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THE ROLE OF TOTAL CURVATURE ON COMPLETE NONCOMPACT RIEMANNIAN 2-MANIFOLDS

BY

KATSUHIRO SHIOHAMA

I. Introduction

The total curvature $c(M)$ of a complete noncompact and oriented Riemannian 2-manifold M is defined by an imporper integral of the Gaussian curvature G with respect to the volume form $d\omega$ induced from the Riemannian structure over M. A classical theorem due to Cohn-Vossen [2] states that if such an M is finitely connected and if the total curvature exists, then

$$c(M) = \int_M G \, d\omega \leq 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. In the case where M is not finitely connected, a result due to A. Huber [4] states that if the total curvature of an infinitely connected M exists, then $c(M) = -\infty$. In contrast with the Gauss–Bonnet theorem for compact case, the total curvature of a noncompact M is not a topological invariant but it depends upon the choice of Riemannian structure. The total curvature should describe a certain property of the Riemannian structure which defines it.

The purpose of the present paper is to investigate a geometric significance on the existence (or non-existence) of total curvatures on complete, finitely connected and noncompact Riemannian 2-manifolds. It is the nature of completeness and noncompactness of M that through each point p on M there passes at least a ray $\gamma\colon [0, \infty) \to M$ with $\gamma(0) = p$. A *Busemann function* $F_\gamma\colon M \to R$ with respect to a ray γ is defined by

$$F_\gamma(x) := \lim_{t \to \infty} [t - d(x, \gamma(t))],$$

where d is the distance function induced from the Riemannian structure of M. Busemann functions play essential roles in the study of complete noncompact manifolds. For instance, they are convex if the sectional curvature is nonnegative and the negative of Busemann functions are convex if the sectional curvature is nonpositive and if M is simply connected. A function $h\colon M \to R$ is said to be an *exhaustion* if $h^{-1}((-\infty, a])$ is compact for all $a \in R$. By definition h is a *non-exhaustion* if it is not exhaustion.

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The present work was motivated by a recent paper [8] of the author in which the following theorem was proved.

THEOREM [8]. *Let M be a complete, noncompact and oriented Riemannian 2-manifold of nonnegative Gaussian curvature. Then the total curvature exists in* $[0, 2\pi]$ *and:*

(1) $c(M) \leq \pi$ *if and only if all Busemann functions on M are non-exhaustions.*

(2) $c(M) > \pi$ *if and only if all Busemann functions on M are exhaustions.*

As a direct consequence of the theorem, M cannot admit both exhaustion and non-exhaustion Busemann functions simultaneously if $G \geq 0$. Thus the behavior of Busemann function is in some sense controlled by the total curvature.

The phenomenon stated above will suggest a role of total curvature in general case. It will be anticipated from the above observation that the existence of a total curvature on M will imply that M will not simultaneously admit both exhaustion and non-exhaustion Busemann functions. However as is seen in Section 2, this phenomenon also depends on the end structure of M. In fact, Proposition 2.1 states that if an n-dimensional ($n \geq 2$) complete noncompact Riemannian manifold N has more than one end then any Busemann function on N takes its infimum to be $-\infty$, and hence is a non-exhaustion.

From now let M be a connected, complete, noncompact, finitely connected and oriented Riemannian 2-manifold with one end. Sufficient conditions for a Busemann function to be a non-exhaustion are proved in Section 4. And sufficient conditions for a Busemann function to be an exhaustion are proved in Section 5. Summing up these arguments we have:

MAIN THEOREM. *Assume that the total curvature exists on M.*

(1) *If* $c(M) < (2\chi(M) - 1)\pi$, *then all Busemann functions are non-exhaustions.*

(2) *If* $c(M) > (2\chi(M) - 1)\pi$, *then all Busemann functions are exhaustions.*

In the case where $c(M) = (2\chi(M) - 1)\pi$, it is shown in Section 6 that there are surfaces M_1, M_2 and M_3 in Euclidean 3-space E^3 with the same total curvature $(2\chi(M) - 1)\pi$ such that their Gaussian curvatures are nonpositive outside compact sets, and all Busemann functions on M_1 (M_2 respectively) are non-exhaustions (exhaustions respectively). There are both exhaustion and non-exhaustion Busemann functions on M_3. This example of M_3 was recently discovered by N. H. Kuiper. There is an example of a complete metric on R^2 with respect to which the total curvature does not exist and on which there are both exhaustion and non-exhaustion Busemann functions.

Recently some relations between the total curvature and the mass of rays emanating from an arbitrary fixed point on M have been investigated in [6] and [7].

The idea of the proof is to construct a 1-parameter (not necessarily continuous) family of (possibly broken) geodesics along an arbitrary fixed ray, contained in a domain homeomorphic to a closed half cylinder. Each member of the family has the property that it has the minimum length among all loops at a point on the ray in the closed half cylinder. The idea of this construction is based on the technique developed by Cohn-Vossen in [2] and is used here to investigate the relation between the total curvature and the behavior of Busemann functions.

The author would like to express his hearty thanks to N. H. Kuiper who provided M_3 and some other interesting examples of surfaces with singularities.

II. Busemann functions and ends

In this section let N be a complete noncompact Riemannian n-manifold and let γ: $[0, \infty) \to N$ be a ray. Fundamental properties of Busemann functions are investigated in [1]. The Lipschitz continuity property is stated here for later use. It follows by definition that $|F_\gamma(x) - F_\gamma(y)| \leq d(x, y)$ for every point x, y on N. Thus F_γ is Lipschitz continuous with the Lipschitz constant 1, and hence it is differentiable except on a set of measure zero. Busemann functions play an essential role in the study of complete noncompact Riemannian manifolds because they are convex (respectively subharmonic) if the sectional curvature (repectively Ricci curvature) is nonnegative, and they are concave if N is simply connected and if the sectional curvature is nonpositive.

If K_1, K_2 are compact sets of N such that $K_1 \subset K_2$, then every component of $N - K_2$ is contained in a unique component of $N - K_1$. An end of N is an element of the inverse limit system {components of $N - K$; K compact} indexed by compact sets of N and directed by the inclusion relation as indicated above.

PROPOSITION 2.1. *Let N be a complete noncompact Riemannian n-manifold. If N has nore than one end, then every Busemann function F_γ on N satisfies $\inf_N F_\gamma = -\infty$, and hence it is non-exhaustion.*

Proof. Let γ: $[0, \infty) \to N$ be a ray and set $p = \gamma(0)$.

There is a compact set $K \subset N$ such that $p \in K$ and such that $N - K$ has at least two unbounded components, say U and V, where U is chosen so as to satisfy $\gamma([0, \infty)) \subset K \cup U$. Take any point x in V and fix. For each $t > 0$ let τ_t: $[0, s_t] \to N$ be a minimizing geodesic with $\tau_t(0) = x$, $\tau_t(s_t) = \gamma(t)$. Clearly

$\tau_t([0, s_t]) \cap K \neq \emptyset$ holds for every $t > 0$. If y_t is a point on $\tau_t([0, s_t]) \cap K$, then

$$t - d(x, \gamma(t)) = t - d(x, y_t) - d(y_t, \gamma(t)) \leq t - d(x, K) - d(y_t, \gamma(t)).$$

Therefore if $\{y_{t_k}\}$ tends to a point $y \in K$ for some divergent sequence $\{t_k\}$, then

$$F_\gamma(x) \leq \lim_{k \to \infty} [t_k - d(y_{t_k}, \gamma(t_k))] - d(x, K) = F_\gamma(y) - d(x, K).$$

Since F_γ takes minimum on K and since for any positive R x can be chosen such that $d(x, K) > R$, $\inf_N F_\gamma = -\infty$. Thus F_γ is non-exhaustion.

III. Basic construction

From now let M be a 2-dimensional, complete, connected, noncompact, oriented and finitely connected Riemannian manifold with one end. M is homeomorphic to a compact 2-manifold with one point removed. Thus there is a compact set $K \subset M$ such that $M - \operatorname{int}(K)$ has a unique unbounded component and is homeomorphic to a closed half cylinder $S^1 \times [0, \infty)$, where $\operatorname{int}(K)$ is by definition the interior of K. The boundary ∂K of K is replaced by a simply closed geodesic polygon which is homotopic to ∂K. Let P_0 be such a simply closed geodesic polygon and let K_0 be a compact set bounded by P_0 and let $U_0 := M - \operatorname{int}(K_0)$. U_0 is homeomorphic to a closed half cylinder $S^1 \times [0, \infty)$.

To investigate the relations between the total curvature and the behavior of Busemann functions on M, it is useful to construct a 1-parameter (not necessarily continuous) family of simply closed geodesic polygons $\{P_t\}$ in U_0 each of which is freely homotopic to P_0 and which is obtained along a fixed ray. Let $\gamma\colon [0, \infty) \to M$ be an arbitrary fixed ray. If $\gamma(0) \in U_0$, then K_0 can be replaced by a larger $K \supset K_0$ whose boundary is a simply closed geodesic polygon P and which satisfies $\gamma(0) \in \partial K$ and

$$\gamma([0, \infty)) \subset U := M - \operatorname{int}(K).$$

If $\gamma(0) \in K_0$, the there is an $a \geq 0$ such that if $\tilde{\gamma}(t) = \gamma(t + a)$, then $\tilde{\gamma}(0) \in \partial K_0$ and $\tilde{\gamma}([0, \infty)) \subset U_0$. Since $F_{\tilde{\gamma}} - F_\gamma = a$, $F_{\tilde{\gamma}}$ is an exhaustion (or a non-exhaustion) if and only if so is F_γ. Therefore in order to check the exhaustion (or non-exhaustion) property of F_γ, one may consider without loss of generality that K, U and $P = \partial K$ are chosen so as to satisfy $\gamma(0) \in P = \partial K$ and $\gamma([0, \infty)) \subset U$. Further additional condition is imposed on the choice of P for later use in Sections 4 and 5.

