \subset T_pM$ is an *umbilic subspace* of x at p if $\langle \alpha(X, Y), \xi \rangle = \lambda \langle X, Y \rangle$, λ a constant, for all $X \in U_p$, all $Y \in T_pM$ and all $\xi \in (T_pM)^\perp$, where $\langle\ ,\ \rangle$ denotes both the Riemannian metric on Q_c and the Riemannian metric on M induced by x. Recently, it was shown (cf. [1]) that if M^n can be isometrically immersed in both Q_c^{n+1} and $\tilde{Q}_{\tilde{c}}^{n+q}$, $\tilde{c} > c$, $q \leq n-3$, then for each $p \in M$ there exists an umbilic subspace $U_p \subset T_pM$ of both immersions with $\dim U_p \geq n - q$. The set of Riemannian metrics which admit locally isometric immersions as above is very large, even if one assumes that the second immersion has flat normal bundle, i.e., the curvature tensor $R^\perp$ of the normal connection vanishes. Namely, it is known (cf. [2]) that non-flat conformally flat hypersurfaces of $\mathbf{R}^{n+1}$, $n \geq 4$, can always be locally isometrically immersed into the sphere $S_{(c)}^{n+1}$, for some c. The question that served as a starting point of this paper was to describe new examples of hypersurfaces of $\mathbf{R}^{n+1}$ which can be immersed into $S_{(c)}^{n+1}$ with $R^\perp \equiv 0$. As we found out, this question is related to the concept of envelopes of a p-parameter family of spheres. To state our results, we need some terminology.

(1.2) Let $x\colon M^n \to Q_c^{n+q}$ be an isometric immersion and assume here and in the sequel that M^n is connected and orientable with a given orientation. Choose a unit normal vector η and denote by A_η the self adjoint map of tangent spaces corresponding to the second fundamental form of x along η, and by $k_1, \ldots, k_n$ the eigenvalues of A_η. In the case that $Q_c^{n+q} = \mathbf{R}^{n+1}$, we denote by N the unit normal to x which gives the orientation of M. We say that $x\colon M^n \to \mathbf{R}^{n+1}$ is a *p-parameter envelope of n-spheres*, $p \leq n-2$, (briefly, p-PES) if at each point of M,

$$k_1 = \cdots = k_{n-p} = \lambda \neq 0 \quad \text{and} \quad k_j \neq \lambda \quad \text{if } n-p+1 \leq j \leq n.$$

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Classically, a p-PES is locally a solution

$$x = x(u_1, \ldots, u_p, t_1, \ldots, t_{n-p})$$

in $\mathbf{R}^{n+1}$ of the system below:

$$(1.3)\qquad \begin{aligned} &\text{(a)}\quad \|x - g\|^2 = r^2, \\ &\text{(b)}\quad \left\langle x - g, \frac{\partial g}{\partial u_j} \right\rangle = -r \frac{\partial r}{\partial u_j}, \quad j = 1, \ldots, p, \end{aligned}$$

where $g\colon L^p \to \mathbf{R}^{n+1}$ is an isometric immersion, $g = g(u_1, \ldots, u_p)$, and $r \in C^\infty(L)$ is a non-vanishing function. Geometrically it means that x is the envelope of the p-parameter family of n-spheres given by (a): the limit of the intersection of neighboring spheres that approach each other are $(n-p)$-spheres that generate the envelope.

(1.4) Let g and r be as above and let $t_1, \ldots, t_{n-p}$ be orthogonal parameters of the unit $(n-p)$-sphere centered at the origin of an euclidean $(n-p+1)$-space. Set

$$x(u_1, \ldots, u_p, t_1, \ldots, t_{n-p}) = g - r\nabla r - r\sqrt{1 - |\nabla r|^2}\,\phi(t_1, \ldots, t_{n-p}),$$

where ∇r is the gradient of r and the vector ϕ has origin at the point $\gamma = g - r\nabla r$ and describes a unit sphere in the affine $(n-p+1)$-plane through γ orthogonal to g.

(1.5) THEOREM. *The hypersurface given by (1.4) satisfies the system (1.3) and is (away from singular points) a p-PES. Conversely, every p-PES satisfies system (1.3) and is locally of the form (1.4), for* $r = 1/\lambda$.

(1.6) Let $x\colon M^n \to \mathbf{R}^{n+1}$ be a p-PES and let D_λ be the smooth distribution defined by taking at each $q \in M$ the $(n-p)$-dimensional eigenspace of A_N corresponding to the eigenvalue λ. We say that x is a *special p-parameter envelope of spheres* (briefly, p-SPES) if the distribution $D_\lambda^\perp$ is integrable.

Now, for each $q \in L$, let $B(q) \subset T_q L$ be the *relative nullity* subspace of the immersion $g\colon L \to \mathbf{R}^{n+1}$ (given by Theo. (1.5)) defined by

$$B(q) = \{X \in T_q L\colon \alpha(X, Y) = 0 \quad \text{for all } Y \in T_q L\},$$

where α stands for the second fundamental form of g.

(1.7) THEOREM. *Let* $x\colon M^n \to \mathbf{R}^{n+1}$ *be a p-PES. Then x is a p-SPES if and only if* $g\colon L^p \to \mathbf{R}^{n+1}$ *has flat normal bundle and* $\nabla r(q) \in B(q)$ *for all* $q \in L$, *where* $r = 1/\lambda$.

The proofs of Theorems (1.5) and (1.7) are presented in Section 2. The main result of this section, Theorem (2.14), has the following consequence:

(1.8) THEOREM. *Let* $x\colon M^n \to \mathbf{R}^{n+1}$, $n \geq 5$, *be a p-PES and assume that* M^n *can be isometrically immersed in* $S^{n+q}_{(c)}$, $2 \leq q < n-p$, *with flat normal bundle. Then x is a p-SPES.*

(1.9) In Section 3 we restrict ourselves to n-dimensional 2-PES, $n \geq 5$, which can be isometrically immersed into $S^{n+2}_{(c)}$ with $R^{\perp} \equiv 0$. Notice that the condition of Theorem (1.8) holds and so x is a 2-SPES. Now, assume that the *index of relative nullity* of $g\colon L^2 \to \mathbf{R}^{n+1}$, defined by $\mu(q) = \dim B(q)$, is constant. By Theorem (1.7), if $\nabla r(q) \neq 0$, then $\mu(q) \neq 0$, which implies that the Gaussian curvature $K_L(q)$ is zero. On the other hand, if $\mu \equiv 0$ then $\nabla r \equiv 0$ and so r must be constant. *Therefore* $x(M)$ *must be a hypersurface of* $\mathbf{R}^{n+1}$ *of one of the following types*:

Type I. A normal bundle of spheres with radius r a constant over a surface with $\mu \equiv 0$ (or $K_L \neq 0$ at every point) in $\mathbf{R}^{n+1}$.

Type II. A 2-SPES where $g(L^2)$ is a flat ruled surface in $\mathbf{R}^{n+1}$ without umbilic points (see (3.22)).

Type III. A 2-SPES where $g(L^2)$ is part of a 2-plane in $\mathbf{R}^{n+1}$.

(1.10) THEOREM. *Let* $x_0\colon M^n \to \mathbf{R}^{n+1}$, $n \geq 5$, *be a simply connected hypersurface of type I or II, and let* $x_1\colon M^n \to S^{n+2}_{(c)}$ *be an isometric immersion with* $R^{\perp} \equiv 0$. *Then there exists an isometric homotopy*

$$\tilde{x}_t\colon [0, 1] \times M \to \mathbf{R}^{n+3}$$

between $\tilde{x}_0 = i_0 \circ x_0$ *and* $\tilde{x}_1 = i_1 \circ x_1$, *where*

$$i_0\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+3} \quad \textit{and} \quad i_1\colon S^{n+2}_{(c)} \to \mathbf{R}^{n+3}$$

are the standard inclusions.

Theorem (1.10) shows that if a 2-SPES of type I or II admits an isometric immersion into $S^{n+2}_{(c)}$ with $R^{\perp} \equiv 0$, then some kind of weak rigidity (see [4], [7]) remains and the immersion cannot be too complicated. For 2-SPES of type I we were able to obtain a general result, Theorem (3.9), which in particular implies the following:

(1.11) THEOREM. *Let* $x\colon M^n \to \mathbf{R}^{n+1}$, $n \geq 5$, *be a normal bundle of spheres with radius r a constant over a surface* $g\colon L^2 \to \mathbf{R}^{n+1}$ *of one of the following types:*

(a) *a rotation surface in* $\mathbf{R}^3$;
(b) *a cone in* $\mathbf{R}^{n+1}$;
(c) *a product of curves in* $\mathbf{R}^{n+1}$.

Then M^n *can be locally isometrically immersed in* $S^{n+2}_{(c)}$ *with* $R^{\perp} \equiv 0$, *for* $0 < c < (1/r)^2$.

