ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 4, Winter 1984

TAME KUMMER EXTENSIONS AND STICKELBERGER CONDITIONS

BY

L. N. CHILDS[1]

In this paper we show that rings of integers of tame Kummer extensions of algebraic number fields K with Galois group G, cyclic of odd prime power order, need not represent classes in the class group $\mathrm{Cl}\,(O_K G)$ which are images under the action of Stickelberger elements.

More explicitly, let l be an odd prime, G a cyclic group of order l^n, $\Delta = \mathrm{Aut}\,(G)$. Let

$$\theta = \frac{1}{l^n} \sum_{\delta \in \Delta} t_n(\delta)\delta^{-1},$$

where δ in Δ acts on σ in G by $\delta(\sigma) = \sigma^{t_n(\delta)}$, $0 < t_n(\delta) < l^n$, $(t_n(\delta), l) = 1$.

Let $J = \mathbf{Z}\Delta \cap \mathbf{Z}\Delta\theta$, the Stickelberger ideal [9, page 27].

Let R be the ring of integers of an algebraic number field K containing $Q(\zeta)$, ζ a primitive l^n-th root of unity. Let $\mathrm{Cl}\,(RG)$ denote the group of isomorphism classes of rank one projective RG-modules. Then there is an action of Δ on $\mathrm{Cl}\,(RG)$ induced by the action of Δ on G. Let $\overline{RG}$ be the maximal order of RG,

$$\overline{RG} = \sum_{\chi \in \hat{G}} Re_\chi \quad (\hat{G} = \mathrm{Hom}\,(G, \mathbf{C})) \quad \text{where} \quad e_\chi = \frac{1}{l^n} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma.$$

The action of Δ on G induces an action of Δ on $\overline{RG}$ by $\delta(e_\chi) = e_{\chi\delta^{-1}}$, so that $\delta(\sum a_\chi e_\chi) = \sum_\chi a_{\chi\delta} e_\chi$. Then we have an induced action of Δ on $\mathrm{Cl}\,(\overline{RG}) = \sum_\chi \mathrm{Cl}\,(R)e_\chi$.

Let $\mathscr{A}$ denote either RG or $\overline{RG}$. We are interested in knowing whether rings of integers of tame extensions L of K with group G yield elements in $\mathrm{Cl}\,(\mathscr{A})^J$, where $\mathrm{Cl}\,(\mathscr{A})^J$ is generated by the elements A^ζ for $A \in \mathrm{Cl}\,(\mathscr{A})$ and ζ in J. For $n = 1$, L. McCulloh [10] has shown this is so.

In this paper we show that for $n = 2$ there exists a Kummer extension L of degree l^2 over a number field K so that the class of $S = O_L$ is not in $\mathrm{Cl}\,(RG)^J$. This example shows that McCulloh's description of classes of rings of integers of tame extensions in terms of actions on the class group by Stickelberger elements does not have a straightforward extension from the prime order case to the prime power order case.

Received May 12, 1982.
[1] Partially supported by the National Science Foundation

Our example also shows that if one defines a product on the set of rings of integers of tame extensions by $O_{L_1} \cdot O_{L_2}$ = ring of integers of the Harrison product $L_1 \cdot L_2$ then the map from rings of integers to the class group is not a homomorphism. This is in contrast to the unramified case [6], and further complicates the problem of characterizing the classes of rings of integers of tame extensions.

The approach we take is as follows: given a ring of integers S of a tame extension L of K with group G, if the class of S is in $\mathrm{Cl}\,(RG)^J$, then the class of $\bar{S} = S \otimes_{RG} \overline{RG}$ is in $\mathrm{Cl}\,(\overline{RG})^J = (\sum_{\chi \in G} \mathrm{Cl}\,(R)e_\chi)^J$. By choosing K, L appropriately, we show that this latter situation cannot hold.

Notation. For an integer a, $t_r(a)$ denotes the remainder upon dividing a by l^r; hence $0 \leq t_r(a) < l^r$.