Let $\gamma\colon [0, \infty) \to M$ be an arbitrary fixed ray and choose a compact set K and set $U := M - \operatorname{int}(K)$, $P := \partial U$ which satisfy the above mentioned properties. Let $\tilde{U}$ be the universal Riemannian covering of U and let $\pi\colon \tilde{U} \to U$ be the covering projection. $\tilde{U}$ is homeomorphic to a closed half plane and its boundary consists of a continuous broken geodesics homeomorphic to R. Let

$\hat{U}$ be the fundamental domain with boundary consisting of two rays $\hat{\gamma}_1$, $\hat{\gamma}_2$: $[0, \infty) \to \tilde{U}$, $\pi \circ \hat{\gamma}_1 = \pi \circ \hat{\gamma}_2 = \gamma$, and a broken geodesic $\hat{P}$ with $\pi(\hat{P}) = P$ such that the endpoints of $\hat{P}$ are $\hat{\gamma}_1(0)$ and $\hat{\gamma}_2(0)$. Any two points $\hat{x}$ and $\hat{y}$ in $\hat{U}$ can be joined by a minimizing (possibly broken) geodesic $\hat{\tau}$ in $\hat{U}$; such a $\hat{\tau}$ is called a *segment in* $\hat{U}$ joining $\hat{x}$ and $\hat{y}$. This fact follows from "Verteilungssatz" (see p. 120 of [2]) and its consequences (see also 12 of [2]). If a segment $\hat{\tau}$ is a broken geodesic, then every corner of it is a corner of $\hat{P}$ and the angle of $\hat{\tau}$ at a corner is not smaller than π if it is measured with respect to the unbounded domain. If $\hat{\tau}$ does not intersect $\hat{P}$, then $\hat{\tau}$ is smooth. If $\hat{d}$ is the distance function of $\hat{U}$ induced from the metric of $\hat{U}$ through π, then the length $L(\hat{\tau})$ of a segment in $\hat{U}$ joining $\hat{x}$ and $\hat{y}$ realizes the $\hat{d}$-distance $\hat{d}(\hat{x}, \hat{y})$. Note that

$$\hat{d}(\hat{x}, \hat{y}) \geq d(\pi(\hat{x}), \pi(\hat{y}))$$

for any $\hat{x}, \hat{y} \in \hat{U}$, where d is the distance function on M.

For each $t \geq 0$ let $\hat{\mathscr{P}}_t$ be the set of all segments in $\hat{U}$ joining $\hat{\gamma}_1(t)$ and $\hat{\gamma}_2(t)$. For $i = 1, 2$ and for every $t \geq 0$ let $\lambda_i(t)^+$ and $\lambda_i(t)^-$ be the supremum and the infimum of the angles at $\hat{\gamma}_i(t)$ between $\hat{\gamma}_i$ and $\hat{P}_t \in \hat{\mathscr{P}}_t$. If all elements of $\hat{\mathscr{P}}_t$ do not intersect $\hat{P}$, then $\hat{d}$-distance minimizing property implies that any two elements in $\hat{\mathscr{P}}_t$ intersect only at their endpoints. If two elements of $\hat{\mathscr{P}}_t$ intersect except their endpoints, then all points on the intersection except endpoints belong to $\hat{P}$. Hence there exists a unique element $\hat{P}_t^+ \in \hat{\mathscr{P}}_t$ such that all elements of $\hat{\mathscr{P}}_t$ are in the compact domain of $\hat{U}$ bounded by $\hat{P}$, $\hat{\gamma}_1([0, t])$ and $\hat{\gamma}_2([0, t])$ and $\hat{P}_t^+$. Similarly, there exists a unique element $\hat{P}_t^- \in \hat{\mathscr{P}}_t$ such that all elements of $\hat{\mathscr{P}}_t$ are not contained in the interior of the compact domain bounded by $\hat{P}$, $\hat{\gamma}_1([0, t])$, $\hat{\gamma}_2([0, t])$ and $\hat{P}_t^-$. It follows from the choice of $\hat{P}_t^+$ and $\hat{P}_t^-$ that $\lambda_i(t)^+$ and $\lambda_i(t)^-$ are equal to the angles between $\hat{P}_t^+$ and $\hat{\gamma}_i$, and between $\hat{P}_t^-$ and $\hat{\gamma}_i$ respectively.

LEMMA 3.1. *For every $t > 0$, the angles $\lambda_i(t)^+$ and $\lambda_i(t)^-$ have the following properties:*

$$\lim_{u \uparrow t} \lambda_i(u)^+ = \lim_{u \uparrow t} \lambda_i(u)^- = \lambda_i(t)^-,$$

$$\lim_{u \downarrow t} \lambda_i(u)^+ = \lim_{u \downarrow t} \lambda_i(u)^- = \lambda_i(t)^+.$$

Proof. It follows from the construction of $\hat{P}_t^+$ and $\hat{P}_t^-$ that

$$\lim_{u \uparrow t} \lambda_i(u)^+ \geq \lim_{u \uparrow t} \lambda_i(u)^- \geq \lambda_i(t)^-$$

and

$$\lim_{u \downarrow t} \lambda_i(u)^- \leq \lim_{u \downarrow t} \lambda_i(u)^+ \leq \lambda_i(t)^+.$$

If $\lim_{u \uparrow t} \lambda_i(u)^+ \leq \lambda_i(t)^-$ is proved, then the same method establishes $\lim_{u \downarrow t} \lambda_i(u)^- \geq \lambda_i(t)^+$. And hence only the first inequality is proved.

Suppose there is a monotone increasing sequence $\{u_j\}$ with $\lim_{j\to\infty} u_j = t$ such that $\lim_{j\to\infty} \lambda_i(u_j)^+ > \lambda_1(t)^-$. Let $r > 0$ be the radius of a convex ball around $\hat{\gamma}_1(t)$ in $\hat{U}$. For sufficiently large j choose a point $\hat{q}_j$ on $\hat{P}_j^+$ such that $d(\hat{q}_j, \hat{\gamma}_1(t)) = r/2$. Then the angle

$$\measuredangle(\hat{q}_j, \hat{\gamma}_1(t), \hat{\gamma}_1(u_j))$$

tends to $\lim \lambda_1(u_j)^+$ as $j\to\infty$. Since $\lim \lambda_1(u_j)^+ > \lambda_1(t)^-$, $\hat{q}_j$ does not belong to the compact domain bounded by $\hat{P}$, $\hat{\gamma}_1([0, t])$, $\hat{\gamma}_2([0, t])$ and $\hat{P}_t^-$ for all large j. In particular $\hat{q}_j$ is on $\hat{P}_t^-$. This means that $\hat{P}_j^+ := \hat{P}_{u_j}^+$ intersects $\hat{P}_t^-$ at two interior points, say, $\hat{m}_j$ and $\hat{n}_j$. Since every subarc of a segment is $\hat{d}$-distance minimizing, $\hat{d}(\hat{m}_j, \hat{n}_j)$ is attained by two distinct segments which are proper subarcs of $\hat{P}_j^+$ and $\hat{P}_j^-$, a contradiction. Similarly, $\lim_{u\uparrow t} \lambda_2(u_j)^+ \leq \lambda_2(t)^-$ for any monotone increasing sequence $\{u_j\}$ with $\lim u_j = t$.

Now the functions $t \to L(\hat{P}_t)$ and $t - L(\hat{P}_t)/2$ play important roles in finding sufficient conditions for F_γ to be an exhaustion or a non-exhaustion. The properties of these functions are therefore discussed in the following:

PROPOSITION 3.2. *The function $t \to L(\hat{P}_t)$ for an arbitrary fixed ray $\gamma: [0, \infty) \to M$ has the following properties:*

(1) *It is Lipschitz continuous with Lipschitz constant 2 and is differentiable except on a set of measure zero.*

(2) *It is differentiable at t if and only if $\hat{P}_t^+ = \hat{P}_t^-$; e.g., there is a unique element in $\hat{\mathscr{P}}_t$.*

(3) $L(\hat{P}_t) - L(\hat{P}_{t'}) = \int_{t'}^{t} [\cos \lambda_1(u) + \cos \lambda_2(u)]\, du$, *where* $\lambda_i(u) := \lambda_i(u)^+ = \lambda_i(u)^-$ *almost everywhere.*

(4) *$t - L(\hat{P}_t)/2$ is strictly monotone increasing with t.*

Proof. For every $t, t' > 0$ and for every $\hat{P}_t \in \hat{\mathscr{P}}_t$ and $\hat{P}_{t'} \in \hat{\mathscr{P}}_{t'}$,

$$\hat{\gamma}_1([t, t']) \cup \hat{P}_{t'} \cup \hat{\gamma}_2([t, t'])$$

is a broken geodesic joining $\hat{\gamma}_1(t)$ and $\hat{\gamma}_2(t)$ in $\hat{U}$. Then the triangle inequality implies that

$$|L(\hat{P}_t) - L(\hat{P}_{t'})| \leq 2|t - t'|.$$

It follows from Lemma 3.1 that

$$\lim_{h\downarrow 0} \hat{P}_{t+h}^- = \hat{P}_t^+ \quad \text{and} \quad \lim_{h\downarrow 0} \hat{P}_{t-h}^+ = \hat{P}_t^-,$$

and hence $\hat{P}_{t+h}^-$ is contained in a small tubular neighborhood of $\hat{P}_t^+$ if h is small. Thus the first and second variation formulas for a 1-parameter variation along $\hat{P}_t^\pm$ imply that for sufficiently small $h > 0$,

$$L(\hat{P}_t^-) - L(\hat{P}_{t-h}^+) = h[\cos \lambda_1(t)^- + \cos \lambda_2(t)^-] + o(h),$$

$$L(\hat{P}_{t+h}^-) - L(\hat{P}_t^+) = h[\cos \lambda_1(t)^+ + \cos \lambda_2(t)^+] + o(h).$$

Therefore $L(\hat{P}_t)$ is differentiable at t if and only if $\lambda_i(t)^- = \lambda_i(t)^+$ for $i = 1, 2$. In other words, the function is differentiable at t if and only if $\hat{\mathscr{P}}_t$ consists of a unique element, e.g., $\hat{P}_t^+ = \hat{P}_t^-$. Thus (1) and (2) are proved. (3) is obvious from

$$\frac{d}{dt} L(\hat{P}_t) = \cos \lambda_1(t) + \cos \lambda_2(t).$$

(4) is a straightforward consequence of the triangle inequality.

It should be noted that if the function $L(\hat{P}_t)$ is not differentiable at t, then there exists a compact domain bounded by $\hat{P}_t^+$ and $\hat{P}_t^-$. If $\hat{P}_t^-$ does not intersect $\hat{P}$ and if $L(\hat{P}_t)$ is not differentiable at t, then the compact domain bounded by these segments (say, $\hat{\Omega}_t$) is homeomorphic to a closed 2-disk and the Gauss–Bonnet theorem implies that

$$c(\hat{\Omega}_t) = \int_{\hat{\Omega}_t} G\, d\omega = \sum_{\lambda=1}^{2} [\lambda_i(t)^+ - \lambda_i(t)^-] > 0.$$

Now it follows from the $\hat{d}$-distance minimizing property of $\hat{P}_t$ that if $\hat{P}_t \cap \hat{P} = \emptyset$ for some $t > 0$ and some $\hat{P}_t \in \hat{\mathscr{P}}_t$, then $\hat{P}_{t'} \cap \hat{P} = \emptyset$ for all $t' > t$ and all $\hat{P}_{t'} \in \hat{\mathscr{P}}_{t'}$. Also if $\hat{P}_t \cap \hat{P} \neq \emptyset$ for some $t \geq 0$ and some $\hat{P}_t \in \hat{\mathscr{P}}_t$, then $\hat{P}_{t'} \cap \hat{P} \neq \emptyset$ for all $t' < t$ and all $\hat{P}_{t'} \in \hat{P}_{t'}$. If $\hat{P}_t \cap \hat{P} = \emptyset$, then $P_t := \pi(\hat{P}_t)$ is a simple geodesic loop on M at $\gamma(t)$. Note also that P_t is freely homotopic to $P = \pi(\hat{P})$ for any $\hat{P}_t$. Therefore the following two cases occur for the family $\{\hat{P}_t \in \hat{\mathscr{P}}_t; t \geq 0\}$ of segments in $\hat{U}$.