Finally, for 2-SPES of type II, we obtain the following result:

(1.12) THEOREM. *Let $x: M^n \to \mathbf{R}^{n+1}$, $n \geq 5$, be a 2-SPES of type II. Then M^n can be locally isometrically immersed into $S^{n+2}_{(c)}$ with $R^\perp \equiv 0$, for some $c > 0$, if and only if L^2 is part of a cone or a generalized cylinder in $\mathbf{R}^{n+1}$.*

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2. Proofs of Theorems (1.5), (1.7) and (1.8)

(2.1) *Proof of Theorem (1.5).* Assume that x is given by (1.4). Equation (1.3) (a) is easily verified. To prove (1.3) (b) we rewrite x as follows:

$$x = g - r\nabla r - r\sqrt{1 - |\nabla r|^2} \sum_{k=1}^{n-p+1} \phi^k(t_1, \ldots, t_{n-p})\xi_k(u_1, \ldots, u_p), \tag{2.2}$$

where $\xi_1, \ldots, \xi_{n-p+1}$ is an orthonormal frame normal to g and

$$(\phi^1, \ldots, \phi^{n-p+1})$$

is an orthogonal parametrization of $S^{n-p}(1) \subset \mathbf{R}^{n-p+1}$. A short computation shows that the unit vector field $N = (g - x)/r$ is normal to x. By differentiation of (1.3) (a), we obtain

$$r\frac{\partial r}{\partial u_j} = \left\langle x - g, \frac{\partial x}{\partial u_j} - \frac{\partial g}{\partial u_j}\right\rangle = -\left\langle x - g, \frac{\partial g}{\partial u_j}\right\rangle,$$

which proves (1.3) (b).

To prove that x is a p-PES, we set

$$e_i = \left\|\frac{\partial x}{\partial t_i}\right\|^{-1} \frac{\partial x}{\partial t_i}.$$

Then $\langle e_i, e_j\rangle = \delta_{ij}$ and

$$A_N(e_i) = -\left\|\frac{\partial x}{\partial t_i}\right\|^{-1} \frac{\partial N}{\partial t_i} = \left\|\frac{\partial x}{\partial t_i}\right\|^{-1} \frac{1}{r}\frac{\partial x}{\partial t_i} = \lambda e_i, \quad \lambda = \frac{1}{r}, \quad i = 1, \ldots, n - p.$$

At some point, take an orthonormal basis of eigenvectors $e_1, \ldots, e_{n-p}$, $T_1, \ldots, T_p$, of A_N with

$$A_N(T_j) = \delta_j T_j, \quad j = 1, \ldots, p.$$

We have to show that $\delta_j \neq \lambda$, for all j. Let us write

$$\frac{\partial x}{\partial u_j} = \sum_{i=1}^{n-p} a_{ji} e_i + \sum_{k=1}^{p} b_{jk} T_k. \tag{2.3}$$

On the other hand,

$$\frac{\partial x}{\partial u_j} = V_j - \frac{\partial r}{\partial u_j} N - r \frac{\partial N}{\partial u_j} = V_j^T + rA_N\left(\frac{\partial x}{\partial u_j}\right),$$

where V_j^T denotes the projection on T_M of $V_j = \partial g/\partial u_j$. Then

$$A_N\left(\frac{\partial x}{\partial u_j}\right) = \lambda\left(\frac{\partial x}{\partial u_j} - V_j^T\right). \tag{2.4}$$

Since $\langle V_j, e_i\rangle = 0$, we have $V_j \in \operatorname{span}\{T_1, \ldots, T_p, N\}$ so we can write

$$V_j = V_j^T + \frac{\partial r}{\partial u_j} N = \sum_{k=1}^{p} c_{jk} T_k + \frac{\partial r}{\partial u_j} N. \tag{2.5}$$

By substituting (2.3) and (2.5) into (2.4), we obtain

$$\lambda c_{jk} = b_{jk}(\lambda - \delta_k), \quad 1 \le j, k \le p \tag{2.6}$$

Notice that the regularity of x implies that $|\nabla r| < 1$; thus

$$N = \nabla r + \sqrt{1 - |\nabla r|^2}\phi \notin \operatorname{span}\{V_1, \ldots, V_p\}.$$

It follows that the vectors $V_j^T = V_j - \partial r/\partial u_j\ N$ are linearly independent. Therefore, we cannot have $\delta_k = \lambda$, for some k; otherwise form (2.6) we would obtain $c_{jk} = 0, j = 1, \ldots, p$, which is a contradiction.

Now assume that x is a p-PES. Then it is well known (cf. [5], p. 372) that the distribution D_λ is differentiable and involutive, and that λ is constant along each leaf Σ_λ of D_λ. Moreover, each Σ_λ is an $(n-p)$-umbilic submanifold of M^n and $\mathbf{R}^{n+1}$. This implies that Σ_λ is part of a $(n-p)$-sphere contained in some affine subspace $\mathbf{R}^{n-p+1} \subset \mathbf{R}^{n+1}$. Choose local coordinates

$$(u_1, \ldots, u_p, t_1, \ldots, t_{n-p})$$

for x such that, for each $(u_1, \ldots, u_p)$ fixed, the coordinates $(t_1, \ldots, t_{n-p})$ parametrize a leaf of D_λ. Consider the *focal set* of x relative to λ given by $g = x + rN$, where $r = 1/\lambda$ and N is a unit vector field normal to x. Since $r = r(u_1, \ldots, u_p)$, we obtain

$$\frac{\partial g}{\partial t_i} = \frac{\partial x}{\partial t_i} + r\frac{\partial N}{\partial t_i} = \frac{\partial x}{\partial t_i} + r\left(-\lambda\frac{\partial x}{\partial t_i}\right) = 0,$$

that is, $g = g(u_1, \ldots, u_p)$.

The fact that g is a submanifold of $\mathbf{R}^{n+1}$ follows from [3]. Equations (1.3) are now easily verified. In particular, (1.3) (b) implies that the component N^T of N tangent to g satisfies $N^T = \nabla r$. Thus, we can write

$$N = \nabla r + \sqrt{1 - |\nabla r|^2}\phi,$$

where ϕ is some unit vector in the normal space of g, and the proof follows easily.

(2.7) *Proof of Theorem* (1.7). Suppose that x is a p-SPES. Since D_λ and $D_\lambda^\perp$ are integrable, we can choose local coordinates

$$(u_1, \ldots, u_p, t_1, \ldots, t_{n-p})$$

such that $(t_1, \ldots, t_{n-p})$ are orthogonal coordinates for the unit $(n-p)$-sphere $S^{n-p}(1)$ and $(u_1, \ldots, u_p)$ are coordinates for the leaves of $D_\lambda^\perp$, and put

$$x = g - r\nabla r - r\sqrt{1 - |\nabla r|^2} \sum_{k=1}^{n-p+1} \phi^k \xi_k.$$

For convenience, we choose here the following standard coordinate system for $S^{n-p}(1) \subset \mathbf{R}^{n-p+1}$:

$$\begin{aligned} \phi^1(t_1, \ldots, t_{n-p}) &= \sin t_1, \\ \phi^2(t_1, \ldots, t_{n-p}) &= \cos t_1 \sin t_2, \\ \phi^{n-p+1}(t_1, \ldots, t_{n-p}) &= \cos t_1 \cos t_2 \ldots \cos t_{n-p}. \end{aligned}$$

On one hand, we have

$$\frac{\partial}{\partial t_i}\left\langle \frac{\partial}{\partial u_j}(x-g), x-g \right\rangle = \left\langle \frac{\partial^2}{\partial t_i \partial u_j}(x-g), x-g \right\rangle, \quad 1 \le i \le n-p,\ 1 \le j \le p,$$

where we use the fact that

$$\left\langle \frac{\partial x}{\partial u_j}, \frac{\partial x}{\partial t_i} \right\rangle = \left\langle \frac{\partial g}{\partial u_j}, \frac{\partial x}{\partial t_i} \right\rangle = 0.$$

On the other hand,

$$\frac{\partial}{\partial t_i}\left\langle \frac{\partial}{\partial u_j}(x-g), x-g \right\rangle = \frac{\partial}{\partial t_i}\left(r \frac{\partial r}{\partial u_j} \right) = 0.$$

It follows that

$$\left\langle \sum_{k=1}^{n-p+1} \frac{\partial \phi^k}{\partial t_i} \frac{\partial \xi_k}{\partial u_j}, r\nabla r + r\sqrt{1 - |\nabla r|^2} \sum_{k=1}^{n-p+1} \phi^k \xi_k \right\rangle = 0, \tag{2.8}$$

for all $i = 1, \ldots, n-p$, $j = 1, \ldots, p$.

Take $\phi^k = \sin t_k$, $\phi^l = \cos t_k$ for $1 \le k < l \le n-p+1$ and $\phi^m = 0$ if $m \ne k, l$. Then (2.8) becomes

$$\cos t_k \left\langle \frac{\partial \xi_k}{\partial u_j}, r\nabla r \right\rangle - \sin t_k \left\langle \frac{\partial \xi_l}{\partial u_j}, r\nabla r \right\rangle + r\sqrt{1 - |\nabla r|^2} \left\langle \frac{\partial \xi_k}{\partial u_j}, \xi_l \right\rangle = 0,$$

which implies that

$$\text{(2.9)} \qquad \begin{aligned} &\text{(i)} \quad \left\langle \frac{\partial \xi_k}{\partial u_j}, \nabla r \right\rangle = 0 \\ &\text{(ii)} \quad \left\langle \frac{\partial \xi_k}{\partial u_j}, \xi_l \right\rangle = 0, \quad 1 \le k, l \le n-p+1. \end{aligned}$$

Equation (2.9) (i) is equivalent to $\nabla r(q) \in B(q)$ for all $q \in L$, while equation (2.9) (ii) means that the vector fields $\xi_1, \ldots, \xi_{n-p+1}$ are parallel in the normal connection of g, i.e., $(\nabla_Z \xi_k) = 0$ for all $Z \in TL$ and $k = 1, \ldots, n-p+1$. We conclude that the normal bundle of g must be flat.