1. Description of the class group

Throughout, G is a cyclic group of order l^n, l an odd prime. Let $\mathscr{A}$ denote either RG or $\overline{RG}$. Then $\mathrm{Cl}\,(\mathscr{A})$ may be described as a group of idele classes,

$$\mathrm{Cl}\,(\mathscr{A}) \cong J(KG)/(KG)^* U(\mathscr{A}) \tag{1.1}$$

(cf. [4]); the map is as follows: Let M be a rank one projective $\mathscr{A}$-module. Then $M_{(l)}$, the semilocalization of M "at (l)", is free, so

$$M_{(l)} = \mathscr{A}_{(l)} v$$

for some basis element v. Also, for any prime p prime to (l), $M_p = R_p G u_p$ (note—away from (l), $\overline{RG} = RG$). So $u_p = \alpha_p v$ for some α_p in KG. For $p \mid (l)$, set $\alpha_p = 1$. View α_p in $K_p G$, the completion of KG at p; then the vector of α_p's, (α_p) defines an idele in $J(KG)$. The isomorphism of (1.1) is then defined by sending the class of M to the class of (α_p).

In general, if $\mathscr{A} = RG$ or $\overline{RG}$, R is semilocal, and M is a rank one projective $\mathscr{A}$-module, then M is free, $M = \mathscr{A}v$. If $\mathscr{A} = RG$, then the basis element v generates a normal basis $\{\sigma(v) \mid \sigma \in G\}$. If $\mathscr{A} = \overline{RG}$, then v generates an R-basis $\{w_\chi \mid \chi \in G\}$ of M where $w_\chi = e_\chi v$. Following [5, Section 2], we call a set $\{w_\chi\}$ of non-zero elements of the rank one projective $\overline{RG}$-module M a *Kummer basis* if for all χ, ψ in $\hat{G}$, $e_\psi w_\chi = \delta_{\psi,\chi} w_\chi$ and $\{w_\chi\}$ is an R-basis of M. If $\{w_\chi\}$ is a Kummer basis of M, then $v = \sum w_\chi$ is an $\overline{RG}$-basis of M.

Note that if $\{w_\chi\}$ is a Kummer basis of M, then, since $\sigma e_\chi = \chi(\sigma) e_\chi$, an easy computation shows that

$$R w_\chi = e_\chi M = M^\chi \quad \text{where } M^\chi = \{a \in M \mid \sigma(a) = \chi(\sigma) a \text{ for all } \sigma \text{ in } G\}.$$

When $\mathscr{A} = \overline{RG}$, $\mathrm{Cl}\,(\overline{RG}) \cong \sum_\chi \mathrm{Cl}\,(R)e_\chi$; given local basis elements v, u_p for M as above, the local basis elements corresponding to the component $\mathrm{Cl}\,(R)e_\chi$ are the Kummer basis elements $e_\chi v = w_\chi$ and $e_\chi u_p$. That is, for each p and χ, $e_\chi u_p = \alpha_{p,\chi} w_\chi$ for some $\alpha_{p,\chi} \in K^*$; the idele $(\alpha_{p,\chi})_p$ of $J(K)$ yields, as in (1.1), a class in $J(K)/K^* U(R) \cong \mathrm{Cl}\,(R)$ which is the component of the class of M corresponding to e_χ in $\mathrm{Cl}\,(\overline{RG})$. We shall exploit this use of Kummer bases below.

2. Stickelberger conditions

In [2] we showed that if M is a $Z\Delta$-module, written additively, then a is in M^J iff there exists b in M so that $\alpha a = \alpha\theta b$ for all α in A, the Z-submodule of $Z\Delta$ generated by l^n and $\{\delta - t(\delta) \mid \delta \in \Delta\}$.

In particular, if a is in M^J, then there is some b in M so that

$$l^n a = l^n \theta b = \sum_{\delta \in \Delta} t(\delta)\delta^{-1}(b).$$

Let $\hat{G} = \langle \chi_1 \rangle$ and if $\chi = \chi_1^k$, denote the idempotent e_χ of $\overline{RG}$ by e_k.