Case I. For any $t \geq 0$ and any $\hat{P}_t \in \hat{\mathscr{P}}_t$, $\hat{P}_t$ intersects $\hat{P}$.

Case II. There is a $t_0 \geq 0$ such that if $t > t_0$ then any $\hat{P}_t \in \hat{\mathscr{P}}_t$ does not intersect $\hat{P}$, and there is a $\hat{P}_{t_0} \in \hat{\mathscr{P}}_{t_0}$ which intersects $\hat{P}$.

It will be shown later that Case I corresponds to F_γ being a non-exhaustion and Case II with certain inevitable conditions corresponds to F_γ being an exhaustion.

The following Lemma 3.3 is originally proved by Cohn-Vossen (see Section 5 in [2]), the statement of which is slightly modified for the later use.

LEMMA 3.3. *Let $\sigma: [0, \infty) \to U$ be a ray and let $\{y_j\}$ be a sequence of points in U which converges to a point in U. For any $\varepsilon > 0$, there is a monotone increasing divergent sequence $\{t_j\}$ and a family of segments $\{\tau_j\}$ in U with each τ_j joining y_j to $\sigma(t_j)$ such that the angle at $\sigma(t_j)$ between τ_j and σ is less than ε.*

Proof. Suppose that for each fixed j there is no segment joining y_j to any point $\sigma(t)$ which makes an angle at $\sigma(t)$ not less than ε. Then $d(y_j, \sigma(t + h)) - d(y_j, \sigma(t)) \leq h \cos \varepsilon$ for small $h > 0$ and for all t. Thus

$$d(y_j, \sigma(t_2)) - d(y_j, \sigma(t_1)) \leq (t_2 - t_1) \cos \varepsilon$$

for all $t_2 > t_1$. It follows from the triangle inequality that

$$(t_2 - t_1) - d(y_j, \sigma(t_1)) \leq d(y_j, \sigma(t_2)) \leq (t_2 - t_1) \cos \varepsilon + d(y_j, \sigma(t_1)).$$

Thus for every fixed t_1, this inequality is violated if t_2 is taken sufficiently large, a contradiction. By iterating this procedure, one obtains the desired sequence and a family of segments.

IV. Sufficient conditions for Busemann functions to be non-exhaustions

This section discusses when F_γ becomes a non-exhaustion, dealing mainly with Case I.

From now let $\gamma\colon [0, \infty) \to M$ be an arbitrary fixed ray. Choose a compact set K and $U = M - \operatorname{int}(K)$ such that $P = \partial U$ is a simply closed geodesic polygon and U is homeomorphic to a closed half cylinder and such that $\gamma(0) \in P$. Moreover P is chosen so as to satisfy $F_\gamma(x) \leq 0$ for all $x \in P$; the parameter of γ will be changed if necessary. This is possible because if

$$\max\{F_\gamma(x); x \in P\} = a > 0,$$

then two sides of P with corner $\gamma(0)$ are replaced by minimizing geodesics in a ball of radius a centered at $\gamma(0)$ joining points on the sides in the ball and $\gamma(a)$. By changing the parameter of γ to $t' = t + a$, one obtains the new polygon P and γ which satisfy the desired property. This property is used to prove Lemma 4.1 (1). Let $\hat{U}$, $\hat{P}$ and $\hat{\gamma}_1$, $\hat{\gamma}_2$ be chosen for the new U, P and γ.

LEMMA 4.1. *Let x be any interior point on U and let $\tau\colon [0, l] \to M$ be a minimizing geodesic joining x to a point $\gamma(t)$, $t > 0$. Let $\hat{x} \in \operatorname{int}(\hat{U})$ be such that $\pi(\hat{x}) = x$ and let $\hat{\tau}$ be the lift of τ with $\hat{\tau}(0) = \hat{x}$. Then:*

(1) $\hat{d}(\hat{x}, \hat{P}) = d(x, P)$.

(2) *If $\hat{\tau}\colon [0, l] \to U$ is well defined, then*

$$d(x, \gamma(t)) = l = \hat{d}(\hat{x}, \hat{\gamma}_i(t))$$

for some $i = 1, 2$. Conversely if $d(x, \gamma(t)) = \hat{d}(\hat{x}, \hat{\gamma}_i(t))$ for some $i = 1, 2$, then there is a minimizing geodesic $\sigma\colon [0, l'] \to U$ in M joining x to $\gamma(t)$ such that σ has a complete lift $\hat{\sigma}\colon [0, l] \to \hat{U}$.

(3) *If $\hat{\tau}$ cannot be lifted completely in $\hat{U}$. then $\tau([0, l])$ intersects* int (K) *and for any $i = 1, 2$,*

$$\hat{d}(\hat{x}, \hat{\gamma}_i(t)) \gneqq d(x, \gamma(t)) \geqq d(x, P) + t - L(P)/2.$$

Moreover, if $\hat{d}(\hat{x}, \hat{\gamma}_i(t)) > d(x, \gamma(t))$ for any $i = 1, 2$, then every minimizing geodesic τ joining x to $\gamma(t)$ intersects int (K) *and τ cannot be lifted in $\hat{U}$.*

Proof of (1) *and* (2). It follows from the choice of P that $F_\gamma(y) \leq 0$ for all $y \in P$, and hence $d(\gamma(t), P) = t$ for all $t \geq 0$. In fact, suppose that there is a $t' > 0$ and $y \in P$ with

$$d(\gamma(t'), y) = d(\gamma(t'), P) < t'.$$

Then $F_\gamma(y) = \lim_{t\to\infty} [t - d(y, \gamma(t))] \geq t' - d(y, \gamma(t')) > 0$, a contradiction to the choice of P. This inequality follows from the fact that $t - d(y, \gamma(t))$ is monotone increasing with t. Let σ: $[0, l'] \to M$ be a minimizing geodesic such that $\sigma(0) = x$, $\sigma(l') \in P$ and $l' = d(x, P)$. It follows from the minimizing property of γ and σ in M that $\sigma([0, l))$ cannot intersect $\gamma([0, \infty))$ and P. Let $\hat{\sigma}$ be the lift of σ with $\hat{\sigma}(0) = \hat{x}$. $\hat{\sigma}$ intersects neither $\hat{\gamma}_1$ nor $\hat{\gamma}_2$, and hence $\hat{\sigma}$ is completely lifted in $\hat{U}$. Thus $\hat{\sigma}(l') \in \hat{P}$. Since $L(\sigma) = L(\hat{\sigma}) = l'$, $d(x, P) = \hat{d}(\hat{x}, \hat{P})$. (2) is now obvious.

Proof of (3). Note that $\tau([0, l))$ does not intersect $\gamma([0, \infty))$. Therefore $\tau([0, l])$ intersects P at a point $\tau(s)$, $0 < s < l$. From triangle inequality,

$$d(x, \gamma(t)) = d(x, \tau(s)) + d(\tau(s), \gamma(t)) \geq d(x, P) + (t - d(\gamma(0), \tau(s)))$$
$$\geq d(x, P) + t - L(P)/2.$$

If $d(x, \gamma(t)) < \hat{d}(\hat{x}, \hat{\gamma}_i(t))$ for $i = 1, 2$, then every segment in $\hat{U}$ joining $\hat{x}$ to $\hat{\gamma}_i(t)$ has length $> d(x, \gamma(t))$. Therefore τ cannot be lifted completely in $\hat{U}$.

THEOREM 4.2. *Assume that Case I hold for γ. Then:*

(1) *F_γ is a non-exhaustion.*
(2) *If the total curvature exists, then $c(M) \leq (2\chi(M) - 1)\pi$.*

Proof of (1). Let $R > L(P) + 1$ be a fixed number and let $V_{R-1}(P)$ be the unique unbounded component of $\{x \in U; d(x, P) > R - 1\}$. By Lemma 4.1 every segment $\hat{\tau}$ in $\hat{U}$ joining a point $\hat{x}$ to a point on $\hat{P}$ such that $L(\hat{\tau}) = \hat{d}(\hat{x}, \hat{P})$ intersects neither $\hat{\gamma}_1([0, \infty))$ nor $\hat{\gamma}_2([0, \infty))$, and $\hat{\tau}$ is contained entirely in $\hat{U}$. Therefore

$$\hat{d}(\hat{x}, \hat{P}) = d(\pi(\hat{x}), P).$$

There exists a curve $\hat{c}$: $[0, 1] \to \hat{V}_{R-1}(\hat{P})$ such that $\hat{c}(0) = \hat{\gamma}_1(R)$ and $\hat{c}(1) = \hat{\gamma}_2(R)$. Take a monotone increasing divergent sequence $\{t_j\}$. For large j with $t_j > R$, the function f_j: $[0, 1] \to \mathbf{R}$ with

$$f_j(u) = \hat{d}(\hat{c}(u), \hat{\gamma}_1(t_j)) - \hat{d}(\hat{c}(u), \hat{\gamma}_2(t_j))$$

takes value 0 on the interval. Indeed, by hypothesis, every segment in $\hat{U}$ joining $\hat{\gamma}_1(R)$ to $\hat{\gamma}_2(R)$ intersects $\hat{P}$, and hence

$$f_j(0) \leq (t_j - R) - [R + t_j - L(\hat{P})] \leq -2R + L(\hat{P}).$$

Similarly,

$$f_j(1) \geq [t_j + R - L)\hat{P})] - (t_j - R) = 2R - L(\hat{P}).$$

Continuity of f_j implies the existence of a $u_j \in (0, 1)$ such that $f_j(u_j) = 0$. Set $\hat{x}_j = \hat{c}(u_j)$, $x_j = \pi(\hat{x}_j)$ and $\hat{x} = \hat{c}(u)$, $x = \pi(\hat{x})$, where u is the limit of a convergent subsequence $\{u_k\}$ of $\{u_j\}$. Let τ_k be a minimizing geodesic in M joining x_k to $\gamma(t_k)$, and let $\hat{\tau}_k$ be the lift of τ_k such that $\hat{\tau}_k(0) = \hat{x}_k$. If τ_k intersects P, then

$$L(\tau_k) \geq t_k + R - 1$$

follows from the triangle inequality. Hence $t_k - d(x_k, \gamma(t_k)) \leq -R$. If τ_k does not intersect P, then it is contained entirely in U. This follows from the fact that τ_k and γ are minimizing in M and hence their lifted image in $\hat{U}$ cannot intersect except at the endpoints. Therefore

$$d(x_k, \gamma(t_k)) = \hat{d}(\hat{x}_k, \hat{\gamma}_1(t_k)) \geq L(\hat{P}_{t_k})/2 \geq t_k - L(\hat{P})/2$$

if τ_k does not intersect P. The last inequality follows from $P \cap P_{t_k} \neq \emptyset$ together with the triangle inequality. In any case,

$$t_k - d(x_k, \gamma(t_k)) \leq L(P)/2$$

for all large k. This implies $F_\gamma(x) \leq L(P)/2$ and $d(x, P) > R - 1$. Since R is taken arbitrarily large, the proof of (1) is complete.