For the converse, since g has flat normal bundle, we may choose parallel vector fields $\xi_1, \ldots, \xi_{n-p+1}$ in (2.2). We already know from Theorem (1.5) that $\lambda = 1/r$ is an eigenvalue of A_N with multiplicity exactly $n-p$ with corresponding eigenvectors

$$e_i = \left\|\frac{\partial x}{\partial t_i}\right\|^{-1} \frac{\partial x}{\partial t_i}.$$

Now using the facts that

$$\left(\frac{\partial \xi_k}{\partial u_j}\right)^{\perp} = 0 \quad \text{and} \quad \nabla r \subset B,$$

one can easily verify that

$$\left\langle \frac{\partial x}{\partial u_j}, \frac{\partial x}{\partial t_i} \right\rangle = 0, \quad 1 \le i \le n-p, 1 \le j \le p.$$

It follows that $D_\lambda^\perp$ is (locally) generated by

$$\left\{\frac{\partial x}{\partial u_1}, \ldots, \frac{\partial x}{\partial u_p}\right\},$$

which is obviously integrable. This completes the proof of (1.7).

(2.10) Now we start to prepare for the proof of Theorem (1.8). Assume that M^n can be isometrically immersed in both Q_c^{n+1} and $\tilde{Q}_{\tilde{c}}^{n+q}$, $\tilde{c} > c$, $1 \le q \le n-3$. We already know that for each $p \in M$, there exists an umbilic subspace $U_p \subset T_p M$ for both immersions with $l = \dim U_p \ge n-q$. Assume that the sectional curvature of M at p is not constant and denote by $TM^\perp$, $\widetilde{TM}^\perp$ the normal bundles of the immersions in Q_c^{n+1} and $\tilde{Q}_{\tilde{c}}^{n+q}$ respectively.

(2.11) PROPOSITION. *There exists an orthonormal basis $\{E_1, \ldots, E_n\}$ of $T_p M$ and orthonormal bases $\{N\}$ of $T_p M^\perp$ and $\{\eta_1, \ldots, \eta_q\}$ of $\widetilde{T_p M}^\perp$ such that*

(i)
$$A_N = \begin{bmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \delta_1 & & \\ & & & & \ddots & \\ & & & & & \delta_{n-l} \end{bmatrix};$$

(ii) $$A_{\eta_1} = \begin{bmatrix} \sqrt{\lambda^2 + c - \tilde{c}} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda^2 + c - \tilde{c}} & & & \\ & & & \mu_1 & & \\ & & & & \ddots & \\ & & & & & \mu_{n-l} \end{bmatrix},$$

$$A_{\eta_t} = \begin{bmatrix} 0 & 0 \\ 0 & B^t \end{bmatrix}, \quad 2 \le t \le q;$$

(iii) $$\sum_{t=2}^{q} [\beta_{ii}^t \beta_{jj}^t - (\beta_{ij}^t)^2] = \frac{(c - \tilde{c})(\lambda - \delta_i)(\lambda - \delta_j)}{\lambda^2 + c - \tilde{c}};$$

(iv) $$\sum_{t=2}^{q} \sum_{j=1}^{n-l} (\beta_{ij}^t)^2 \neq 0. \quad 1 \le i \le n - l;$$

where

$$\mu_j = \frac{\lambda\delta_j + c - \tilde{c}}{\sqrt{\lambda^2 + c - \tilde{c}}}, \quad 1 \le j \le n - l,$$

and B^t is an $(n - l) \times (n - l)$ matrix given by $B^t = (\beta_{ij}^t)$.

Proof. The existence of the orthonormal bases satisfying (i) and (ii) above is a consequence of (2.2) of [1]. Now (iii) follows from the Gauss equation. Notice that $\lambda^2 + c - \tilde{c} \neq 0$, otherwise from the Gauss equation again, we have $A_N = \lambda I$, that is, M has constant curvature at p. Finally, (iv) means that the common umbilic subspace contained in the proper space of A_N corresponding to λ has dimension l.

(2.12) COROLLARY. *The common umbilic subspace*

$$U_p = \operatorname{span}\{E_1, \ldots, E_l\}$$

is unique.

Proof. Assume that E_{l+1}, E_{l+2} belong to a common umbilic subspace of both immersions, i.e., $\delta_1 = \delta_2 \neq \lambda$ (so $\mu_1 = \mu_2$), $\beta_{11}^t = \beta_{22}^t$, $\beta_{1j}^t = 0$, $\beta_{i2}^t = 0$, for $2 \le t \le q$ and $(i, j) \neq (2, 1)$. Then the left-hand side of (2.11) (iii) becomes nonnegative while the right-hand side becomes negative, a contradiction.

(2.13) Under the same assumptions and terminology of (2.11), let us denote by V_p the (proper) subspace of T_pM which corresponds to the eigenvalue λ of A_N, that is,

$$V_p = \{X \in T_pM;\ A_N(X) = \lambda X\}.$$

(2.14) THEOREM. *Suppose that M^n can be isometrically immersed in both Q_c^{n+1} and $\tilde{Q}_{\tilde{c}}^{n+q}$, $\tilde{c} > c$, $1 \leq q \leq n-3$. Assume that the second immersion satisfies $\tilde{R}^{\perp} \equiv 0$ and that* $\dim V_p = l$ *is constant. Then the distribution $p \mapsto V_p^{\perp}$ is integrable.*

Proof. Denote by $\tilde{\alpha}$ the second fundamental form of the immersion into $\tilde{Q}_{\tilde{c}}^{n+q}$. On each open subset of M in which U_p has constant dimension, the unit vector field defined by

$$\operatorname{span}\{\eta_1(p)\} = \operatorname{span}\{\tilde{\alpha}(X, X); X \in U_p\}$$

is differentiable. Thus we can choose normal vectors fields $\eta_2, \ldots, \eta_q$ such that $\eta_1, \ldots, \eta_q$ is a differentiable orthonormal frame. By using the fact that $\tilde{R}^{\perp} \equiv 0$ together with Proposition (2.11), it is easy to see that there exists an open and dense subset B of M such that, in each connected component of B, there is a tangent orthonormal frame $E_1, \ldots, E_l$, $T_1, \ldots, T_{n-l}$, which diagonalizes $\tilde{\alpha}$, $V_p = \operatorname{Span}\{E_1, \ldots, E_l\}$ and

$$\text{(i)} \qquad A_N = \begin{bmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \delta_1 & & \\ & & & & \ddots & \\ & & & & & \delta_{n-l} \end{bmatrix}, \quad \delta_j \neq \lambda,\ 1 \leq j \leq n-l;$$

$$\text{(ii)} \qquad A_{\eta_1} =$$

$$\begin{bmatrix} \sqrt{\lambda^2 + c - \tilde{c}} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda^2 + c - \tilde{c}} & & & \\ & & & \dfrac{\lambda\delta + c - \tilde{c}}{\sqrt{\lambda^2 + c - \tilde{c}}} & & \\ & & & & \ddots & \\ & & & & & \dfrac{\lambda\delta_{n-l} + c - \tilde{c}}{\sqrt{\lambda^2 + c - \tilde{c}}} \end{bmatrix},$$

(2.15)

$$A_{\eta_t} = \begin{bmatrix} \gamma_1^t & & & & & \\ & \ddots & & & & \\ & & \gamma_l^t & & & \\ & & & \beta_1^t & & \\ & & & & \ddots & \\ & & & & & \beta_{n-l}^t \end{bmatrix}, \quad 2 \leq t \leq q;$$

(iii) (a) $\sum_{t=2}^{q} \beta_j^t \beta_k^t = \dfrac{(c-\tilde{c})(\lambda-\delta_j)(\lambda-\delta_k)}{\lambda^2+c-\tilde{c}}, \quad 1 \le j \ne k \le n-l;$

(b) $\sum_{t=2}^{q} \gamma_i^t \beta_j^t = 0, \quad 1 \le i \le l,\ 1 \le j \le n-l;$

(c) $\sum_{t=2}^{q} \gamma_i^t \gamma_j^t = 0, \quad 1 \le i \ne j \le l.$

By the Codazzi equation for the hypersurface, we have

$$\langle (\nabla_{T_j} A_N) T_k, E_i \rangle = \langle (\nabla_{T_k} A_N) T_j, E_i \rangle,$$

from which we obtain

$$(\delta_k - \lambda)\langle \nabla_{T_j} T_k, E_i \rangle = (\delta_j - \lambda)\langle \nabla_{T_k} T_j, E_i \rangle. \tag{2.16}$$

Similarly for the other immersion, we have

$$(\beta_k^t - \gamma_i^t)\langle \nabla_{T_j} T_k, E_i \rangle = (\beta_j^t - \gamma_i^t)\langle \nabla_{T_k} T_j, E_i \rangle \quad \text{for } t \ge 2. \tag{2.17}$$