Now consider $M = \mathrm{Cl}\,(\overline{RG}) = \sum_{k=0}^{l^n-1} \mathrm{Cl}\,(R)e_k$. If $a = \sum_k a_k e_k$ is in $\mathrm{Cl}\,(\overline{RG})^J$, then there exists b in $\mathrm{Cl}\,(RG)^J$, then there exists b in $\mathrm{Cl}\,(RG)$ such that $l^n a = l^n\theta b$, that is,

$$\sum_{k=0}^{l^n-1} l^n a_k e_k = l^n\theta \sum_{k=0}^{l^n-1} b_k e_k$$

$$= \sum_{\delta\in\Delta} t(\delta)\delta^{-1}\left(\sum_{k=0}^{l^n-1} b_k e_k\right).$$

Since $\delta^{-1}(\chi_1^k) = \chi_1^k\ \delta = \chi_1^{kt(\delta)} = \chi_1^{t_n(kt(\delta))}$, we have

$$\sum_{k=0}^{l^n-1} l^n a_k e_k = \sum_{k=0}^{l^n-1} \sum_{\delta\in\Delta} t(\delta) b_k e_{t_n(kt(\delta))} \tag{2.1}$$

$$= \sum_{k=0}^{l^n-1} \sum_{\delta\in\Delta} t(\delta) b_{t_n(kt(\delta^{-1}))} e_k.$$

Equating coefficients of e_k for each k, $0 \le k \le l^n - 1$, we have

$$l^n a_k = \sum_{\delta\in\Delta} t(\delta) b_{t_n(kt(\delta^{-1}))} \tag{2.2}$$

where, recall, $t_n(m)$ is the remainder upon dividing m by l^n.

3. Tame extensions

Let L be a tame Galois extension of K with group G, cyclic of order l^n. Let R, S be the rings of integers of K, L, respectively. Then $S_{(l)}$ is unramified over $R_{(l)}$, so there exists v in S so that $S_{(l)} = R_{(l)}Gv$ with $\sum_\sigma \sigma(v) = 1$. For $\chi \in \hat{G}$, let

$$z_\chi = \sum \chi(\sigma)^{-i}\sigma^i(v) = l^n e_\chi v.$$

Then $\sigma(z_\chi) = \chi(\sigma)z_\chi$ and so, if $\chi(\sigma) = \zeta$, $z_\chi^{l^n} = 1 + (1-\zeta)r$ for some r in R; hence $z_\chi^{l^n}$ is a unit in $R_{(l)}$. Let $z_{\chi_1} = z$.

Let $S^\chi = \{s \in S \mid \sigma(s) = \chi(\sigma)s$ for all σ in $G\}$ for χ in $\hat{G}$, and let $\tilde{S} = \sum_{\chi\in G} S^\chi$, the Kummer order of S [5]. Since $S_{(l)} = R_{(l)}Gv$, an easy computation shows that $S_{(l)}^{\chi_1} = R_{(l)}z$; since z^{l^n} is a unit in $R_{(l)}$, if $\chi = \chi_1^k$, $R_{(l)}z_\chi = R_{(l)}z^k$. Hence $\tilde{S}_{(l)} = \sum_k R_{(l)}z^k$.

Let $\bar{S} = S \otimes_{RG} \overline{RG}$.

(3.1) LEMMA. $\tilde{S}_{(l)} \cong \bar{S}_{(l)} = S_{(l)} \otimes_{RG} \overline{RG}$.

Proof. Replace R by $R_{(l)}$, and drop the localization subscript (l) in this proof.

Now $\overline{RG} = \sum_\chi Re_\chi$, and the map $RG \to \overline{RG}$ sends α to $\sum_\chi \alpha e_\chi = \sum_\chi \chi(\alpha)e_\chi$. So $\bar{S} = \sum e_\chi S$. Let ϕ: $\bar{S} \to \tilde{S}$ be multiplication by $l^n = g$, the order of G. Then

$$\phi\left(\sum_\chi e_\chi s_\chi\right) = \sum_\chi \left(\sum_\sigma \chi(\sigma)\sigma^{-1}(s_\chi)\right);$$

and

$$\sum_\sigma \chi(\sigma)\sigma^{-1}(s_\chi) \in S^\chi \quad \text{for each } \chi.$$