Proof of (2). For a monotone increasing divergent sequence $\{t_j\}$ choose a monotone increasing family $\{D_j\}$ of compact domains such that $K \subset D_1$, $\bigcup D_j = M$, ∂D_j is a simply closed geodesic polygon on which $\gamma(t_j)$ lies and ∂D_j is freely homotopic to P. Such a family is obtained by taking a curve $\hat{c}_j$: $[0, 1] \to \hat{V}_{t_j - 1}(\hat{P})$ as in the proof of (1), replacing it by a simple geodesic polygon and finally projecting it by π.

Set $\hat{D}_j := \hat{U} \cap \pi^{-1}(D_j)$ and for each j consider $\hat{U}_j = \hat{U} - \operatorname{int}(D_j)$ instead of $\hat{U}$. The boundary $\partial\hat{D}_j$ takes the place of $\hat{P}$. For each j and each $t > t_j$, either segment $\hat{Q}_{j,t}$ in $\hat{U}_j$ joining $\hat{\gamma}_1(t)$ and $\hat{\gamma}_2(t)$ intersects $\partial\hat{D}_j$ or else there is a $t_j^* \geq t_j$ such that if $t > t_j^*$, then $\hat{Q}_{t,j}$ does not intersect $\partial\hat{D}_j$. It follows from the definition of $\hat{P}_t$ and $\hat{Q}_{j,t}$ that $L(\hat{Q}_{j,t}) \geq L(\hat{P}_t)$ for all $t \geq t_j$. Thus the proof is divided into two cases.

If there is a subsequence $\{t_k\}$ of $\{t_j\}$ such that for each k every segment $\hat{Q}_{k,t}$ in $\hat{U}_k$ joining $\hat{\gamma}_1(t)$ and $\hat{\gamma}_2(t)$ intersects $\partial\hat{D}_k$ for all $t \geq t_k$, then Lemma 3.3 implies that for each k, if ε is any positive number then there is an $s > t_k$ such that the angle of $\pi(\hat{Q}_{k,s})$ at $\gamma(s)$ is less than ε. Recall that every corner of $Q_{k,s} = \pi(\hat{Q}_{k,s})$ except $\gamma(s)$ makes an angle not greater than π, and hence for small ε, $Q_{k,s}$ bounds a convex domain $\tilde{D}_k$ in M. It follows from $D_k \subset \tilde{D}_k$ that $\bigcup \tilde{D}_k = M$. Since the total curvature exists,

$$c(M) = \lim c(D_k) \leq (2\chi(M) - 1)\pi + \varepsilon.$$

The proof is complete since ε is arbitrary.

In the next case assume that there is a subsequence $\{t_k\}$ of $\{t_j\}$ such that for each k there is a $t_k^* \geq t_k$ such that for any $t > t_k^*$ and for any segment $\hat{Q}_{k,t}$ in $\hat{U}_k$, $\hat{Q}_{k,t}$ does not intersect $\partial\hat{U}_k$. Since the distance restricted to $\hat{U}_k$ is not less than $\hat{d}$, $L(\hat{Q}_{k,t}) \geq L(\hat{P}_t)$ for all $t \geq t_k$. For an arbitrary fixed $\varepsilon > 0$, there is an $s_k > t_k^*$ such that $\hat{Q}_{k,s_k}$ makes angles with $\hat{\gamma}_1$ and $\hat{\gamma}_2$ less than $\varepsilon/2$. In fact, if $\mu_i(t)$ is the angle between $\hat{\gamma}_i$ and $\hat{Q}_{k,t}$ at $\hat{\gamma}_i(t)$, where $L(\hat{Q}_{k,t})$ is differentiable at t and if $\mu_1(t) \geq \varepsilon/2$ or $\mu_2(t) \geq \varepsilon/2$ for all $t \geq t_k^*$, then Proposition 3.2 (3) implies that

$$L(\hat{Q}_{k,t}) - L(\hat{Q}_{k,t_k^*}) \leq (t - t_k^*)(\cos(\varepsilon/2) + 1).$$

But on the other hand, $L(\hat{Q}_{k,t}) \geq L(\hat{P}_t)$ for all $t \geq t_k$, and hence

$$L(\hat{Q}_{k,t}) - L(\hat{Q}_{k,t_k^*}) \geq L(\hat{P}_t) - L(\hat{Q}_{k,t_k^*}) > (2t - L(\hat{P})) - L(\hat{Q}_{k,t_k^*}).$$

This leads to a contradiction for all sufficiently large t.

It should be noted that Case II does not necessarily imply that F_γ is an exhaustion. This is because the family of compact domains $\{D_t^+\}$ bounded by P_t^+ does not cover the whole M even when Case II holds for γ.

THEOREM 4.3. *Let γ: $[0, \infty) \to M$ be a fixed ray and K, $U = M - \operatorname{int}(K)$ and $P = \partial U$ be taken as before. Assume that Case II holds for γ. Then F_γ is a non-exhaustion if either*

(a) *there is a point $z \in M - \bigcup_t D_t^+$,*

or

(b) $\lim_{t\to\infty} [t - L(P_t)/2] < \infty$.

Moreover, in addition to the above assumption, if the total curvature exists, then $c(M) \leq (2\chi(M) - 1)\pi$.

Remark. It follws from the triangle inequality that under the assumption of Case II for γ, (a) implies (b). However the converse is not true in general even if the total curvature exists.

Proof in case (a). It follows by hypothesis that there is a $t_0 \geq 0$ such that every P_t in $\mathscr{P}_t$ does not intersect P for all $t > t_0$. Thus every P_t in $\mathscr{P}_t$ is a geodesic loop at $\gamma(t)$ for all $t > t_0$. The midpoint m_t of a fixed P_t is not necessarily the cut point to $\gamma(t)$ (It will be shown in Lemma 5.1 that m_t is the cut point to $\gamma(t)$ along P_t if $\bigcup_t D_t^+ = M$.) It follows from the hypothesis that if b: $[0, 1] \to M$ is a curve with $b(0) \in P$ and $b(1) = z$, then every P_t intersects $b([0, 1])$ for all $t > 0$.

Assume in the first case that there is a monotone increasing divergent sequence $\{t_j\}$ such that for each j the midpoint m_j of some $P_j := P_{t_j}$ in $\mathscr{P}_{t_j}$ is not a cut point to $\gamma(t_j)$ along P_j. Then every minimizing geodesic in M joining m_j to $\gamma(t_j)$ intersects P. For a fixed large R, choose a curve $\hat{c}$: $[0, 1] \to \hat{c}_{R-1}(\hat{P})$ such that $\hat{c}(0) = \hat{\gamma}_1(R)$, $\hat{c}(1) = \hat{\gamma}_2(R)$ and $c = \pi \circ \hat{c}$ is freely homotopic to P. Such a curve is obtained as in the proof of Theorem 4.2 (1).

Lemma 4.1, (1) implies that $d(c(u), P) > R - 1$ for all $u \in [0, 1]$. For each j with $t_j > R$ there is a point $\hat{x}_j$ on $\hat{c}$ ([0, 1]) such that

$$\hat{d}(\hat{x}_j, \hat{\gamma}_1(t_j)) = \hat{d}(\hat{x}_j, \hat{\gamma}_2(t_j)).$$

If a minimizing geodesic in M joining $x_j = \pi(\hat{x}_j)$ and $\gamma(t_j)$ intersects P, then

$$d(x_j, \gamma(t_j)) \geq d(x_j, P) + d(\gamma(t_j), P) > t_j + R - 1.$$

If a minimizing geodesic in M joining x_j to $\gamma(t_j)$ does not intersect P, then it has a complete lift in $\hat{U}$ and Lemma 4.1 (2) implies

$$d(x_j, \gamma(t_j)) = \hat{d}(\hat{x}_j, \hat{\gamma}_i(t_j)) \geq L(P_j)/2, \quad i = 1, 2.$$

If $\hat{m}'_j$ is a point on the intersection of $\hat{b}([0, 1])$ and $\hat{P}_j$, where $\hat{b}$ is the lift of b in $\hat{U}$, then the triangle inequality implies

$$L(\hat{P}_j) \geq 2t_j - [\hat{d}(\hat{m}'_j, \hat{\gamma}_1(0)) + \hat{d}(\hat{m}'_j, \hat{\gamma}_2(0))] \geq 2t_j - C$$

where $C = L(b) + L(P)$. In any case,

$$t_j - d(x_j, \gamma(t_j)) \leq t_j - \min\{t_j + R - 1, t_j - C/2\}.$$

Thus, if x is the limit of a subsequence $\{x_k\}$ of $\{x_j\}$, then $F_\gamma(x) \leq C/2$ and $d(x, P) > R - 1$. This proves that F_γ is a non-exhaustion.

Assume in the next case that there is a $t' > t_0$ such that for any $t > t'$ and for any P_t in $\mathscr{P}_t$ the midpoint m_t of P_t is the cut point to $\gamma(t)$ along P_t. Choose a curve $\hat{c}$: $[0, 1] \to \hat{V}_{R-1}(\hat{P})$ and let $\{t_j\}$ be a monotone divergent sequence as before. For each j with $t_j > R$ let y_i be a point on the intersection of $P_j \in \mathscr{P}_{t_j}$ with $b([0, 1])$ and let $\hat{x}_j$ be a point on $\hat{c}([0, 1])$ with

$$\hat{d}(\hat{x}_j, \hat{\gamma}_1(t_j)) = \hat{d}(\hat{x}_j, \hat{\gamma}_2(t_j))$$

as before. If $\lim x_k = x \in c([0, 1])$ and $\lim y_k = y \in b([0, 1])$ for a subsequence $\{t_k\}$ of $\{t_j\}$, then

$$t_k - d(x_k, \gamma(t_k)) \leq t_k - L(P_k)/2 \leq t_k - d(y_k, \gamma(t_k)).$$

Hence

$$F_\gamma(x) = \lim [t_k - d(x_k, \gamma(t_k))] \leq \lim [t_k - d(y_k, \gamma(t_k))] = F_\gamma(y).$$

Since R is any, this implies that $F_\gamma^{-1}((-\infty, F_\gamma(y)])$ is noncompact and the proof is complete.