We claim that $\langle \nabla_{T_j} T_k, E_i \rangle = 0$, if $j \ne k$. In fact, if $\langle \nabla_{T_j} T_k, E_i \rangle \ne 0$ then $\langle \nabla_{T_k} T_j, E_i \rangle \ne 0$ by (2.16). Now, from (2.15) (iii) and (2.17) it follows quickly that

$$\sum_{t=2}^{q} (\gamma_i^t)^2 (\langle \nabla_{T_j} T_k, E_i \rangle - \langle \nabla_{T_k} T_j, E_i \rangle) = 0, \tag{2.18}$$

and

$$\sum_{t=2}^{q} (\beta_k^t)^2 \langle \nabla_{T_j} T_k, E_i \rangle = \frac{(c-\tilde{c})(\lambda-\delta_j)(\lambda-\delta_k)}{\lambda^2+c-\tilde{c}} \langle \nabla_{T_k} T_j, E_i \rangle. \tag{2.19}$$

First suppose that, at some point, $\sum_{t=2}^{q} (\gamma_i^t)^2 = 0$. From (2.17), we have

$$\beta_k^t \langle \nabla_{T_j} T_k, E_i \rangle = \beta_j^t \langle \nabla_{T_k} T_j, E_i \rangle.$$

Hence

$$\sum_{t=2}^{q} \beta_k^t \beta_j^t \langle \nabla_{T_j} T_k, E_i \rangle \langle \nabla_{T_k} T_j, E_i \rangle \ge 0.$$

By using (2.15) (iii), we get

$$\frac{(c-\tilde{c})(\lambda-\delta_j)(\lambda-\delta_k)}{\lambda^2+c-\tilde{c}} \langle \nabla_{T_j} T_k, E_i \rangle \langle \nabla_{T_k} T_j, E_i \rangle > 0. \tag{2.20}$$

On the other hand, (2.16) says that

$$(\lambda-\delta_j)(\lambda-\delta_k)\langle \nabla_{T_j} T_k, E_i \rangle \langle \nabla_{T_k} T_j, E_i \rangle > 0,$$

which contradicts (2.20), since

$$\frac{c-\tilde{c}}{\lambda^2+c-\tilde{c}} < 0.$$

Now, suppose that $\sum_{t=2}^{q} (\gamma_i^t)^2 \neq 0$. From (2.16) and (2.18), we obtain

$$\langle \nabla_{T_j} T_k, E_i \rangle = \langle \nabla_{T_k} T_j, E_i \rangle$$

and $(\lambda - \delta_j)(\lambda - \delta_k) > 0$, which contradicts (2.19). This proves the claim and completes the proof of Theorem (2.14).

(2.21) *Proof of Theorem* (1.8). At a point $r \in M$ we have that

$$k_1 = \cdots = k_{n-p} = \mu \quad \text{and} \quad k_j \neq \mu, \quad n - p + 1 \leq j \leq n.$$

Let $W_r = \{X \in T_r M;\ A_N(X) = \mu X\}$. Since $U_r \subset V_r$ and $q < n - p$, we have

$$\dim V_r \geq \dim U_r \geq n - q > p,$$

so $V_r \cap W_r \neq \emptyset$. It follows that $V_r = W_r$ for all $r \in M$ and the theorem follows from Theorem (2.14).

3. Proofs of Theorems (1.10), (1.11) and (1.12)

(3.1) LEMMA. *Let $x\colon M^n \to \mathbf{R}^{n+1}$, $n \geq 5$, be a 2-SPES whose second fundamental form has eigenvalues*

$$k_1 = \cdots = k_{n-2} = \lambda,\ \lambda_{n-1} = \delta_1,\ \lambda_n = \delta_2,\ \delta_1 \neq \delta_2.$$

Then M^n can be locally isometrically immersed in $S^{n+2}(c)$ with $R^\perp \equiv 0$, for some $0 < c < \lambda^2$, if and only if the following conditions hold:

(3.2) $$E_i\left(\frac{T_j(\lambda)}{\lambda - \delta_j}\right) = 0, \quad 1 \leq i \leq n-2,\ 1 \leq j \leq 2;$$

(3.3) *There exist functions $c_1, c_2 \in C^\infty(M)$ such that*

(i) $$c_1 c_2 = -\frac{c}{\lambda^2 - c}$$

(ii) $$E_i(c_j) = 0, \quad 1 \leq i \leq n-2,\ 1 \leq j \leq 2;$$

(iii) $$T_k(\beta_j) = \frac{(\beta_k - \beta_j)}{\delta_k - \delta_j} T_k(\delta_j) + \frac{(\lambda\delta_j - c)\beta_k}{(\lambda^2 - c)(\lambda - \delta_k)} T_k(\lambda),$$

for $1 \leq j \neq k \leq 2$, where $\beta_j = c_j(\lambda - \delta_j)$;

(iv) $$T_1\left(\frac{c_2 T_2(\lambda)}{\sqrt{\lambda^2 - c}}\right) - \frac{c_2 T_2(\lambda)}{(\delta_2 - \delta_1)\sqrt{\lambda^2 - c}} T_1(\delta_2) = T_2\left(\frac{c_1 T_1(\lambda)}{\sqrt{\lambda^2 - c}}\right) - \frac{c_1 T_1(\lambda)}{(\delta_1 - \delta_2)\sqrt{\lambda^2 - c}} T_2(\delta_1),$$

where $E_1, \ldots, E_{n-2}, T_1, T_2$ is an orthonormal frame field for TM such that $A_N(E_i) = \lambda E_i$, $A_N(T_j) = \delta_j T_j$, $1 \le i \le n-2$, $1 \le j \le 2$, and N is unit vector field normal to $x(M)$.

Proof. The Codazzi equation for x is equivalent to the following set of equations

$$(3.4)\quad \begin{aligned} &(1)\quad E_i(\lambda) = 0, \quad 1 \le i \le n-2, \\ &(2)\quad \langle \nabla_{E_i} E_j, T_k \rangle = 0, \quad 1 \le i \ne j \le n-2, 1 \le k \le 2, \\ &(3)\quad T_k(\lambda) = (\lambda - \delta_k)\langle \nabla_{E_i} E_i, T_k \rangle, \quad 1 \le k \le 2, 1 \le i \le n-2; \\ &(4)\quad E_i(\delta_j) = (\delta_j - \lambda)\langle \nabla_{T_j} T_j, E_i \rangle \quad 1 \le i \le n-2, 1 \le j \le 2; \\ &(5)\quad (\delta_j - \delta_k)\langle \nabla_{E_i} T_j, T_k \rangle = (\delta_k - \lambda)\langle \nabla_{T_j} T_k, E_i \rangle \quad 1 \le j \ne k \le 2; \\ &(6)\quad T_j(\delta_k) = (\delta_j - \delta_k)\langle \nabla_{T_k} T_j, T_k \rangle, \quad 1 \le j \ne k \le 2. \end{aligned}$$

Suppose there exists an immersion $y\colon M^n \to S^{n+2}(c)$ with $R^\perp \equiv 0$. It follows easily from Proposition (2.11) that there exists an orthonormal frame η_1, η_2 normal to y such that

$$A_{\eta_1} = \begin{bmatrix} \sqrt{\lambda^2 - c} & & & & \\ & \ddots & & & \\ & & \sqrt{\lambda^2 - c} & & \\ & & & \dfrac{\lambda\delta_1 - c}{\sqrt{\lambda^2 - c}} & \\ & & & & \dfrac{\lambda\delta_2 - c}{\sqrt{\lambda^2 - c}} \end{bmatrix},$$

$$A_{\eta_2} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \beta_1 & \\ & & & & \beta_2 \end{bmatrix},$$

where

$$\beta_1\beta_2 = \frac{-c(\lambda - \delta_1)(\lambda - \delta_2)}{\lambda^2 - c}.$$

Then, the Codazzi equation for y is equivalent to (3.4) from (1) to (5) and

$$(3.5)\quad \begin{aligned} &(1)\quad \beta_j\langle \nabla^\perp_{T_k}\eta_1, \eta_2 \rangle = T_k\left(\frac{\lambda\delta_j - c}{\sqrt{\lambda^2 - c}}\right) - \frac{\lambda(\delta_k - \delta_j)}{\sqrt{\lambda^2 - c}}\langle \nabla_{T_j} T_k, T_j \rangle, \\ &\qquad\qquad 1 \le k \ne j \le 2; \\ &(2)\quad \langle \nabla^\perp_{E_i}\eta_2, \eta_1 \rangle = 0; \\ &(3)\quad \beta_j\langle \nabla_{E_i} E_i, T_j \rangle = \sqrt{\lambda^2 - c}\,\langle \nabla^\perp_{T_j}\eta_2, \eta_1 \rangle; \\ &(4)\quad E_i(\beta_j) = \beta_j\langle \nabla_{T_j} T_j, E_i \rangle; \end{aligned}$$

(5) $(\beta_j - \beta_k)\langle \nabla_{E_i} T_j, T_k\rangle = \beta_k \langle \nabla_{T_j} T_k, E_i\rangle, \quad 1 \le j \ne k \le 2;$

(6) $T_k(\beta_j) = (\beta_k - \beta_j)\langle \nabla_{T_j} T_k, T_j\rangle + \dfrac{\lambda\delta_j - c}{\sqrt{\lambda^2 - c}} \langle \nabla^{\perp}_{T_k} \eta_2, \eta_1\rangle, \quad j \ne k.$