Clearly ϕ is 1-1. To show ϕ is onto, let $S = RG\alpha$; then $\bar{S} = \sum_\chi Re_\chi \alpha$. If $b \in S^\chi$, $b = \sum_\tau a_\tau \tau(\alpha)$, then

$$\sigma(b) = \sum a_{\sigma^{-1}\tau}\tau(\alpha) = \sum_\chi \chi(\sigma)a_\tau \tau(\alpha).$$

Hence $a_{\sigma^{-1}\tau} = \chi(\sigma)a_\tau$ for all σ, τ, so

$$b = a_1 \sum_\tau \chi(\tau^{-1})\tau(\alpha) = a_1 g e_\chi \alpha.$$

Then $b = \phi(a_1 e_\chi \alpha)$ in ϕS.

It follows easily that $\{z^k/l^n \mid k = 0, 1, \ldots, l^n - 1\}$ is a Kummer basis for $\bar{S}_{(l)}$.

4. The strategy

Let K be a number field containing $Q(\zeta)$, ζ a primitive l^n root of unity, and let $R = O_K$.

(4.1) PROPOSITION. *If d in R such that $d \equiv 1 \pmod{(1-\zeta)^e}$ where e is sufficiently large, and $L = K[z]$, $z^{l^n} = d$, then L is unramified at all primes p of R dividing (l).*

Proof. It suffices to show that d is an l^n-th power in K_p, the completion of K at p, for then p will split completely in L. But for e sufficiently large, the exponential and logarithm functions may be defined, and an l^n-th root of d may be obtained as exp $((\log d)/l^n)$: see [8], Chapter V, 3.6, page 151.

Now restrict to $n = 2$, and assume K contains a primitive l^2 root of unity.

Let $(d) = \mathscr{P}_1^{q_1}\mathscr{P}_2^{q_2}, \ldots, \mathscr{P}_r^{q_r}$. Suppose $(q_i, l) = 1$. Let $L = K[z]$, $z^{l^2} = d$. Then $\{z^i/l^2\}$ is a Kummer basis for $\bar{S}_Q$ at all primes $Q \neq \mathscr{P}_1, \ldots, \mathscr{P}_r$, and in particular at (l).

Pick a prime $\mathcal{P}_i = \mathcal{P}$ and drop the subscript i. Let π be a uniformizing parameter at $\mathcal{P}$, i.e., $\mathcal{P}R_{\mathcal{P}} = \pi R_{\mathcal{P}}$. Then $z^{l^2} = \pi^q u_1$ for some unit u_1 in $R_{\mathcal{P}}$. Let $qh = 1 + l^2 s$, and let $w = z^h/\pi^s$. Then

$$w^{l^2} = \frac{(z^{l^2})^h}{\pi^{sl^2}} = \frac{\pi^{qh}}{\pi^{sl^2}}\, u_2 = \pi u_2,$$

u_2 a unit of $R_{\mathcal{P}}$.

Thus w is a root of the Eisenstein polynomial $x^{l^2} - \pi u_2$, so $O_{L,\mathcal{P}} = S_{\mathcal{P}} = R_{\mathcal{P}}[w]$ and $\{w^i \mid 0 \le i < l^2\}$ is a Kummer basis for $S_{\mathcal{P}}$ as an $R_{\mathcal{P}}G$-module (cf. [1]); moreover,

$$w^q = \frac{z^{1+l^2 s}}{\pi^{sq}} = \frac{z\pi^{sq}}{\pi^{sq}}\, u = uz$$

for some unit u of $R_{\mathcal{P}}$. So for $1 \le s \le l-1$,

$$S_{\mathcal{P}}^{\chi_1^{ls}} = \{a \in S \mid \sigma(a) = \chi_1^{ls}(\sigma)a\} = S_{\mathcal{P}} \cap Kz^{ls} = R_{\mathcal{P}}\, w^{t_2(qls)},$$

where $t_2(m) =$ remainder on dividing m by l^2. The ls-components $(\alpha_{\mathcal{P},ls})$, $s = 1, \ldots, l-1$, of the idele associated to $\bar{S}$ at $\mathcal{P}$, satisfy

$$w^{t_2(qls)} = \alpha_{\mathcal{P},ls}(z^{ls}/l^2)$$

and $\alpha_{\mathcal{P},ls}$ is obtained as follows:

$$\begin{aligned} w^{t_2(qls)} &= w^{qls}(w^{t_2(qls) - qls}) \\ &= (uz)^{ls} w^{l^2[(t_2(qls) - qls)/l^2]}. \end{aligned}$$

