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LIFTING HOMOMORPHISMS OF MODULES

BY

ROBERT M. GURALNICK

Throughout this note R will denote a commutative noetherian ring with an ideal P. Let Λ be a module finite R-algebra and M and N finitely generated Λ-modules. Obviously, if $M \simeq N$, then $M_k = M/P^kM \simeq N_k$ for each $k > 0$. The reverse implication has been studied in the case R is a discrete valuation ring and P is the maximal ideal of R. Maranda [4] and Higman [3] considered the case where Λ is an order in a separable algebra and M and N are R-free and proved that there exists a positive integer k (depending only on Λ) such that $M_k \simeq N_k$ implies $M \simeq N$. Reiner [5] extended this result and proved

THEOREM (Reiner). *Let R be a dvr with quotient field K. If Λ is an order in an arbitrary finite dimensional K-algebra and M and N are R-free Λ-modules, then $M_k \simeq N_k$ for all $k > 0$ implies $M \simeq N$.*

In this article, we extend this result in several directions. We only assume R is a local noetherian ring, P is its maximal ideal, Λ is module finite over R, and M and N are finitely generated. Moreover, we show that it suffices to check if $M_k \simeq N_k$ for a sufficiently large k instead of all k. It is easy to see that the k can no longer be chosen to depend only on Λ; in our result, k depends on M and N. The main ingredient in the proof is the following weak version of the Artin-Rees Lemma (see [1, p. 197]).

LEMMA. *If L is a finitely generated R-module with a submodule L_0, then there exists a nonnegative integer e such that $L_0 \cap P^{e+j}L \subset P^jL_0$ for all $j > 0$.*

Our first result shows that homomorphisms from M_k to N_k can be lifted to homomorphisms from M to N provided k is large and we don't insist they agree completely on M_k.

THEOREM A. *There exists a nonnegative integer $e = e_P(M, N)$ such that if*

$$\sigma \in \operatorname{Hom}_\Lambda(M_{f+e}, N_{f+e}) \quad \text{for } f > 0,$$

then there exists $\tau \in \operatorname{Hom}_\Lambda(M, N)$ with τ and σ inducing the same maps from M_f to N_f.

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Proof. Choose presentations $\Lambda^{(n)}/A\Lambda^{(t)} = M$ and $\Lambda^{(n)}/B\Lambda^{(t)} = N$ where A and $B \in \Lambda_{n\times t}$, the set of $n \times t$ matrices over Λ. Define T: $\Lambda_n \times \Lambda_t \to \Lambda_{n\times t}$ by $T(U,V) = UA - BV$. Here Λ_n is the ring of $n \times n$ matrices over Λ. By the Artin-Rees Lemma, there exists a nonnegative integer e satisfying

$$\operatorname{im} T \cap P^{e+f}\Lambda_{n\times t} \subset P^f \operatorname{im} T. \tag{$*$}$$

Now if $\sigma \in \operatorname{Hom}_\Lambda(M_{f+e}, N_{f+e})$, choose $U_0 \in \Lambda_n$ inducing σ on

$$M_{f+e} = \Lambda^{(n)}/(A\Lambda^{(t)} + P^{f+e}\Lambda^{(n)}).$$

Since $U_0A\Lambda^{(t)}B\Lambda^{(t)} + P^{f+e}\Lambda^{(n)}$, we can find V_0 with $U_0A \equiv BV_0(\operatorname{mod} P^{f+e})$ Thus

$$T(U_0, V_0) \in P^{e+f}\Lambda_{n\times t}$$

and so by $(*)$, $T(U_0,V_0) = T(U_1,V_1)$ for some $(U_1,V_1) \in P^f(\Lambda_n \times \Lambda_t)$. Set $U' = U_0 - U_1$ and $V' = V_0 - V_1$. Then $T(U',V') = 0$, and so U' induces a Λ-homomorphism τ of M into N. Since $U \equiv U' \operatorname{mod} P^f$, τ and σ induce the same maps from M_f to N_f as desired.

For the next two corollaries, assume R is local and P is its maximal ideal. An easy consequence of the theorem is our promised generalization of Reiner's theorem.

COROLLARY 1. *Let* $l = \max\{e_P(M,N), e_P(N,M)\}$. *If* $M_{l+1} \simeq N_{l+1}$, *then* $M \simeq N$.

Proof. Choose inverse isomorphisms α_1 and α_2 between M_{l+1} and N_{l+1}. By the theorem, we can find $\beta_1 \in \operatorname{Hom}_\Lambda(M,N)$ and $\beta_2 \in \operatorname{Hom}_\Lambda(N,M)$ such that $\alpha_1 = \beta_1$ in $\operatorname{Hom}_\Lambda(M_1,N_1)$ and $\alpha_2 = \beta_2$ in $\operatorname{Hom}_\Lambda(N_1,M_1)$. Hence $\beta_1\beta_2$ induces an automorphism of M_1 and thus $\beta_1\beta_2$ is a surjective endomorphism by Nakayama's Lemma. As M is noetherian, this implies $\beta_1\beta_2 \in \operatorname{Aut}_\Lambda(M)$. Similarly $\beta_2\beta_1 \in \operatorname{Aut}_\Lambda(N)$. Hence β_1 is an isomorphism from M to N as desired.