We will begin by checking conditions (3.3). Define

$$c_j = \frac{\beta_j}{\lambda - \delta_j}, \quad j = 1, 2, \tag{3.6}$$

in a way that c_1, c_2 satisfy (3.3) (i). From equations (3.4) (4) and (3.5) (4), we have

$$\frac{E_i(\beta_j)}{\beta_j} = \frac{E_i(\delta_j)}{\delta_j - \lambda},$$

which with (3.6) and $E_i(\lambda) = 0$, gives (3.3) (ii). By substituting equations (3.4) (3), (3.4) (6) and (3.5) (3) into (3.5) (6), we obtain (3.3) (iii). A straightforward computation using the Codazzi equation shows that

$$R^{\perp}(T_1, T_2)\eta_2 = \Bigg[T_1\left(\frac{c_2 T_2(\lambda)}{\sqrt{\lambda^2 - c}}\right) - T_2\left(\frac{c_1 T_1(\lambda)}{\sqrt{\lambda^2 - c}}\right) - \frac{1}{(\delta_2 - \delta_1)\sqrt{\lambda^2 - c}} (c_1 T_1(\lambda) T_2(\delta_1) - c_2 T_2(\lambda) T_1(\delta_2)) \Bigg] \eta_1,$$

which with the Ricci equation gives (3.3) (iv).

Finally, a similar procedure shows that

$$0 = R^{\perp}(E_i, T_j)\eta_2 = \frac{c_j}{\sqrt{\lambda^2 - c}} \left(E_i\, T_j(\lambda) - \frac{E_i(\delta_j)}{\lambda - \delta_j}\, T_j(\lambda) \right) \eta_1,$$

which immediately gives (3.2).

Now suppose that (3.2) and (3.3) hold and let

$$\beta_j = c_j(\lambda - \delta_j), \quad j = 1, 2.$$

Let P be a 2-dimensional vector bundle over M with a Riemannian metric, say $\langle\!\langle\ ,\ \rangle\!\rangle$, where each fiber is generated by an orhtonormal frame η_1, η_2. Define in P a connection $\tilde{\nabla}$, compatible with $\langle\!\langle\ ,\ \rangle\!\rangle$, by

$$\begin{aligned} &\tilde{\nabla}_{E_i} \eta_j = 0, \quad 1 \le i \le n - 2, \quad 1 \le j \le 2, \\ &\tilde{\nabla}_{T_j} \eta_1 = -\frac{c_j}{\sqrt{\lambda^2 - c}}\, T_j(\lambda)\eta_2, \quad j = 1, 2. \end{aligned} \tag{3.7}$$

Let $\tilde{R}$ be the curvature tensor of $\tilde{\nabla}$ and $\tilde{\alpha}$ be the symmetric section of the bundle Hom $(TM \times TM, P)$ given by

$$\tilde{\alpha}(E_i, E_j) = \delta_{ij}\sqrt{\lambda^2 - c}\,\eta_1, \quad 1 \le i, \quad j \le n-2,$$

$$\tilde{\alpha}(T_j, T_k) = \delta_{jk}\left(\frac{\lambda\delta_j - c}{\sqrt{\lambda^2 - c}}\,\eta_1 + \beta_j\eta_2\right), \quad 1 \le j, \quad k \le 2, \tag{3.8}$$

$$\tilde{\alpha}(E_i, T_j) = 0, \quad 1 \le i \le n-2, \quad 1 \le j \le 2.$$

By a direct computation we can see that $\tilde{\alpha}$ satisfies the Gauss equation for an immersion into $S^{n+2}(c)$. Equations (3.5) from (1) to (5) also hold and (3.5) (6) is a consequence of (3.3) (iii). Then the Codazzi equation is also verified by $\tilde{\nabla}$ and $\tilde{\alpha}$. By using equations (3.2) and (3.3) (iv), we can see that $\tilde{R}$ vanishes. Since $\tilde{\alpha}$ is diagonalized by $E_1, \ldots, E_{n-2}, T_1, T_2$, we conclude that the Ricci equation holds. Hence by the fundamental theorem of submanifolds (see [6], p. 80), there exists a local isometric immersion $y: M^n \to S^{n+2}(c)$ such that we may identify the normal bundle of y with P. Also the metric induced on the normal bundle of y coincides with the metric of P, and the second fundamental form as well as the normal connection of y coincide with $\tilde{\alpha}$ and $\tilde{\nabla}$, respectively. This completes the proof of the lemma.

(3.9) THEOREM. *Let $x: M^n \to \mathbf{R}^{n+1}$, $n \ge 5$, be a normal bundle of spheres with radius r a constant over an umbilic free surface $g: L^2 \to \mathbf{R}^{n+1}$ with $R_g^\perp \equiv 0$. Then M^n can be locally isometrically immersed into $S^{n+2}(c)$, $0 < c < (1/r)^2$, with $R^\perp \equiv 0$ if and only if there exist smooth functions $a(u_1)$ and $b(u_2)$ such that*

$$a(u_1)G(u_1, u_2) + b(u_2)E(u_1, u_2) = a(u_1)b(u_2), \tag{3.10}$$

where (u_1, u_2) are principal coordinates for g and E, $F = 0$, G are the coefficients of the first fundamental form of g.

Proof: By the results of Section 2, we can write x as

$$x(u_1, u_2, t_1, \ldots, t_{n-2}) = g(u_1, u_2) - rN(u_1, u_2 t_1, \ldots, t_{n-2})$$

where

$$N = \sum_{k=1}^{n-1} \phi^k(t_1, \ldots, t_{n-2})\xi_k(u_1, u_2)$$

and $\xi_1, \ldots, \xi_{n-1}$ are parallel orthonormal fields normal to g. We have

$$\frac{\partial x}{\partial u_j} = \left(1 + \frac{r}{\psi_j}\langle\alpha(v_j, v_j), N\rangle\right)v_j, \quad j = 1, 2,$$

$$\frac{\partial x}{\partial t_i} = r\sum_{k=1}^{n-1}\frac{\partial\phi^k}{\partial t_i}\,\xi_k, \quad i = 1, \ldots, n-2, \tag{3.11}$$

$$\frac{\partial N}{\partial u_j} = -\frac{1}{\psi_j}\langle\alpha(v_j, v_j), N\rangle v_j,$$

where $\psi_1 = E$, $\psi_2 = G$, $v_j = \partial g/\partial u_j$ and α is the second fundamental form of g. It follows easily from (3.11) that $(u_1, u_2, t_1, \ldots, t_{n-2})$ are orthonormal coordinates which diagonalize the second fundamental form of x. Therefore

$$\lambda = \frac{1}{r} = -\frac{\left\langle \dfrac{\partial x}{\partial t_i}, \dfrac{\partial N}{\partial t_i} \right\rangle}{\left\| \dfrac{\partial x}{\partial t_i} \right\|^2}, \quad i = 1, \ldots, n-2,$$

$$\delta_j = -\frac{\left\langle \dfrac{\partial x}{\partial u_j}, \dfrac{\partial N}{\partial u_j} \right\rangle}{\left\| \dfrac{\partial x}{\partial u_j} \right\|^2} = \frac{\langle \alpha(v_j, v_j), N\rangle}{\psi_j + r\langle \alpha(v_j, v_j), N\rangle}, \quad j = 1, 2. \tag{3.12}$$

Notice that if $\delta_1 = \delta_2$, it follows that

$$\left\langle \frac{\alpha(v_1, v_1)}{E} - \frac{\alpha(v_2, v_2)}{G}, N \right\rangle = 0, \tag{3.13}$$

which means that g has umbilic points. Then $\delta_1 \neq \delta_2$ and Lemma (3.11) applies. Since λ is constant, we only have to determine under what conditions there exist c_1, c_2 which satisfy equations (3.3) (i), (ii) and (iii). Taking

$$E_i = \left\| \frac{\partial x}{\partial t_i} \right\|^{-1} \frac{\partial x}{\partial t_i}, \quad T_j = \left\| \frac{\partial x}{\partial u_j} \right\|^{-1} \frac{\partial x}{\partial u_j},$$

equation (3.3) (iii) becomes

$$\frac{\partial c_j}{\partial u_k} = \frac{(\lambda - \delta_k)}{(\delta_k - \delta_j)(\lambda - \delta_j)} \frac{\partial \delta_j}{\partial u_k} (c_k - c_j), \quad j \neq k. \tag{3.14}$$

But by (3.12), we have

$$\lambda - \delta_j = \frac{\psi_j}{r(\psi_j + r\langle \alpha_j, N\rangle)},$$

and

$$\frac{\partial \delta_j}{\partial u_k} = \frac{\psi_j \langle \nabla^{\perp}_{v_k} \alpha_j, N\rangle - \dfrac{\partial \psi_j}{\partial u_k} \langle \alpha_j, N\rangle}{(\psi_j + r\langle \alpha_j, N\rangle)^2},$$

where $\alpha_j = \alpha(v_j, v_j)$. It follows that

$$\frac{(\lambda - \delta_k)}{(\lambda - \delta_j)(\delta_k - \delta_j)} \frac{\partial \delta_j}{\partial u_k} = \frac{\psi_k}{\psi_j} \frac{\left\langle \psi_j \nabla^{\perp}_{v_k} \alpha_j - \dfrac{\partial \psi_j}{\partial u_k} \alpha_j, N \right\rangle}{\langle \psi_j \alpha_k - \psi_k \alpha_j, N\rangle} \tag{3.15}$$