So, recalling that l^2 is a unit mod $\mathcal{P}$, there is a unit u of $R_{\mathcal{P}}$ so that

$$\alpha_{\mathcal{P},ls} = u\pi^{[(t_2(qls) - qls)/l^2]} = u\pi^{[(t_1(qs) - qs)/l]}.$$

If we have an idele which has (up to local unit factors) local components $\pi_i^{r_i}$ at $\mathcal{P}_i$, $i = 1, \ldots, r$, and 1 elsewhere, its image in Cl (R) under the isomorphism of (1.1) (with $G = (1)$) is the class of the ideal $\prod_{i=1}^r \mathcal{P}_i^{r_i}$. Hence, the image of $\bar{S}$ in

$$M = \sum_{s=1}^{l-1} \mathrm{Cl}\,(R)e_{ls}$$

(the part of Cl $(\overline{RG})$ corresponding to e_{ls}, $s = 1, \ldots, l-1$) is

$$\mathcal{A} = \sum_{s=1}^{l-1} \mathcal{A}_s e_{ls} \quad \text{where} \quad \mathcal{A}_s = \prod_{i=1}^{r} \mathcal{P}_i^{[(t_1(q_i s) - q_i s)/l]}.$$

Now M is a $Z\Delta$-direct summand of $\mathrm{Cl}\,(\overline{RG})$. If $\mathscr{A}$ is in M^J, then there exists $\mathscr{B} = \sum_{s=1}^{l-1} \mathscr{B}_s e_{ls}$ in M so that $\mathscr{A}^{l^2} = \mathscr{B}^{l^2\theta}$; in particular, following (2.2),

$$\mathscr{A}_s^{l^2} = \prod_{\delta\in\Delta} \mathscr{B}^{t_2(\delta)}_{t_1(st_2(\delta^{-1}))} \quad (s = 1, \ldots, l-1)$$
$$= \prod_{\delta\in\Delta} \mathscr{B}^{t_2(\delta)}_{t_1(st_1(\delta^{-1}))} \quad (s = 1, \ldots, l-1).$$

For $s = 1$,

$$\mathscr{A}_1^{l^2} = \prod_{\delta\in\Delta} \mathscr{B}^{t_2(\delta)}_{t_1(\delta^{-1})};$$

For $s = l-1$,

$$\mathscr{A}_{l-1}^{l^2} = \prod_{\delta\in\Delta} \mathscr{B}^{t_2(\delta)}_{t_1((l-1)t_1(\delta^{-1}))}.$$

But $(l-1)t_1(\delta^{-1}) \equiv l^2 - t_2(\delta^{-1}) \pmod l$, so

$$t_1((l-1)t_1(\delta^{-1})) = t_1(l^2 - t_2(\delta^{-1})),$$

and so

$$\mathscr{A}_{l-1}^{l^2} = \prod_{\delta\in\Delta} \mathscr{B}^{t_2(\delta)}_{t_1(l^2 - t_2(\delta^{-1}))} = \prod_{\delta\in\Delta} \mathscr{B}^{l^2 - t_2(\delta)}_{t_1(\delta^{-1})}$$

Multiplying, get

$$(4.2) \qquad (\mathscr{A}_1\mathscr{A}_{l-1})^{l^2} = \prod_{\delta\in\Delta} \mathscr{B}^{l^2}_{t_1(\delta^{-1})}$$

Since for each s, $1 \le s \le l-1$, there are l elements of $\Delta \cong (\mathbf{Z}/l^2\mathbf{Z})^*$ with $t_1(\delta^{-1}) = s$, we get

$$(\mathscr{A}_1\mathscr{A}_{l-1})^{l^2} = \left(\prod_{s=1}^{l-1} \mathscr{B}_s\right)^{l^3} = \mathscr{C}^{l^3}$$

for some ideal $\mathscr{C}$. Now

$$\mathscr{A}_1 = \prod_{i=1}^{r} \mathscr{P}_i^{(t_1(q_i) - q_i)/l}$$

and

$$(4.3) \qquad \mathscr{A}_{l-1} = \prod_{i=1}^{r} \mathscr{P}_i^{(t_1(q_i(l-1)) - q_i(l-1))/l}.$$