Proof in Case (b). With the same notation as in the first proof,

$$t_k - d(x_k, \gamma(t_k)) \leq t_k - L(P_k)/2.$$

If $\lim [t - L(P_t)/2] = \kappa$, then Proposition 3.2, (4) implies that $F_\gamma(x) \leq \kappa$. Since R is arbitrary, $F_\gamma^{-1}((-\infty, \kappa])$ is noncompact. The proof is complete.

For the proof of the final statement of Theorem 4.3 in the case of (a), let $\hat{P}_\infty$ be a straight line obtained as the limit of $\{\hat{P}_j\}$. Let $\theta(t_j)$ be the angle of the corner of P_j at $\gamma(t_j)$. If $\hat{V} \subset \hat{U}$ is the domain bounded by $\hat{P}_\infty$, it is homeomorphic to a closed half plane. Set $V = \pi(\hat{V})$. Since the total curvature exists,

$$c(M) = c(V) + c(M - V)$$

where $M - V = \bigcup D_j$ and D_j is the compact domain bounded by P_j. Then it has already been proved by Cohn-Vossen in [3] that $c(V) \leq 0$. From

$$c(D_j) = 2\chi(M) - (\pi - \theta(t_j))$$

and $\lim \theta(t_j) = 0$ (Lemma 3.3), $c(M) \leq c(M - V) = (2\chi(M) - 1)\pi$.

For the proof of the final statement in the case of (b), it suffices to check $\lim \theta(t_j) = 0$. But this is a direct consequence of

$$t - L(P_t)/2 \geq \int_0^t [1 - \cos \theta(u)/2]\, du + L(P_0).$$

V. Sufficient conditions for Busemann functions to be exhaustions

Let γ: $[0, \infty) \to M$ be a fixed ray and let K, $U = M - \operatorname{int}(K)$, $P = \partial U$ be defined as in the previous section. In view of Theorems 4.2 and 4.3, one considers the following three assumptions:

(1) Case II holds for γ,
(2) $\bigcup_t D_t = M$,
(3) $g(t) = t - L(P_t)/2$ is unbounded above.

It follows from definition of Busemann functions that

$$|F_\gamma(x) - F_\gamma(y)| \leq d(x, y) \quad \text{for all } x, y \in M.$$

Therefore if $x \in P_t$ for some $P_t \in \mathscr{P}_t$, then

$$|F_\gamma(x) - F_\gamma(\gamma(t))| \leq d(x, \gamma(t)) \leq L(P_t)/2.$$

Thus from $F_\gamma(\gamma(t)) = t$, one has $F_\gamma(x) \geq t - L(P_t)/2$ if $x \in P_t$. It follows from this inequality that if $g(t)$ is unbounded above, then Case II holds for γ, and $\bigcup_t D_t = M$. In fact, the above inequality for any $x \in P_t$ implies that $d(P, P_t)$ is unbounded above.

In order to find sufficient conditions for F_γ to be an exhaustion, it is necessary to construct complete lifts of minimizing geodesics joining $\gamma(t)$ to points on P_s for $s > t$. The following Lemmas 5.1 and 5.2 show the existence of complete lifts of minimizing geodesics on M.

LEMMA 5.1. *Under assumptions* (1) *and* (2), *there exists* $t_* \geq t_0$ *such that for any* $t \geq t_*$ *and any* $\hat{P}_t \in \hat{\mathscr{P}}_t$, *if* $\hat{m}_t$ *is the midpoint of* $\hat{P}_t$ *then* $\hat{d}(\hat{m}_t, \hat{\gamma}_i(t)) = d(m_t, \gamma(t)) = L(P_t)/2$ *for* $i = 1, 2$, *where* $m_t = \pi(\hat{m}_t)$. *Moreoer if* $t' \geq t \geq t_*$, *then*

$$\min\{\hat{d}(\hat{m}_t, \hat{\gamma}_1(t')), \hat{d}(\hat{m}_t, \hat{\gamma}_2(t'))\} = d(m_t, \gamma(t'))$$

and there is a minimizing geodesic τ in M joining m_t and $\gamma(t')$ whose image is in $M - \operatorname{int}(D_t)$ and which has a complete lift in $\hat{U}$, where D_t is the compact domain bounded by P_t.

Proof. Suppose that there is a divergent sequence $\{t_j\}$ and for each j there is a $\hat{P}_j \in \hat{\mathscr{P}}_{t_j}$ such that if $\hat{m}_j$ is the midpoint of $\hat{P}_j$, then

$$\hat{d}(\hat{m}_j, \hat{\gamma}_i(t_j)) > d(m_j, \gamma(t_j)) \quad \text{for } i = 1, 2.$$

If τ_j is a minimizing geodesic joining m_j and $\gamma(t_j)$, then Lemma 4.1 (3) implies that τ_j intersects P and

$$d(m_j, \gamma(t_j)) \geq d(m_j, P) + t_j - L(P)/2.$$

It follows from Proposition 3.2 (3) that if $\theta(t) = \lambda_1(t) + \lambda_2(t)$ (where the function $t \to L(P_t)$ is differentiable at t), then

$$L(P_j)/2 \leq \int_0^{t_j} \cos(\theta(u)/2)\, du + L(P_0)/2.$$

On the other hand $L(P_j)/2 = \hat{d}(\hat{m}_j, \hat{\gamma}_i(t_j)) > d(m_j, \gamma(t_j))$. Combining these inequalities with $L(P_0) \leq L(P)$ yields

$$\int_0^{t_j} [1 - \cos(\theta(u)/2]\, du < L(P) - d(m_j, P) \leq L(P) - d(P_j, P).$$

But the right hand side of the latter inequality is negative for all large j, a contradiction. Thus the existence of t_* is proved.

Next, for every $t' \geq t \geq t_*$ and every $P_t \in \mathscr{P}_t$ let $\tau'\colon [0, l'] \to M$ be a minimizing geodesic with $\tau'(0) = m_t$, $\tau'(l') = \gamma(t')$. Each subarc of P_t with endpoint m_t and $\gamma(t)$ is minimizing from the first part of the proof of the lemma. Hence $\tau'([0, l']) \cap P_t = \{m_t\}$. It follows from $\gamma(t') \in M - D_t$ that $\tau'([0, l'])$ is contained entirely in $M - \operatorname{int}(D_t)$, and thus τ' has a complete lift in $\hat{U} - \operatorname{int}(\hat{D}_t)$. This proves the desired equality.

LEMMA 5.2. *Under assumptions* (1) *and* (2), *let $s > t_*$ be a non-differentiable point of the function $t \to L(P_t)$ and let $\hat{\Omega}_s \subset \hat{U}$ be the domain* bounded *by $\hat{P}_s^-$ and $\hat{P}_s^+$. If $\hat{y}$ is a point on* int $\hat{\Omega}_s$ *with*

$$\hat{d}(\hat{y}, \hat{\gamma}_1(s)) = \hat{d}(\hat{y}, \hat{\gamma}_2(s)),$$

then

$$d(y, \gamma(s)) = \hat{d}(\hat{y}, \hat{\gamma}_i(s)) \quad \text{for } i = i, 2,$$

where $y = \pi(\hat{y})$. Moreover if $s' \geq s \geq t_*$, then

$$d(y, \gamma(s')) = \min\{\hat{d}(\hat{y}, \hat{\gamma}_1(s)), \hat{d}(\hat{y}, \hat{\gamma}_2(s))\}$$

and every minimizing geodesic τ' joining y to $\gamma(s')$ has its image in $M - D_s^-$ and has a complete lift in $\hat{U}$, where D_s^- is the compact domain bounded by P_s^-.

Proof. Let $\tau\colon [0, l] \to M$ be a minimizing geodesic with $\tau(0) = y$, $\tau(l) = \gamma(s)$. If τ has a complete lift $\hat{\tau}\colon [0, l] \to \hat{U}$ with $\hat{\tau}(0) = \hat{y}$, $\hat{\tau}(l) = \hat{\gamma}_i(s)$ for some $i = 1, 2$, then the desired equality holds and the convexity of $\hat{\Omega}_s$ implies $\hat{\tau}([0, l]) \subset \hat{\Omega}_s$. Suppose that τ does not have a complete lift. Then τ intersects P, and hence τ intersects P_s^- at an interior point q of P_s^-. Thus q and $\gamma(s)$ are joined by two distinct minimizing geodesics, one of which is a proper subarc of τ. This is a contradiction.

Let $\tau'\colon [0, l'] \to M$ be a minimizing geodesic with $\tau'(0) = y$, $\tau'(l') = \gamma(s')$. For $i = 1, 2$ let $\hat{\tau}_i\colon [0, l] \to \hat{U}$ be a segment in $\hat{U}$ with $\hat{\tau}_i(0) = \hat{y}$, $\hat{\tau}_i(l) = \hat{\gamma}_i(s)$. Then $\hat{\tau}_1[0, l]) \cup \hat{\tau}_2([0, l])$ divides $\hat{\Omega}_s$ into two components. If τ' intersects P, then τ' also intersects one of the $\tau_1 = \pi(\hat{\tau}_1)$ and $\tau_2 = \pi(\hat{\tau}_2)$ at an interior point. But this contradicts the minimizing property of τ_i and τ'. Since $\tau'([0, l'])$ is contained entirely in $M - D_s^-$, τ' has a complete lift in $U - D_s^-$. Therefore

$$d(y, \gamma(s')) = \min \{\hat{d}(\hat{y}, \hat{\gamma}_1(s')), \hat{d}(\hat{y}, \hat{\gamma}_2(s'))\}$$

and the proof is complete.

THEOREM 5.3. *Let $\gamma\colon [0, \infty) \to M$ be a ray. Assume that the total curvature exists. If $g(t) = t - L(P_t)/2$ is unbounded above, then*

(1) *$c(M) \geq (2\chi(M) - 1)\pi$ and*
(2) *F_γ is an exhaustion.*

Proof of (1). Let $\{t_j\}$ be a monotone divergent sequence such that for each j, $g'(t_j) > 0$. If D_j is the compact domain bounded by P_{t_j} and if $\theta(t_j)$ is the angle of P_{t_j} at $\gamma(t_j)$, then $\theta(t_j) < \pi$ follows from $g'(t_j) > 0$. Since $g(t)$ is unbounded, $d(P_{t_j}, P) \to \infty$ as $j \to \infty$, and hence $\bigcup D_j = M$. Since

$$c(D_j) = (2\chi(M) - 1)\pi + \theta(t_j),$$

the proof of (1) is complete.