By the Codazzi equation for g, we have

$$\nabla^{\perp}_{v_k}\alpha_j - 2\langle \nabla_{v_k} v_j, v_j\rangle \frac{\alpha_j}{\psi_j} = -\langle \nabla_{v_j} v_k, v_j\rangle \frac{\alpha_j}{\psi_j} - \left\langle \nabla_{v_j} v_j, v_k \frac{\alpha_k}{\psi_k}\right\rangle,$$

from which we get

$$\langle \nabla_{v_j} v_j, v_k\rangle = \frac{\psi_k\left\langle \psi_j \nabla^{\perp}_{v_k}\alpha_j - \dfrac{\partial \psi_j}{\partial u_k}\alpha_j, \xi\right\rangle}{\langle \psi_k \alpha_j - \psi_j \alpha_k, \xi\rangle}$$

for any vector ξ normal to g. Then we can rewrite (3.15) as

$$\text{(3.16)} \qquad \begin{aligned} \frac{\lambda - \delta_k}{(\lambda - \delta_j)(\delta_k - \delta_j)} &= \frac{1}{\psi_j}\langle \nabla_{v_j} v_j, v_k\rangle \\ &= \frac{1}{\psi_j}\langle v_j, \nabla_{v_k} v_j\rangle = \frac{1}{2\psi_j}\frac{\partial \psi_j}{\partial u_k}. \end{aligned}$$

From (3.16), equation (3.14) is equivalent to

$$\frac{\partial c_1}{\partial u_2} = \frac{1}{2E}\frac{\partial E}{\partial u_2}(c_2 - c_1), \quad \frac{\partial c_2}{\partial u_1} = \frac{1}{2G}\frac{\partial G}{\partial u_1}(c_1 - c_2).$$

Now, by using $c_1 c_2 = -c/(\lambda^2 - c)$, we obtain

$$\text{(3.17)} \quad \begin{aligned} &(1) \quad \frac{\partial c_1}{\partial u_2} = -\frac{1}{2E}\frac{\partial E}{\partial u_2}\frac{1}{c_1}\left(c_1^2 + \frac{c}{\lambda^2 - c}\right), \\ &(2) \quad \frac{\partial c_1}{\partial u_2} = \frac{1}{2G}\frac{\partial G}{\partial u_1}\frac{c_1(\lambda^2 - c)}{c}\left(c_1^2 + \frac{c}{\lambda^2 - c}\right). \end{aligned}$$

The solutions of (3.17) (1) and (2) are of the types

$$\text{(3.18)} \qquad c_1^2 = \frac{\bar{a}(u_1)}{E} = \frac{c}{\lambda^2 - c}$$

and

$$\text{(3.19)} \qquad c_1^2 = \frac{c}{(\lambda^2 - c)}\frac{\bar{b}(u_2)G}{(1 - \bar{b}(u_2)G)},$$

respectively, where $\bar{a}(u_1)$, $\bar{b}(u_2)$ are arbitrary functions. By comparing (3.18) and (3.19) and taking

$$a = \left(\frac{\lambda^2 - c}{c}\right)\bar{a}, \; b = \frac{1}{\bar{b}},$$

we obtain (3.10), which completes the proof of (3.9).

(3.20) *Remark.* Take $E(0, 0) = G(0, 0) = 1$. A short computation shows that

$$a(u_1) = \frac{dE(u_1, 0)}{d - G(u_1, 0)}, \quad b(u_2) = \frac{dG(0, u_2)}{d - (d-1)E(0, u_2)}, \tag{3.21}$$

where $d > 1$ is a constant such that $a > 1$, $b > 1$.

(3.22) At this point it is convenient to review some definitions.

(1) A ruled surface $g\colon L^2 \to \mathbf{R}^{n+1}$ with no umbilic points can be given by $g(s, t) = c(s) + tv(s)$, where $c(s)$ is a curve in $\mathbf{R}^{n+1}$ and v is a normal vector field with $|\dot{c}| = |v| = 1$ (here $\cdot = d/ds$). The condition that L^2 is flat is equivalent to the existence of a function $\psi(s)$ such that $\dot{v} = \psi\dot{c}$; i.e., v is parallel along c.

(2) By a product of curves in $\mathbf{R}^{n+1}$ we mean a surface

$$g(s, t) = (\gamma(s), \phi(t))$$

where $\gamma(s)$ and $\phi(t)$ are curves in $\mathbf{R}^l$ and $\mathbf{R}^{n-l+1}$ respectively, with $\mathbf{R}^l \times \mathbf{R}^{n-l+1} = \mathbf{R}^{n+1}$.

(3) A generalized cylinder is a particular case of a product of curves, namely when one of the curves is a straight line.

(3.23) *Proof of Theorem* (1.11). (a) Let g be given in principal coordinates (u_1, u_2) by

$$g(u_1, u_2) = (x(u_1)\cos u_2,\ x(u_1)\sin u_2,\ \xi(u_1)),$$

where $x \neq 0$ and $(x')^2 + (y')^2 = 1$. Let e_3 be an unit vector normal to g in $\mathbf{R}^3 \subset \mathbf{R}^{n+1}$ and let $e_4, \ldots, e_{n+1}$ be constant orthonormal vector fields in $\mathbf{R}^{n+1}$, normal to g and e_3.

By (3.12), we have

$$\delta_j = \frac{\phi^1 k_j}{1 + r\phi^1 k_j}, \quad j = 1, 2,$$

where k_1, k_2 denote the principal curvatures of g in $\mathbf{R}^3$. Thus $\delta_1 = \delta_2$ if and only if $k_1 = k_2$. We have two cases to consider.

First case. Suppose that g has no open subset of umbilic points. Taking

$$b(u_2) = b_0 > 1 \quad \text{and} \quad a(u_1) = \frac{b_0}{b_0 - x(u_1)^2},$$

where b_0 is constant, one can easily see that (3.10) holds on an open and dense subset of L^2. By continuity it holds everywhere. Thus, the proof follows from Theorem (3.9).

Second case. Suppose that $g(L^2)$ is a (piece of a) sphere, say $S^2(1) \subset \mathbf{R}^3$:

$$g(u_1, u_2) = (\cos u_1 \cos u_2, \cos u_1 \sin u_2, \sin u_1), \quad -\pi/2 < u_1 < \pi/2.$$

The frame

$$E_i = \left\|\frac{\partial x}{\partial t_i}\right\|^{-1} \frac{\partial x}{\partial t_i}, \quad T_j = \left\|\frac{\partial x}{\partial u_j}\right\|^{-1} \frac{\partial x}{\partial u_j}, \quad 1 \le i \le n-2, \quad 1 \le j \le 2,$$

arising from (3.11) gives $A_N(E_i) = \lambda E_i$, $A_N(T_j) = \delta_j T_j$, where

$$\lambda = \frac{1}{r}, \quad \delta_1 = \delta_2 = \frac{\phi^1}{1 + r\phi^1}.$$

Let $d > 1$ be a constant and define $\beta_j = c_j(\lambda - \delta_j)$, where

$$(c_1(u_1))^2 = k \frac{\cos^2 u_1}{d - \cos^2 u_1}, \quad c_2 = -\frac{k}{c_1} \quad \text{and} \quad k = \frac{c}{\lambda^2 - c}.$$

By an argument similar to that of the second part of Lemma (3.1), it is not difficult to conclude that the claimed immersion in $S^{n+2}(c)$ exists.

(b) A cone in $\mathbf{R}^{n+1}$ is a flat ruled surface $g(s, t) = c(s) + tv(s)$ such that $\psi(s)$ defined by $\dot{v} = \psi\dot{c}$ is a non-zero constant. Clearly (s, t) are principal coordinates and $E = (1 + t\psi)^2$, $F = 0$, $G = 1$.

If the curvature $k(s)$ of $c(s)$ does not vanish in an open interval, we proceed as in the first case of (a), just taking

$$a(s) = a_0 > (1 + t\psi)^2, \quad b(t) = \frac{a_0}{a_0 - (1 + t\psi)^2}.$$

If $k \equiv 0$, we proceed analogously to the second case of (a).

(c) A similar argument as in (a) and (b) above.

This completes the proof of Theorem (1.11).

(3.24) *Remark.* Let $x: M^n \to \mathbf{R}^{n+1}$, $n \ge 5$, be a normal bundle of spheres with radius r over a flat ruled surface $g(s, t) = c(s) + tv(s)$ without umbilic points. Suppose that there exists an isometric immersion $y: M^n \to S^{n+2}_{(c)}$ with $R^\perp \equiv 0$. Then ψ must be constant. In fact, in this case by Theorem (3.9) there exist functions $a(s)$, $b(s)$ such that

$$a(s) + b(t)(1 + t\psi(s))^2 = a(s)b(t).$$

By doing $t = 0$ in the above equation, we see that a is constant. Thus

$$(1 + t\psi(s))^2 = \frac{a(b(t) - 1)}{b(t)},$$

from which it follows that ψ must be a constant. Observe that the case $\psi \equiv 0$ is included in part (c) of Theorem (1.11).