So

$$\mathscr{A}_1\mathscr{A}_{l-1} = \prod_{i=1}^{r} \mathscr{P}_i^{1-q_i}$$

Hence (4.2) becomes

$$(4.4) \qquad \left(\prod_{i=1}^{r} \mathscr{P}_i^{1-q_i}\right)^{l^2} = \mathscr{C}^{l^3}.$$

(4.5) PROPOSITION. *Let* $K \subset \mathbf{Q}(\zeta)$, ζ *a primitive* l^2*-root of unity. If there exists* $d \equiv 1 \pmod{(1-\zeta)^m}$, *m sufficiently large, such that*

$$(d) = \mathcal{P}_1^{q_1}, \ldots, \mathcal{P}_r^{q_r},$$

and $(\prod_{i=1}^r \mathcal{P}_i^{1-q_i})^{l^2}$ *is not the class of an* l^3 *power in* $\mathrm{Cl}(O_K)$, *then there exists a tame extension L of K, namely* $L = K(d^{1/l^2})$, *with Galois group G cyclic of order* l^2 *so that the class of* O_L *is not in* $\mathrm{Cl}(O_K G)^J$.

It suffices to choose $m \geq l(2l-1)$.

5. An example

Let K be a number field containing a l^2 root of unity ζ such that $\mathrm{Cl}(O_K)$ has a cyclic direct summand of degree l^3. Such a field can be found by a result of Sonn [11].

Let e be as in (4.1) (for $n = 2$) and $(1-\zeta)^e = \mathcal{m}$.

Let $I_{\mathcal{m}}$ be the subgroup of ideals of K prime to $\mathcal{m}$, $S_{\mathcal{m}}$ the subgroup of principal ideals (d), $d \equiv 1 \pmod{\mathcal{m}}$. Then $I_{\mathcal{m}}/S_{\mathcal{m}}$ is a finite group mapping surjectively onto $\mathrm{Cl}(O_K)$.

By Dirichlet's theorem [7, p. V-3] every class in $I_{\mathcal{m}}/S_{\mathcal{m}}$ contains infinitely many primes ideals of K.

Let $\mathcal{A}$ be a class in $I_{\mathcal{m}}/S_{\mathcal{m}}$ whose image in $\mathrm{Cl}(O_K)$ generates the cyclic direct summand of degree l^3. Suppose $\mathcal{A}$ has order k in $I_{\mathcal{m}}/S_{\mathcal{m}}$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_{k-1}$ be primes in $\mathcal{A}$.

Let $(d) = \mathcal{P}_1^2 \mathcal{P}_2, \ldots, \mathcal{P}_{k-1}$ with $d \equiv 1 \pmod{\mathcal{m}}$.

Let $L = K[z]$, $z^{l^2} = d$. Then, by (4.1), L is a tame extension of K.

So Proposition (4.4) yields the equation $\mathcal{P}_1^{-l^2} = \mathcal{C}^{l^3}$, which cannot be solved since $\mathcal{P}_1$ generates a cyclic direct summand of $\mathrm{Cl}(O_K)$ of order l^3. Hence:

(5.1) THEOREM. *There exists a number field K and a tame Galois extension L of K with Galois group G cyclic of order* l^2 *for which the class in* $\mathrm{Cl}(O_K G)$ *of* O_L *is not in* $\mathrm{Cl}(O_K G)^J$.

(5.2) *Remark.* Let $T(R, G)$ denote the set of R-algebras S which are integral closures of R in Galois extensions L of K with group G, which are tamely ramified. Let $N(R, G)$ be the subset consisting of S such that L/K is unramified (at all finite primes).