Here is the idea of the proof of (2). Suppose that F_γ is non-exhaustion. Then a contradiction is derived by showing that the total curvature does not exist. This is achieved by constructing a family of disjoint compact domains in $\hat{U}$ each of which is bounded by a geodesic loop, on which the curvature integral is greater than π. If F_γ is a non-exhaustion, then there is a $c \in R$ and a divergent sequence $\{x_j\}$ of points on M such that $F_\gamma(x_j) \leq c$ for all j. There is a number j_0 such that if $j > j_0$, then there is an $s_j > 0$ such that $x_j \notin P_t$ for all P_t in $\mathscr{P}_t$ and for all $t \geq t_0$, and in particular x_j is contained in the interior of $\Omega_j = \pi(\hat{\Omega}_{s_j})$, and $\lim s_j = \infty$. In fact, suppose that x_j is on some P_{u_j} for all large j. As has already been noted at the beginning of this section, $F_\gamma(z) \geq g(t)$ holds for any z on P_t and for all $P_t$in $\mathscr{P}_t$. Thus

$$c \geq F_\gamma(x_j) \geq u_j - d(x_j, \gamma(u_j)) \geq g(u_j).$$

Since g is strictly monotone increasing by Proposition 3.2, this implies that $\{u_j\}$ is bounded above, and $\{x_j\}$ is bounded, a contradiction. Thus $x_j \in$

int (Ω_j) for all $j > j_0$, and g is not differentiable at s_j. The desired domains bounded by geodesic loops may be found in $\hat{\Omega}_j$'s or around them as follows.

Proof of (2). Fix a j with $s_j > t_*$ such that $g(s_j) > c + 1$ and fix $k > j$. Recall that $\hat{x}_j$ is contained in the interior of $\hat{\Omega}_j = \hat{\Omega}_{s_j}$ which is bounded by the segments $\hat{P}_j^+ := \hat{P}_{s_j}^+$ and $\hat{P}_j^- := \hat{P}_{s_j}^-$. Let $\hat{y}$ be the midpoint of $\hat{P}_j^-$, and for $i = 1, 2$ let $\hat{\sigma}_i$: $[0, l_i] \to \hat{U}$ be segments such that $\hat{\sigma}_i(0) = \hat{y}$, $\hat{\sigma}_i(l_i) = \hat{\gamma}_i(s)$. By Lemma 5.1,

$$\hat{\sigma}_i([0, l_i]) \subset \hat{U} - \operatorname{int}(\hat{D}_j^-) \quad \text{for } i = 1, 2.$$

Thus $\hat{\Omega}_j$ is divided into three components by the subarc of $\hat{P}_j^-$, $\hat{P}_j^+$ and the subarcs of $\hat{\sigma}_1([0, l_1])$ and $\hat{\sigma}_2([0, l_2])$.

If $\hat{x}_j$ is contained in the compact domain bounded by the geodesic triangle with vertices $\hat{\gamma}_i(s_j)$, $\hat{y}$ and the intersection $\hat{P}_j^+ \cap \hat{\sigma}_i([0, l_i])$ for some $i = 1, 2$, then the circumference of the triangle with sides

$$\hat{\sigma}_i([0, l_i]), \quad \hat{\gamma}_i([s_j, s_k])$$

and a subarc of $\hat{P}_j^-$ is less than $2\hat{d}(\hat{x}_j, \hat{\gamma}_i(s_k))$. In fact,

$$2\hat{d}(\hat{x}_j, \hat{\gamma}_i(s_k)) \geq 2d(x_j, \gamma(s_k)) \geq 2\,|F_\gamma(x_j) - F_\gamma(\gamma(s_k))| = 2(s_k - F_\gamma(x_j)).$$

The circumference of the triangle is less than

$$2\{(s_k - s_j) + L(P_j)/2\} = 2(s_k - g(s_j)) < 2(s_k - c - 1) < 2(s_k - F_\gamma(x_j)).$$

This inequality makes it possible to proceed to the standard length-decreasing deformation of this triangle with the base point $\hat{\gamma}_i(s_k)$ fixed. No length-decreasing curve of this deformation passes through $\hat{x}_j$, and the limit $\hat{\beta}_j$ of the deformation exists and is a geodesic loop at $\hat{\gamma}_i(s_k)$.

If $\hat{x}_j$ belongs to the interior of the compact domain bounded by the geodesic triangle whose sides are on $\hat{P}_j^+$, $\hat{\sigma}_1([0, l_1])$ and $\hat{\sigma}_2([0, l_2])$, then

$$\hat{d}(\hat{x}_j, \hat{\gamma}_1(s_k)) + \hat{d}(\hat{x}_j, \hat{\gamma}_2(s_k)) > \hat{d}(\hat{y}, \hat{\gamma}_1(s_k)) + \hat{d}(\hat{y}, \hat{\gamma}_2(s_k)).$$

In fact, one has

$$\sum_{i=1}^{2} \hat{d}(\hat{x}_j, \hat{\gamma}_i(s_k)) - \sum_{i=1}^{2} \hat{d}(\hat{y}, \hat{\gamma}_i(s_k)) \geq 2(s_k - c) - \sum_{i=1}^{2} \hat{d}(\hat{y}, \hat{\gamma}_i(s_k)).$$

It follows from triangle inequality that

$$\sum_{i=1}^{2} \hat{d}(\hat{y}, \hat{\gamma}_i(s_k)) \leq 2(s_k - s_j) + L(\hat{P}_j^-).$$

Therefore

$$2(s_k - c) - \sum_{i=1}^{2} \hat{d}(\hat{y}, \hat{\gamma}_i(s_k)) \geq 2(g(s_j) - c) > 2.$$

The latter inequality ensures that the broken geodesic

$$\hat{\sigma}_1([0, l_1]) \cup \hat{\sigma}_2([0, l_2])$$

is continuously deformed by length-decreasing deformation of curves with base points $\hat{\gamma}_1(s_k)$ and $\hat{\gamma}_2(s_k)$ fixed. Consider a compact domain $\hat{D}_k^-$ which is bounded by $\hat{\gamma}_1([0, s_k])$, $\hat{P}$, $\hat{\gamma}_2([0, s_k])$ and two segments joining $\hat{x}_j$ to $\hat{\gamma}_1(s_k)$ and $\hat{\gamma}_2(s_k)$. Then there is a (not minimizing) geodesic $\hat{S}$ joining $\hat{\gamma}_1(s_k)$ to $\hat{\gamma}_2(s_k)$ whose length realizes the infimum of all curve lengths joining the same endpoints in the domain. Moreover $\hat{S}$ has the properties: (1) $\hat{x}_j \notin \hat{S}$; (2) the sum of distances from any two points on $\hat{S}$ to $\hat{x}_j$ is greater than the length of the subarc of $\hat{S}$ determined by the two points; (3) the compact domain bounded by $\hat{S}$ and $\hat{P}_j^+$ contains $\hat{x}_j$ in its interior.

On the other hand since $\hat{x}_j$ is not on any $\hat{P}_j \in \hat{\mathscr{P}}_j$,

$$\hat{d}(\hat{x}_j, \hat{\gamma}_1(s_j)) + \hat{d}(\hat{x}_j, \hat{\gamma}_2(s_j)) > L(\hat{P}_j).$$

Thus there is a sufficiently small $\kappa > 0$ such that if $\hat{z}$ is the midpoint of $\hat{P}_j^+$, then for any $s \in [s_j - \kappa, s_j]$,

$$\hat{d}(\hat{x}_j, \hat{\gamma}_1(s)) + \hat{d}(\hat{x}_j, \hat{\gamma}_2(s)) > \hat{d}(\hat{z}, \hat{\gamma}_1(s)) + \hat{d}(\hat{z}, \hat{\gamma}_2(s))$$

and such that $\hat{x}_j$ is in the interior of a compact domain bounded by $\hat{\gamma}_1([0, s])$, $\hat{P}$, $\hat{\gamma}_2([0, s])$ and two segments joining $\hat{z}$ to $\hat{\gamma}_1(s)$ and $\hat{\gamma}_2(s)$. Then the standard length-decreasing deformation procedure is again applied to find a (not minimizing) geodesic $\hat{T}$ joining $\hat{\gamma}_1(s)$ to $\hat{\gamma}_2(s)$ which attains the infimum of all curvelengths which have the common endpoints and which are contained in the domain bounded by $\hat{\gamma}_1([s, \infty))$, $\hat{\gamma}_2([s, \infty))$ and the two segments joining $\hat{x}_j$ to $\hat{\gamma}_1(s)$ and $\hat{\gamma}_2(s)$. Moreover $\hat{T}$ has the following properties: (1) $\hat{x}_j \notin \hat{T}$; (2) the sum of distances from any two points on $\hat{T}$ to $\hat{x}_j$ is greater than the length of the subarc of $\hat{T}$ determined by the two points; (3) the compact domain bounded by $\hat{T}$ and $\hat{S}$ contains $\hat{x}_j$ in its interior.

Let $\hat{D}$ be the domain bounded by $\hat{S}$ and $\hat{T}$ and let $\hat{q}$ and $\hat{r}$ be the corner of $\hat{D}$. Let $\hat{a}, \hat{b}$: $[0, 1] \to \hat{U}$ be the geodesics with $\hat{a}(0) = \hat{b}(0) = \hat{q}$, $\hat{a}(1) = \hat{b}(1) = \hat{r}$ and $\hat{a}([0, 1]) \subset \hat{S}$, $\hat{b}([0, 1]) \subset \hat{T}$. Then

$$\hat{d}(\hat{q}, \hat{r}) = \min \{L(\hat{a}), L(\hat{b})\}.$$

In fact, let $\hat{c}$: $[0, 1] \to \hat{U}$ be a minimizing geodesic with $\hat{c}(0) = \hat{q}$, $\hat{c}(1) = \hat{r}$. Then $\hat{x}_j \notin \hat{c}([0, 1])$ follows from the above mentioned property (2). Unless $\hat{c}$ coincides with $\hat{a}$ or $\hat{b}$, then $\hat{c}((0, 1))$ is contained in the interior of one of the three domains which are bounded by $\hat{\gamma}_1([s, \infty))$, $\hat{T}$, $\hat{\gamma}_2([s, \infty))$ and $\hat{\gamma}_1([0, s])$, $\hat{P}$, $\hat{\gamma}_2([0, s])$, $\hat{S}$ and $\hat{D}$. But in any case this contradicts the length minimizing properties of $\hat{S}$ and $\hat{T}$ in the corresponding domains.