(3.25) *Proof of Theorem* (1.12). Since the case of r constant follows from Theorem (1.11) and Remark (3.24), we will assume here that ∇r does not vanish on an open subset.

By assumption, we have

$$V_1 = \frac{\partial g}{\partial s} = (1 + t\psi)\dot{c}, \quad V_2 = \frac{\partial g}{\partial t} = v, \tag{3.26}$$

so $\langle V_1, V_2 \rangle = 0$. Equation (2.9) (i) says that

$$\left(\frac{\partial}{\partial s} \nabla r\right)^{\perp} = 0, \quad \left(\frac{\partial}{\partial t} \nabla r\right)^{\perp} = 0,$$

which is equivalent to

$$\frac{\partial r}{\partial s} (\ddot{c})^{\perp} = 0.$$

But $(\ddot{c})^{\perp}$ never vanishes because g is umbilic free, so $r = r(t)$. Then (2.2) becomes

$$x = c + (t - rr')v - r\sqrt{1 - (r')^2} \sum_{k=1}^{n-1} \phi^k \xi_k, \tag{3.27}$$

where $r' = dr/dt$ and $\xi_1, \ldots, \xi_{n-1}$ are parallel in the normal connection of g. By (3.26) the tangent planes to g are constant along the rulings, therefore we can choose ξ_k in such a way that $\xi_k = \xi_k(s)$, $k = 1, \ldots, n - 1$. Thus

$$\frac{\partial x}{\partial s} = (1 + (t - rr')\psi - r\sqrt{1 - (r')^2}\Omega)\dot{c},$$

$$\frac{\partial x}{\partial t} = (t - rr')' - (r\sqrt{1 - (r')^2})' \sum_{k=1}^{n-1} \phi^k \xi_k,$$

where

$$\Omega(s, t_1, \ldots, t_{n-2}) = -\left\langle \sum_{k=1}^{n-1} \phi^k \xi_k, \ddot{c} \right\rangle.$$

So,

$$\left\langle \frac{\partial x}{\partial s}, \frac{\partial x}{\partial t} \right\rangle = 0.$$

On the other hand

$$\left\langle \frac{\partial^2 x}{\partial s\, \partial t}, N \right\rangle = 0,$$

since $\partial^2 x/\partial s\,\partial t$ is parallel to $\dot{c}$ and

$$N = r'v + \sqrt{1-(r')^2}\sum_{k=1}^{n-1}\phi^k\xi_k$$

is orthogonal to $\dot{c}$.

Now, a direct computation shows that

$$\delta_1 = \frac{-\Omega\sqrt{1-(r')^2} - r'\psi}{1 + t\psi - rr'\psi - r\sqrt{1-(r')^2}\Omega}, \quad \delta_2 = \frac{-r''}{1-(r')^2 - rr''}.$$

Notice that Ω cannot vanish on an open subset of M, otherwise $(\ddot{c})^{\perp} \equiv 0$. Therefore, since $\delta_2 = \delta_2(t)$, there exists an open and dense subset U of M where $\delta_1 \neq \delta_2$ and δ_1 does not vanish on U.

Now suppose that there exists an isometric immersion of M_n into $S^{n+2}(c)$ with $R^{\perp} \equiv 0$. We want to show that ψ is constant. For this it is sufficient to show that ψ is constant in U where Lemma (3.1) applies. By equation (3.3) (iii), we have that

$$\text{(3.28)} \qquad \frac{c_2(\lambda-\delta_2) - c_1(\lambda-\delta_1)}{\delta_2-\delta_1}\frac{\partial\delta_1}{\partial t} = \frac{\partial}{\partial t}(c_1(\lambda-\delta_1)) - \frac{\lambda\delta_1 - c}{\lambda^2 - c}c_2\lambda'.$$

A straightforward computation shows that

$$\text{(3.29)} \qquad \frac{\partial\delta_1}{\partial t} = \frac{\psi\sqrt{1-(r')^2} - r'\Omega}{\sqrt{1+(r')^2}(\Omega\sqrt{1-(r')^2} - r'\psi)}(1-(r')^2 - rr'')(\delta_1-\delta_2)\delta_1.$$

Since λ, δ_2 and $\|\partial x/\partial t\|$ depend only on t, and r' is not identically zero on an open subset, it follows from (3.3) (iv) that $c_2 = c_2(t)$. Thus, from (3.3) (i) we have that $c_1 = c_1(t)$. This remark jointly with equations (3.28) and (3.29) gives

$$\text{(3.30)} \qquad \left[\frac{\psi\sqrt{1-(r')^2} - r'\Omega}{\sqrt{1-(r')^2}(\Omega\sqrt{1-(r')^2} + r'\psi)}(1-(r')^2 - rr'') \right.$$
$$\left. \times (c_1 - c_2)\left(\frac{1}{r} - \delta_2\right) + B\right]\frac{(-\Omega\sqrt{1-(r')^2} - r'\psi)}{(1 + t\psi - rr'\psi - r\sqrt{1-(r')^2}\Omega)} = A,$$

where

$$A = (c_1\lambda)' + \frac{cc_2\lambda'}{\lambda^2 - c}, \quad B = Ar - \frac{r'}{r}(c_2 - c_1), \quad \lambda = 1/r.$$

Equation (3.30) is equivalent to

$$\text{(3.31)} \qquad \Omega\left[(c_2 - c_1)\left(\frac{1}{r} - \delta_2\right)(r'' + (r')^2 - 1)r' - B(1-(r')^2)\right.$$
$$\left. + Ar(1-(r')^2)\psi\right] + \sqrt{1-(r')^2}\left[(c_1 - c_2)\left(\frac{1}{r} - \delta_2\right)(rr'' + (r')^2 - 1)\psi\right.$$
$$\left. - Br'\psi - A(1 + t\psi - rr'\psi)\right] = 0.$$

It is not hard to verify that the coefficient of Ω in (3.31) is identically zero. So, since $\sqrt{1-(r')^2} \neq 0$, we get

$$(c_1 - c_2)\left(\frac{1}{r} - \delta_2\right)(rr'' + (r')^2 - 1) - Br'\psi - A(1 + t\psi - rr'\psi) = 0.$$

By substituting the values of δ_2 and B in the above equation, we obtain

$$\psi((c_2 - c_1) - tAr) = Ar. \tag{3.32}$$

Notice that $(c_2 - c_1) - tA_r \neq 0$ if t is small. Since $\psi = \psi(s)$, $r = r(t)$, $c_j = c_j(t)$ and $A = A(t)$, then (3.32) says that ψ must be constant, which proves the first part of the theorem.

Suppose now that $g: L^2 \to \mathbf{R}^{n+1}$ is part of a cone or a cylinder and let us check conditions (3.2) and (3.3).

By equation (3.4) (3), we have

$$\begin{aligned} E_k\left(\frac{T_1(\lambda)}{\lambda - \delta_1}\right) &= E_k\langle \nabla_{E_i} E_i, T_1\rangle \\ &= E_k\left(\left\|\frac{\partial x}{\partial t_i}\right\|^{-2} \left\|\frac{\partial x}{\partial s}\right\|^{-1} \left\langle \frac{\partial^2 x}{\partial t_i^2}, \frac{\partial x}{\partial s}\right\rangle\right) \\ &= 0 \end{aligned}$$

and

$$E_k\left(\frac{T_2(\lambda)}{\lambda - \delta_2}\right) = 0,$$

so equation (3.2) holds.

In the present case, equation (3.3) reduces to

$$\frac{\lambda}{2}\frac{d(c_1^2)}{dt} + \left(\lambda' + \frac{\lambda\psi}{1 + t\psi}\right)c_1^2 - \frac{c}{\lambda^2 - c}\left(\frac{c\lambda'}{\lambda^2 - c} - \frac{\lambda\psi}{1 + t\psi}\right) = 0, \tag{3.33}$$

where we set

$$c_2 = \frac{-c}{c_1(\lambda^2 - c)}$$

in (3.28).

Equation (3.33) can be explicitly solved by the change of variable

$$z = \lambda^2(1 + t\psi)^2 c_1^2.$$

The solution is

$$c_1 = \pm\sqrt{-\frac{c}{\lambda^2 - c} + \frac{\theta_0}{\lambda^2(1 + t\psi)^2}},$$

where θ_0 is a constant such that the term under the root sign is positive. By Lemma (3.1), we have a 1-parameter family of isometric immersions of M^n into $S^{n+2}(c)$ with $R^\perp = 0$. This completes the proof of Theorem (1.12).

(3.34) Let $x_0 \colon M^n \to \mathbf{R}^{n+k}$, $x_1 \colon M^n \to \mathbf{R}^{n+l}$, $1 \leq k \leq l$, be isometric immersions. An *isometric homotopy* x_t between x_0 and x_1 is a continuous map $x \colon [0, 1] \times M^n \to \mathbf{R}^{n+l}$ which satisfies the following conditions:

(i) $x(0, q) = x_0(q)$, for all $q \in M$;
(ii) $x(1, q) = x_1(q)$, for all $q \in M$;
(iii) for each $t \in [0, 1]$, the mapping $x_t \colon M^n \to \mathbf{R}^{n+l}$ given by $x_t(q) = x(t, q)$ is an isometric immersion.