There is a multiplication (Harrison product) on Galois extensions L/K, given by

$$L_1 \cdot L_2 = (L = \otimes_K L_2)^{DG} \quad \text{where} \quad DG = \{(\sigma, \sigma^{-1}) \in G \times G\}.$$

This induces a multiplication on $T(R, G)$ by letting $S_1 \cdot S_2$ be the integral closure of R in $L_1 \cdot L_2$. This multiplication on $N(R, G)$ makes $N(R, G)$ into an abelian group, and Garfunkel and Orzech [6] have shown that the map τ: $T(R, G) \to \mathrm{Cl}(RG)$, $\tau(S)$ the class of S in $\mathrm{Cl}(RG)$, is a homomorphism when restricted to $N(R, G)$. But the example of (5.1) shows that τ need not be a homomorphism on $T(R, G)$. For if L is the quotient field of S and L is a Galois extension of K with group G, cyclic of order l^2, then L^{l^2} is the trivial Galois extension, $L^{l^2} \cong \mathrm{Hom}\,(G, K)$. Hence the l^2-fold product of S with itself in $T(R, G)$ is isomorphic to $\mathrm{Hom}\,(G, R)$, which is trivial in $\mathrm{Cl}\,(RG)$. But taking the example of (5.1) for S, if the class of S, raised to the l^2 power, were trivial in $\mathrm{Cl}\,(RG)$, then the class of $(S \otimes_{RG} \overline{RG})$, raised to the l^2 power in $\mathrm{Cl}\,(\overline{RG}) = \sum \mathrm{Cl}\,(R)e_k$, would be trivial. But the image of $(S \otimes_{RG} \overline{RG}$ in $\mathrm{Cl}\,(R)e_{l(l-1)}$ can be obtained from (4.3) with $q_1 = 2$:

$$\mathscr{A}_{l-1} = \mathscr{P}_1^{[t_1(2l-1)-2(l-1)]/l} = \mathscr{P}_1^{-1}.$$

So if the l^2 power of the class of S were trivial in $\mathrm{Cl}\,(RG)$, then the l^2 power of the class of $\mathscr{P}_1^{-1}$ would be trivial in $\mathrm{Cl}\,(R)$. But we chose $\mathscr{P}_1$ in the example of (5.1) so that the class of $\mathscr{P}_1^{-l^2}$ is non-trivial. Hence:

(5.3) If multiplication in $T(R, G)$ is defined by letting $S_1 \cdot S_2$ be the integral closure of R in $L_1 \cdot L_2$, the Harrison product of L_1 and L_2, then the "take the class" map from $T(R, G)$ to $\mathrm{Cl}\,(RG)$ need not be a homomorphism.

References

1. B. J. Birch, "Cyclotomic fields and Kummer extensions" in *Algebraic number theory*, J. W. S. Cassels and A. Frohlich, eds, Thompson, Washington, D.C., 1967, pp. 85–93.
2. L. N. Childs, *A Stickelberger condition on cyclic Galois extensions*, Canadian J. Math., vol. 34 (1982), pp. 686–690.
3. ———, *Abelian Galois extensions of rings containing roots of unity*, Illinois J. Math., vol. 15 (1971), pp. 273–280.
4. A. Frohlich, *Locally free modules over arithmetic orders*, J. Reine Angew. Math., vol. 274/275 (1975), pp. 112–129.
5. ———, *The module structure of Kummer extensions over Dedekind domains*, J. Reine Angew. Math., vol. 209 (1962), pp. 39–53.
6. G. Garfinkel and M. Orzech, *Galois extensions as modules over the group ring*, Canadian Math. J., vol. 22 (1970), pp. 242–248.
7. K. Iwasawa, "Class fields" in *Seminar on complex multiplication*, A. Borel, et al. ed., Springer Lecture Notes in Math., vol. 21, 1966, chapter V.
8. G. J. Janusz, *Algebraic number fields*, Academic Press, New York, 1973.
9. S. Lang, *Cyclotomic fields*, Springer-Verlag, New York, 1978.
10. L. McCulloh, "A Stickelberger condition on Galois module structure for Kummer extension of prime degree" in *Algebraic number fields*, A. Frohlich, ed., Academic Press, London, 1977, pp. 561–588.
11. J. Sonn, *Class groups and Brauer groups*, Israel J. Math., vol. 34 (1979), pp. 97–105.

State University of New York at Albany
Albany, New York