The final step of the proof of (2) in this case is to find a geodesic loop in $\hat{D}$ which has basepoint at $\hat{a}(1/2)$ or $\hat{b}(1/2)$. Recall that $\hat{D}$ is contained in $\hat{\Omega}_j$, where j is fixed. In order to find the desired geodesic loop, the length-decreasing deformation proceeds in $\hat{D}$, and this is possible when $\hat{d}(\hat{q}, \hat{x}_j) + \hat{d}(\hat{r}, \hat{x}_j)$ is greater than $L(\hat{a})$ or $L(\hat{b})$.

Without loss of generality one may now assume that $\hat{d}(\hat{q}, \hat{r}) = L(\hat{a}) < L(\hat{b})$. This is because the arguments developed below will show that the case $L(\hat{b}) \leq L(\hat{a})$ is covered by this case. Then property (2) for a, together with what is assumed above, ensures the existence of a small positive h such that if h' is in $[0, h)$, then

$$\hat{d}(\hat{x}_j, \hat{b}(h')) + \hat{d}(\hat{x}_j, \hat{b}(1-h')) > \hat{d}(\hat{b}(h'), \hat{a}(1/2)) + \hat{d}(\hat{b}(1-h'), \hat{a}(1/2)).$$

Thus the same procedure as before implies the existence of a geodesic $\hat{a}_{h'}$: $[0, 1] \to \hat{D}$ which has the minimum length among all curves joining $\hat{b}(h')$ and $\hat{b}(1-h')$ and they are in the subdomain of $\hat{D}$ bounded by $\hat{a}([0, 1])$, $\hat{b}([0, h'])$, $\hat{b}([1-h', 1])$ and two segments joining $\hat{x}_j$ to $\hat{b}(h')$ and $\hat{b}(1-h')$. For every h' in $[0, h)$ this $\hat{a}_{h'}$ has the same properties (1), (2) and (3) as do $\hat{S}$ and $\hat{T}$. Let $h_0 \in (0, 1/2]$ be the supremum of the set of all parameters h' each $\hat{a}_{h'}$ of which has the properties (1), (2) and (3) in the above stated subdomain of $\hat{D}$. If $h_0 = 1/2$, then $\hat{a}^{1/2}$ is clearly a geodesic loop in $\hat{D}$ at $\hat{b}(1/2)$ and the desired domain is obtained.

Suppose that h_0 is less than 1/2. Then a contradiction is derived as follows. It follows from the choice of h_0 that

$$\hat{d}(\hat{b}(h_0), \hat{x}_j) + \hat{d}(\hat{b}(1-h_0), \hat{x}_j) = L(\hat{a}_{h_0}).$$

On the other hand, property (2) for $\hat{b}$ implies that

$$L(\hat{a}_{h_0}) > L(\hat{b} \,|\, [h_0, 1-h_0]).$$

Therefore there exists an h_1 in $[0, h_0)$ such that

$$L(\hat{b} \,|\, [h_1, 1-h_1]) = L(\hat{a}_{h_1}).$$

Because $h_1 < h_0$, the subdomain in $\hat{D}$ which is bounded by

$$\hat{a}_{h_1}([0, 1]) \quad \text{and} \quad \hat{b}([h_1, 1-h_1])$$

contains $\hat{x}_j$ in its interior. Moreover it follows that

$$\hat{d}(\hat{b}(h_1), \hat{b}(1-h_1)) = L(\hat{b} \,|\, [h_1, 1-h_1]) = L(\hat{a}_{h_1}).$$

For simplicity let $\hat{a}_1 := \hat{a}_{h_1}$. The same argument as before shows that there is a small positive u such that if u' is in $[0, u)$, then

$$\hat{d}(\hat{a}_1(u), \hat{x}_j) + \hat{d}(\hat{a}_1(1-u), \hat{x}_j) > \hat{d}(\hat{a}_1(u), \hat{b}(1/2)) + \hat{d}(\hat{a}_1(1-u), \hat{b}(1/2)).$$

Thus there exists a geodesic $\hat{b}_u$: $[0, 1] \to \hat{D}$ with

$$\hat{b}_u(0) = \hat{a}_1(u), \quad \hat{b}_u(1) = \hat{a}_1(1-u)$$

such that $L(\hat{b}_u)$ is the infimum of all lengths of curves joining the common endpoints $\hat{a}_1(u)$ and $\hat{a}_1(1-u)$ and they are contained in the subdomain of $\hat{D}$ bounded by

$$\hat{a}_1([0, u]), \quad \hat{a}_1([1-u, 1]), \quad \hat{b}([h_1 - h_1])$$

and two segments joining $\hat{x}_j$ to $\hat{a}_1(u)$ and $\hat{a}_1(1-u)$. By the construction of $\hat{b}_u$, $L(\hat{b}_u) < \hat{d}(\hat{x}_j, \hat{b}_u(0)) + \hat{d}(\hat{x}_j, \hat{b}_u(1))$ and $\hat{b}_u$ has properties (1), (2) and (3). Let u_0 be the supremum of all parameters u in $[0, 1/2)$ each $\hat{b}_u$ of which has properties (1), (2), (3) in the subdomain. Then the length of $\hat{b}_{u_0}$ is equal to

$$\hat{d}(\hat{x}_j, \hat{a}_1(u_0)) + \hat{d}(\hat{x}_j, \hat{a}_1(1-u_0)).$$

From the property (2) for $\hat{a}_1$ this equality means that the length of the limit geodesic is greater than $L(\hat{a}_1 \,|\, [u_0, 1-u_0])$ and hence there is a $u_1 \in (0, u_0)$ such that

$$L(\hat{a}_1 \,|\, [u_1, 1-u_1]) = L(\hat{b}_1) = \hat{d}(\hat{a}_1(u_1), \hat{a}_1(1-u_1)),$$

where $\hat{b}_1 := \hat{b}_{u_1}$. Thus the subdomain of $\hat{D}$ bounded by

$$\hat{b}_1([0, 1]) \quad \text{and} \quad \hat{a}_1([u_1, 1-u_1])$$

contains $\hat{x}_j$ in its interior and hence $\hat{a}_1 \neq \hat{b}_1$. Moreover it follows from triangle inequality that

$$\begin{aligned} L(\hat{b} \,|\, [h_1, 1-h_1]) &= L(\hat{a}_1) \\ &= L(\hat{a}_1 \,|\, [0, u_1]) + L(\hat{b}_1) + L(\hat{a}_1 \,|\, [u_1, 1-u_1]) \\ &> L(\hat{b}_1 \,|\, [h_1, 1-h_1]). \end{aligned}$$

But this is a contradiction.

The above argument shows that there are countably many disjoint domains bounded by geodesic loops or geodesic biangles on each of which the curvature integral is greater than $\pi/2$, and hence the total curvature does not exist. This contradiction is derived by supposing that F_γ is a non-exhaustion. Thus the proof is complete.

As a direct consequence of Theorems 4.2, 4.3 and 5.3 one has:

COROLLARY. *Assume that F_γ is an exhaustion. Then,* (1) *Case II holds for γ,* (2) $\bigcup_t D_t = M$, (3) *$g(t) = t - L(P_t)/2$ is unbounded and* (4) *if $c(M)$ exists, then $C(M) \geq (2\chi(M) - 1)\pi$.*

Proof of the main theorem. Assume that the total curvature exists and

$$c(M) < (2\chi(M) - 1)\pi.$$

Then a contrapositive of the above corollary implies that every Busemann function on M is a non-exhaustion. Assume that $c(M) > (2\chi(M) - 1)\pi$, and suppose that F_γ is a non-exhaustion for some ray γ. Then a contrapositive of Theorem 5.3 implies that $g(t)$ is bounded above, and Theorem 4.3, (b) implies that if Case II holds for γ then

$$c(M) \leq (2\chi(M) - 1)\pi,$$

and Theorem 4.2 implies that if Case I holds for γ, then

$$c(M) \leq (2\chi(M) - 1)\pi,$$

a contradiction.

VI. Examples in the case that $c(M) = (2\chi(M) - 1)\pi$

In this section the discussion is divided into two cases; the simple case is the one where $G \geq 0$ outside a compact set of M.

PROPOSITION 6.1. *Assume that $G \geq 0$ outside a compact set of M and assume that $c(M) = (2\chi(M) - 1)\pi$. Then all Busemann functions are non-exhaustions.*

Proof. Suppose that F_γ is an exhaustion for some ray $\gamma\colon [0, \infty) \to M$. Then, the corollary in the preceeding section implies that Case II holds for γ. Hence for $t > t_0$, P_t is a geodesic loop at $\gamma(t)$ and

$$c(D_t) = (2\chi(D) - 1)\pi + \theta(t).$$

It follows from $G \geq 0$ outside a compact set that if $t_0' > t_0$ and if $G \geq 0$ outside $D_{t_0'}$, then the function $t \to \theta(t)$ is monotone non-decreasing for $t > t_0'$. Therefore

$$c(M) = \lim_{t \to \infty} c(D_t) \geq (2\chi(M) - 1)\pi + \theta(t_0'),$$

a contradiction.

In the case where $G \leq 0$ outside a compact set and $c(M) = (2\chi(M) - 1)\pi$, the situation is more complicated, as shown in the examples below.

Example 1. Let M_1 be an immersed surface in E^3, K_1 a compact set, and $U_1 = M_1 - \operatorname{int}(K_1)$ be isometric to a portion of a flat right cone which is obtained by rotating the half line with the slope $\sqrt{3}$. Then $c(M) = (2\chi(M) - 1)\pi$ is obvious. Every Busemann function on M_1 is a non-exhaustion. In fact, for any fixed ray $\gamma\colon [0, \infty) \to M_1$ there is a constant b such that $\gamma([b, \infty))$ is contained in the right cone. Then $F_\gamma^{-1}(\{b\})$ contains a ray in U_1 which starts at $\gamma(b)$ and is orthogonal to $\gamma(b)$. Therefore F_γ is a non-exhaustion. Furthermore, there is a point $z \in M_1 - \bigcup_{t \geq 0} D_t$ and $g(t)$ is bounded above.

Example 2. An immersed surface M_2 in E^3 is constructed in such a way that $G \leq 0$ outside a compact set, $c(M) = (2\chi(M) - 1)\pi$, and all Busemann functions on M_2 are exhaustions. The construction of a closed half cylinder $U_2 \subset M_2$ is done as follows.

First of all consider a non-differentiable surface U_2' of revolution around the z-axis whose profile curve is a union of line segments in the (xz)-plane with monotone decreasing slopes which tends to $\sqrt{3}$; it is expressed as $z = h(x)$, $x \geq 1$ and $h(1) = 1$,

$$h(x) = (x - x_j) \cot \theta_j + h(x_j) \quad \text{for } x_j \leq x \leq x_{j+1},$$

where $\{x_j\}$ is a divergent monotone sequence with $x_1 = 1$ and $\{\theta_j\}$ is monotone increasing such that $\lim \theta_j = \pi/6$. This curve is later approximated by a smooth curve such that U_2 is obtained by rotating it around z-axis.