(3.35) *Proof of Theorem* (1.10). For each point p in M, we know (cf. Section 2) that we can choose an orthonormal basis $\{E_1, \ldots, E_{n-2}, T_1, T_2\}$ of T_pM and orthonormal bases $\{N\}$ and $\{\eta_1, \eta_2\}$ for the normal spaces of x_0 and x_1 at p respective.y such that

$$A_N = \begin{bmatrix} \lambda & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \delta_1 & \\ & & & & \delta_2 \end{bmatrix},$$

$$A_{\eta_1} = \begin{bmatrix} \sqrt{\lambda^2 - c} & & & & \\ & \ddots & & & \\ & & \sqrt{\lambda^2 - c} & & \\ & & & \dfrac{\lambda\delta_1 - c}{\sqrt{\lambda^2 - c}} & \\ & & & & \dfrac{\lambda\delta_2 - c}{\sqrt{\lambda^2 - c}} \end{bmatrix},$$

$$A_{\eta_2} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & \beta_1 & \\ & & & & \beta_2 \end{bmatrix}, \quad \beta_1 \beta_2 = \frac{-c(\lambda - \delta_1)(\lambda - \delta_2)}{\lambda^2 - c}.$$

Notice that if $\delta_1 = \delta_2$, then $\beta_1 \beta_2 < 0$, so $\beta_1 \neq \beta_2$. It follows that the above choices can be performed in a differentiable way on M.

Let η_3 denote a unit normal to i_1 and define an orthonormal frame normal to $\tilde{x}_1$ by

$$\zeta_1 = \frac{\sqrt{\lambda^2 - c}}{\lambda}\eta_1 + \frac{\sqrt{c}}{\lambda}\eta_3,$$
$$\zeta_2 = \frac{\sqrt{c}}{\lambda}\eta_1 - \frac{\sqrt{\lambda^2 - c}}{\lambda}\eta_3,$$
$$\zeta_3 = \eta_2.$$

It follows that

$$A_{\zeta_1} = A_N,\ A_{\zeta_2} = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \dfrac{\sqrt{c}(\delta_1 - \lambda)}{\sqrt{\lambda^2 - c}} & \\ & & & \dfrac{\sqrt{c}(\delta_2 - \lambda)}{\sqrt{\lambda^2 - c}} \end{bmatrix},\quad A_{\zeta_3} = A_{\eta_2}.$$

For each $t \in [0, 1]$, let us define a Riemannian connection $\tilde{\nabla}$ on the normal bundle $TM^{\perp}$ of $\tilde{x}_1$ and a section $\tilde{A}$ of the bundle

$$\operatorname{Hom}(TM \times TM^{\perp}, TM)$$

by

$$\tag{3.36} \begin{aligned} \tilde{\nabla}_X \zeta_1 &= t\nabla_X^{\perp}\zeta_1, \\ \tilde{\nabla}_X \zeta_i &= t\langle \nabla_X^{\perp}\zeta_i, \zeta_1\rangle\zeta_1 + \langle \nabla_X^{\perp}\zeta_i, \zeta_j\rangle\zeta_j, \quad 2 \le i \neq j \le 3, \\ \tilde{A}_{\zeta_1} &= A_{\zeta_1}, \\ \tilde{A}_{\zeta_i} &= tA_{\zeta_i}, \quad 2 \le i \le 3, \end{aligned}$$

where $\nabla^{\perp}$ denotes the Riemannian connection of $TM^{\perp}$.

(3.37) *Assertion.* For each $t \in [0, 1]$, we have:

(i) $\tilde{\nabla}$ and $\tilde{A}$ satisfy the Gauss and the Codazzi equations;
(ii) $\tilde{\nabla}$ and $\tilde{A}$ satisfy the Ricci equation if and only if

$$T_1(\lambda)T_2(\lambda) = 0.$$

To prove (i), we first observe that for $j = 2, 3$,

$$\tilde{A}_{\tilde{\nabla}_X\zeta_j}(Y) = tA_{\tilde{\nabla}_X\zeta_j}(Y), \quad (\nabla_X \tilde{A}_{\zeta_j})(Y) = t(\nabla_X A_{\zeta_j})(Y).$$

Thus,

$$(\nabla_X \tilde{A}_{\zeta_j})(Y) - \tilde{A}_{\perp_X\zeta_j}(Y) = t[(\nabla_X A_{\zeta_j})(Y) - A_{\nabla_X^{\perp}\zeta_j}(Y)],$$

and the result follows from the Codazzi equation for $\tilde{x}_1$. When $j = 1$, we proceed as follows. From the Codazzi equation for x_0, we have

$$(\nabla_X \tilde{A}_{\zeta_1})(Y) = (\nabla_Y \tilde{A}_{\zeta_1})(X).$$

Then, from the Codazzi equation for $\tilde{x}_1$, we obtain

$$A_{\nabla_X \tilde{\nabla}|\zeta_1}(Y) = A_{\nabla_Y \perp|\zeta_1}(X).$$

So

$$\tilde{A}_{\tilde{\nabla}|_X \zeta_1}(Y) = t^2 A_{\nabla_X \zeta_1}(Y) = t^2 A_{\nabla_Y \zeta_1}(X) = \tilde{A}_{\perp|_Y \zeta_1}(X),$$

since $\nabla_X^\perp \zeta_1$ has only components on ζ_1 and ζ_2. This proves (i).

Observe that $\tilde{\nabla}$ and $\tilde{A}$ satisfy the Ricci equation if and only if $\tilde{R} \equiv 0$. But

$$\tilde{R}(X, Y)\zeta_1 = tR^\perp(X, Y)\zeta_1 = 0,$$

since $R^\perp \equiv 0$. So, to prove (ii), it is enough to verify under what conditions $\langle \tilde{R}(X, Y)\zeta_2, \zeta_3 \rangle = 0$.

A long but not difficult computation shows that

$$\langle \tilde{R}(X, Y)\zeta_2, \zeta_3 \rangle = (t^2 - 1)[\langle \nabla_Y^\perp \zeta_2, \zeta_1 \rangle \langle \nabla_X^\perp \zeta_1, \zeta_3 \rangle - \langle \nabla_X^\perp \zeta_2, \zeta_1 \rangle \langle \nabla_Y^\perp \zeta_1, \zeta_3 \rangle].$$

But, since $\nabla_{E_j}^\perp \zeta_1 = 0$, we may assume that $X = T_1$, $Y = T_2$.

From

$$\nabla_{T_j}^\perp \zeta_1 = T_j\left(\frac{\sqrt{\lambda^2 - c}}{\lambda}\right)\eta_1 + \frac{\sqrt{\lambda^2 - c}}{\lambda} \nabla_{T_j}^\perp \eta_1 + T_j\left(\frac{\sqrt{c}}{\lambda}\right)\eta_3,$$

we obtain

$$\langle \nabla_{T_j}^\perp \zeta_1, \zeta_2 \rangle = T_j\left(\frac{\sqrt{\lambda^2 - c}}{\lambda}\right)\frac{\sqrt{c}}{\lambda} - T_j\left(\frac{\sqrt{c}}{\lambda}\right)\frac{\sqrt{\lambda^2 - c}}{\lambda} = \frac{\sqrt{c}}{\lambda\sqrt{\lambda^2 - c}}\, T_j(\lambda),$$

and

$$\langle \nabla_{T_j}^\perp \zeta_1, \zeta_3 \rangle = -\frac{\beta_j}{\lambda(\lambda - \delta_j)}\, T_j(\lambda) = -\frac{c_j}{\lambda}\, T_j(\lambda).$$

Therefore

$$\langle \tilde{R}(T_1, T_2)\zeta_2, \zeta_3 \rangle = \frac{\sqrt{c}}{\lambda^2\sqrt{\lambda^2 - c}}(c_2 - c_1)T_1(\lambda)T_2(\lambda),$$

and then $\tilde{R} \equiv 0$ iff $T_1(\lambda)T_2(\lambda) = 0$. This completes the proof of the assertion.

(3.38) Now assume that $T_1(\lambda)T_2(\lambda) = 0$. Then (cf. [7]) for each $t \in [0, 1]$, there exists an isometric immersion f_t: $M^n \to \mathbf{R}^{n+3}$ whose normal bundle is identified with the normal bundle of $\tilde{x}_1$, and whose normal connection and second fundamental form are identified with $\tilde{\nabla}$ and $\tilde{\alpha}$, respectively. Furthermore, the mapping $(t, p) \mapsto f_t(p)$ is a continuous map from $[0, 1] \times M^n$ into $\mathbf{R}^{n+3}$. For $t = 1$, we have $f_1 = \tilde{x}_1$ (up to a rigid motion of $\mathbf{R}^{n+3}$). For $t = 0$, (3.36) shows that the normal space of f_0 has constant dimension one and is parallel. Therefore, $f_0(M)$ is contained in some $(n+1)$-dimensional affine subspace of $\mathbf{R}^{n+3}$. It then follows that $f_0 = \tilde{x}_0$ (up to a rigid motion of $\mathbf{R}^{n+3}$). To conclude the proof of (1.10), we only have to observe that the hypersurfaces of Type I or II satisfy $T_1(\lambda)T_2(\lambda) = 0$.

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