Write $N|M$ if N is isomorphic to a summand of M. Let l be defined as above.

COROLLARY 2. *If* $N_{l+1}|M_{l+1}$, *then* $N|M$.

Proof. Since $N_{l+1}|M_{l+1}$, there exist

$$\alpha_1 \in \operatorname{Hom}_\Lambda(N_{l+1}, M_{l+1}) \quad \text{and} \quad \alpha_2 \in \operatorname{Hom}_\Lambda(M_{l+1}, N_{l+1})$$

so that $\alpha_2\alpha_1 \in \mathrm{Aut}_\Lambda(N_{l+1})$. By the theorem, we can find

$$\beta_1 \in \mathrm{Hom}_\Lambda(N, M) \quad \text{and} \quad \beta_2 \in H_\Lambda(M, N)$$

so that $\beta_2\beta_1$ induces the same automorphism as $\alpha_2\alpha_1$ on N_1. Hence $\beta_2\beta_1 \in \mathrm{Aut}_\Lambda(N)$ and so $M \simeq N \oplus \ker\beta_2$ as desired.

Note that Corollary 2 does not imply that if M_{l+1} is decomposable, then M is. Indeed, this is easily seen to be false. A version of this is true if R is complete. We first need a preliminary lemma.

LEMMA. *There exists a nonnegative integer $g = g_P(M, N)$ such that if*

$$\sigma \in \mathrm{Hom}_\Lambda(M, N) \quad \textit{and} \quad \sigma(M) \subset P^{f+g}N,$$

then $\sigma \in P^f\mathrm{Hom}_\Lambda(M, N)$.

Proof. Choose generators $m_1, \ldots, m_n$ for M. Define an R-linear map

$$\theta\colon \mathrm{Hom}_\Lambda(M, N) \to N^{(n)}$$

by $\theta(\sigma) = (\sigma(m_1), \ldots, \sigma(m_n))$.

Again, by the Artin-Rees Lemma, there exists a nonnegative integer g such that

$$\mathrm{im}\,\theta \cap P^{f+g}N^{(n)} \subset P^f\mathrm{im}\,\theta$$

for all $f > 0$. Thus if $\sigma(M) \subset P^{f+g}N$, $\theta(\sigma) = \theta(\tau)$ for some $\tau \in P^f\mathrm{Hom}_\Lambda(M, N)$. Hence $\sigma(m_i) = \tau(m_i)$ and so $\sigma = \tau \in P^f\mathrm{Hom}_\Lambda(M, N)$.

THEOREM B. *Let R be a noetherian local ring with maximal ideal P which is complete with respect to the P-adic topology. Let Λ be a module finite R-algebra and M a finitely generated Λ-module. Set $\nu = \nu(M) = e_P(M, M) + g_P(M, M)$. If $M_{\nu+1}$ is decomposable, then so is M.*

Proof. Let $E = \mathrm{End}_\Lambda(M)$. If $M_{\nu+1}$ is decomposable, there exists a nontrivial idempotent $\alpha \in \mathrm{End}_\Lambda(M_{\nu+1})$. By Theorem A, we can choose $\beta \in \mathrm{End}_\Lambda(M)$ such that β and α agree on $M_{\nu+1-e}$. Thus

$$(\beta - \beta^2)(M) \subset P^{\nu+1-e}M = P^{g+1}M.$$

Hence by the lemma, $\beta - \beta^2 \in PE$. Note since β and α induce the same endomorphism of M_1, β is neither the zero map nor an isomorphism on M_1. Hence neither β nor $1 - \beta \in PE$. This shows E/PE has a nontrivial idempotent. Since E is a module finite R-algebra and R is complete, this implies E has a nontrivial idempotent δ. Hence $M = \delta(M) \oplus \ker\delta$ is decomposable.

Special cases of Theorem B have been obtained by Maranda, Heller, and Reiner. For references, see [2, Section 76] and [5].

In general, it seems fairly difficult to determine the integers $e_P(M, N)$ and $g_P(M, N)$. However, if R is a discrete valuation ring with prime $P = (\pi)$, Λ is R-torsion-free (for e) and M and N are R-torsion-free (for g), then one can obtain upper bounds by computing the Smith normal form for the maps T and θ constructed above. For example, if T has invariants $\pi^{a_1}, \ldots, \pi^{a_r}, 0, \ldots, 0$, then $e_P(M, N) \leq \max\{a_i\}$.

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University of Southern California
Los Angeles, California