It turns out that for given $\{\theta_j\}$ with the above properties, $\{x_j\}$ is chosen in such a way that if γ is a profile curve of U_2 (and hence it is a ray on M_2), then $g_\gamma(t) = t - L(P_t)/2$ is unbounded. If $g_\gamma(t)$ is unbounded for this γ, then every Busemann function on M_2 is an exhaustion. In fact let σ: $[0, \infty) \to M_2$ be a ray. Without loss of generality assume that $\sigma([0, \infty))$ is contained in U_2. Let τ: $[0, \infty) \to U_2$ be a profile curve with $\sigma(0) = \tau(0)$. Then g_τ is unbounded for τ. For every $t \geq 0$ let P_t and Q_t be geodesic loops at $\tau(t)$ and $\sigma(t)$ which are obtained in the basic construction along τ and σ. If r is the distance function in E^3 to z-axis, then $r(\tau(t) \geq r(\sigma(t))$ for all $t \geq 0$. And since the graph of the profile curve is strictly increasing, $L(P_t) \geq L(Q_t)$ holds for all $t \geq 0$. Thus $g_\tau(t) \leq g_\sigma(t)$. Since $g_\tau(t)$ is divergent by assumption, so is g_σ.

It remains to find for a given $\{\theta_j\}$ a $\{x_j\}$ so that if γ': $[0, \infty) \to U'_2$ is a profile curve then $g_{\gamma'}$ is divergent. Thus sufficient conditions for $\{x_j\}$ with the desired properties are discussed. Let γ': $[0, \infty) \to U'_2$ be a profile curve with $x(\gamma(0)) = z(\gamma(0)) = 1$, $y(\gamma(0)) = 0$. For every j let l_j be the length of the line segment on γ' between x_j and x_{j+1}. Then

$$l_j = (x_{j+1} - x_j)/\sin \theta_j.$$

In order to simplify the construction and make it possible to find the desired smooth approximation U_2 of U'_2, we must choose l_j so that if $s_j = \sum_{k=1}^{j} l_k$ then P_{s_j} lies in the portion of the right cone $z = h(x)$, $x_j \leq x \leq x_{j+1}$ for each j. This is guaranteed if

$$\text{(1)} \quad \frac{(1 + \sum_{k=1}^{j-1} l_k \sin \theta_k)}{\sin \theta_j} < \left[\frac{(1 + \sum_{n=1}^{j} l_k \sin \theta_k)}{\sin \theta_j}\right] \times \cos (\pi \sin \theta_j) \quad \text{for } j = 1, 2, \ldots.$$

In fact, when the piece of the right cone between x_j and x_{j+1} is cut open and developed in R^2, (1) is implied by the condition that the chord joining $\gamma'_1(s_{j+1})$ and $\gamma'_2(s_{j+1})$ does not touch the inside circle.

For a given positive monotone increasing $\{\theta_j\}$ with $\lim \theta_j = \pi/6$, it is possible to choose a divergent sequence $\{l_j\}$ with

$$l_1 = 1 + \csc \theta_1(\sec (\pi \sin \theta_1) - 1).$$

This is because (1) reduces to

$$l_j > \left[\frac{(1 + \sum_{k=1}^{j-1} l_k \sin \theta_k)}{\sin \theta_j}\right] (\sec (\pi \sin \theta_j) - 1) \quad \text{for } j \geq 2.$$

To check the divergent property of $g_{\gamma'}$ on U'_2, each piece of the right cone is developed into the (uv)-plane so that $\gamma'_1([s_j, s_{j+1}])$ is symmetric to $\gamma'_2([s_j, s_{j+1}])$ with respect to the v-axis and

$$\lim v(\gamma'_i(s)) = \lim v(\gamma'_i(s)).$$

Note that γ'_1 and γ'_2 are not continuous at every $s = s_j$ (when developed) but every $\gamma'_i([s_j, s_{j+1}])$ is a line segment with slope $\cot(\pi \sin \theta_j)$ and they are placed in such a way that $s \to v(\gamma'_i(s))$ is continuous. Then it follows from the monotone increasing property of $\{\theta_j\}$ that for every j,

$$\lim_{s \downarrow s_j} u(\gamma'_i(s)) < \lim_{s \uparrow s_j} u(\gamma'_i(s)).$$

This inequality makes it possible to give an upper bound for $L(P_{s_j})$:

$$L(P_{s_j})/2 < \sum_{k=1}^{j} l_k \sin(\pi \sin \theta_k) + \sec \theta_1 \sin(\pi \sin \theta_1).$$

Hence

$$g_{\gamma'}(s_j) > \sum_{k=1}^{j} l_k(1 - \sin(\pi \sin \theta_k)) - \sec \theta_1 \sin(\pi \sin \theta_1).$$

If $\{l_j\}$ is chosen so that

$$l_j > [j(1 - \sin(\pi \sin \theta_{j-1}))]^{-1}, \quad j \geq 2,$$

then

$$g_{\gamma'}(s_j) > \sum_{k=1}^{j} k^{-1} - \sec \theta_1 \sin(\pi \sin \theta_1),$$

and thus $g_{\gamma'}$ is divergent. For a given positive monotone increasing sequence $\{\theta_j\}$ with $\lim \theta_j = \pi/6$, it is possible to choose $\{l_j\}$ such that the above two inequalities are satisfied for $j = 1, 2, \ldots$.

Finally, U'_2 is approximated by U_2 in such a way that for every j there is an $s'_j < s_j$ which is sufficiently close to s_j such that $P_{s_{j'}}$ on U_2 is contained entirely in the portion of the right cone between x_j and x_{j+1}. Then $P_{s_{j'}}$ coincides with some $P_{s_{j''}}$ on U'_2. But because the approximation is sufficiently close, $s''_j - s'_j$ is uniformly bounded. Therefore $g_\gamma(s'_j)$ on U_2 is divergent. This completes the construction on M_2.

Example 3 (N. H. Kuiper). We will construct a complete metric on R^2 with respect to which $G \leq 0$ outside a compact set so that the total curvature is π and on which both exhaustion and non-exhaustion Busemann functions exist.

First of all a C^1-surface M_r in E^3 is defined as the boundary of a metric r-ball in E^3 (where $r > 0$ is small) around a flat board

$$W_0 = \{(x, y, z); \quad x \geq 0, y \geq (\log(x + 1))/2\}$$

in the (xy)-plane. This surface can be approximated by M_3 without changing the properties mentioned above. The Gauss normal map of this surface gives the total curvature and is equal to 2π (around the origin) $+(-\pi)$ (along the curve $y = \frac{1}{2}\log(x+1) = \pi$.

Consider the curves c_0, c: $[1, \infty) \to E^3$ and M_r where

$$c_0(x) = (x, (\log(x+1))/2, 0)$$

and $c(x)$ is the point of intersection of M_r with the (xy)-plane such that $c(x) - c_0(x)$ is normal to $c_0'(x)$. Then $\|c_0(x) - c(x)\| = r$ and the difference of their lengths between 1 and x satisfies

$$L(c_0 \mid [1, x]) - L(c \mid [1, x]) = r \int_1^x \kappa(u)\, du,$$

where $\kappa > 0$ is the curvature of c_0. By definition of c_0 it follows that

$$0 \leq L(c_0 \mid [1, x]) - L(c \mid [1, x]) \leq \pi r/6 \quad \text{for } x > 1.$$

$G = r^{-2}$ in a neighborhood of $(-\sqrt{3}\,r/2, -r/2, 0)$, $G \leq 0$ in the r-ball around c, and $G = 0$ outside these neighborhoods. Elementary calculus shows that $c([1, \infty))$ is the image of a geodesic ray on M_r (M_r is a smooth surface around the curve). Thus the computations show that if $p = (0, y, r) \in M_r$ is an arbitrary fixed point with $y > 0$, then $d(p, c,(x))$, the distance from $c(x)$ to the half line passing through p and parallel to y-axis, equals x. Hence

$$\begin{aligned} L(c \mid [1, x]) - d(p, c(x)) &\leq L(c_0 \mid [1, x]) + \pi r/6 - x \\ &= \int_0^x [\{1 + (4(u+1))^{-2}\}^{1/2} - 1]\, du + \pi r/6. \end{aligned}$$

This means that the Busemann function with respect to this ray is a non-exhaustion. If γ is a ray on M_r which is parallel to y-axis, then F_γ is an exhaustion because $F_\gamma((-\infty, a])$ is contained in the compact set $\{q \in M_r;\ y(q) = y(\gamma(a))\}$.

Example 4. We will construct a surface M_4 in E^3 so that (1) the total curvature does not exist, (2) there are both exhaustion and non-exhaustion Busemann functions. Let $\theta \in (\pi/3, \pi/2)$ be a fixed number and consider the following right cone in E^3:

$$\begin{aligned} \{(x, y, z);\quad & x = r\cos\theta\cos\psi,\ y = r\cos\theta\sin\psi, \\ & z = r\sin\theta,\ r > 0,\ 0 \leq \theta < 2\pi\}. \end{aligned}$$

For a monotone divergent sequence $\{s_j\}$ let

$$p_j = (s_j \cos\theta, 0, s_j \sin\theta)$$

and at each point p_j on the cone hang down a string of length

$$s_j \cos(\pi \sin \theta).$$

This figure can be approximated by a smooth surface M_4 withoug singularity such that each string is replaced by a sufficiently narrow flat cylinder with a cap. Therefore the distance function on M_4 is approximately measured on the original right cone.

Since the integral of the positive part of Gaussian curvature on M_4 is ∞, the total curvature does not exist on M_4.

If the figure is cut open along the half line

$$s \to (s \cos \theta, 0, s \sin \theta)$$

and developed into the (u, v)-plane so that it is symmetric with respect to the v-axis then every string has its endpoint on the u-axis. The Busemann function with respect to this half line is a non-exhaustion because for any $a > 0$, the function's a-level set is the union of the segment of length $2a \tan(\pi \sin \theta)$ and the points which are the intersections of the line $v = a$ with strings. However all the other Busemann functions are exhaustions.

Remark. It should be noted that if M is non-orientable with one end, then all the results obtained above are valid. For a non-orientable M with one end, $c(M)$ is defined as the half of the total curvature on its orientable double cover.

Kuiper also constructed a surface of revolution in E^3, homeomorphic to R^2, with a singularity on which all Busemann functions are exhaustions and so that the total curvature is π. For the definition of total curvature of a surface with a singularity, see [5].

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Kyushu University
Fukuoka, Japan