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THE ISOMORPHISM PROBLEM FOR INCIDENCE RINGS

BY

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Introduction

Let P be a locally finite pre-ordered set, hereafter shortened to pre-ordered set, and let R be a ring (with identity). The incidence ring $I(P, R)$ consists of all functions $f\colon P \times P \to R$ such that $f(x, y) = 0$ whenever $x \nleq y$. If $f, g \in I(P, R)$ and $r \in R$ then $f + g$, rf and fg are defined by the equations

$$(f+g)(x,y) = f(x,y) + g(x,y), \qquad (rf)(x,y) = rf(x,y),$$
$$(fg)(x,y) = \sum_{x \le z \le y} f(x,z)g(z,y).$$

Let C be a class of pre-ordered sets. We say that a ring R respects order in C if whenever $P, Q \in C$ and $I(P, R) \approx I(Q, R)$ then $P \approx Q$. If C is the class of all pre-ordered (partially ordered) sets then we say that R respects preorder (partial order). Given C, our objective is to determine conditions on a ring R which imply that R respects order in C. This isomorphism problem, as well as its analogue in the theory of group rings [cf. 10], has attracted considerable attention lately (cf. [3], [7], [8], [9], [11], [12]).

Our main result is a solution of the isomorphism problem for incidence rings over products of indecomposable commutative rings. We will also study conditions under which commutative rings respect order in the class of *finite* connected pre-ordered sets.

For unexplained notation, see [12]. Given a disjoint family $\{P_i\}$ of pre-ordered sets, the disjoint union pre-ordered set $\dot{\bigcup} P_i$ is obtained by declaring $x \le y$ iff there exists an i such that $x, y \in P_i$ and $x \le y$ in P_i. For any cardinal number n and pre-ordered set P, let nP be the disjoint union of Pn times. We shall use the fact that any preordered set P is order isomorphic to some $\dot{\bigcup} n_i K_i$ where $\{K_i\}$ is a family of connected pre-ordered sets such that $K_i \not\approx K_j$ for $i \neq j$. It will be convenient to allow $n_i = 0$. In this event $n_i K_i = \emptyset$.

Section 1

In this section we shall prove that indecomposable commutative rings respect pre-order. We begin with some lemmas.

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LEMMA 1.1. *Let R be a commutative ring and $N \in M_n(R)$. If each entry of N is nilpotent then N is nilpotent.*

Proof. Recall that any ideal in R which is generated by a finite number of nilpotent elements is nilpotent. Let I be the ideal generated by the entries of N. Then there exists an S such that $I^S = 0$. We now check that $N^S = 0$. Each entry of N^S is a sum of terms of the form $x_1 \dots x_S$ where each $x_i \in I$. Hence $x_1 \dots x_S \in I$ so $N^S = 0$.

LEMMA 1.2. *Suppose R is an indecomposable commutative ring and that E is an idempotent in $M_n(R)$. If for every $A, B \in M_n(R)$ $EABE = EAEBE$ then $E = 0$ or $E = 1$.*

Proof. We first prove the lemma when R is a field. Define a map φ: $M_n(R) \to M_n(R)$ by $A \mapsto EAE$. The hypotheses imply that φ is a homomorphism. Since $M_n(R)$ is simple, $\ker\varphi = 0$ or $\ker\varphi = M_n(R)$. If $\ker\varphi = M_n(R)$ then $E1E = 0$ so $E = 0$. If $\ker\varphi = 0$ then from $\varphi(1) = \varphi(E)$ we conclude that $E = 1$.

Next we prove the lemma when R is an integral domain. So let R be an integral domain and K its quotient field. Given $A, B \in M_n(K)$ there exist $r, s \neq 0$, $r, s \in R$ such that $rA, sB \in M_n(R)$. From the equation $E(rA)(sB)E = E(rA)E(sB)E$ we see that $rsEABE = rsEAEBE$. So for any $A, B \in M_n(K)$, $EABE = EAEBE$. Therefore $E = 0$ or $E = 1$.

Now let R be any indecomposable commutative ring, let P be a prime ideal in R, and consider the quotient map π_P: $M_n(R) \to M_n(R/P)$. Then for all $A, B \in M_n(R)$,

$$\pi_P(E)\pi_P(A)\pi_P(B)\pi_P(E) = \pi_P(E)\pi_P(A)\pi_P(E)\pi_P(B)\pi_P(E)$$

and $\pi_P(E) = \pi_P(E)^2$. Therefore $\pi_P(E)$ satisfies the hypotheses of the lemma in $M_n(R/P)$. Since R/P is an integral domain, $\pi_P(E) = 0$ or $\pi_P(E) = 1$. In either case the off-diagonal entries of E are in P. Since P is arbitrary, the off-diagonal entries of E are in $\operatorname{rad} R$, the nilradical of R, and hence are nilpotent. Write $E = D + N$ where D is a diagonal matrix and N has nilpotent entries, and consider the quotient map π: $M_n(R) \to M_n(R/\operatorname{rad} R)$. $\pi(E)$ is idempotent in $M_n(R/\operatorname{rad} R)$ and has the form

$$\begin{bmatrix} [d_1] & & \\ & \ddots & \\ & & [d_n] \end{bmatrix}$$

where $[d_i]^2 = [d_i]$. Since idempotents $[d_i]$ can be lifted to idempotents $\operatorname{mod} \operatorname{rad} R$ [1, Prop. 27.1], there exist idempotents $e_i \in R$, $i = 1, \dots, n$, such that $e_i - d_i \in \operatorname{rad} R$. Since R is indecomposable $e_i = 0$ or $e_i = 1$. Suppose for some i, $e_i = 0$. Then for any prime ideal P, $\pi_P(E) = 0$ so the entries of E are all nilpotent. Lemma 1.1 implies that $E = 0$. Assume, then, that $e_i = 1$ for each i. Then $E = 1 + N$ where each entry of N is nilpotent. $1 - E$ is then an idempotent so $1 - E = 0$, or $E = 1$.

Throughout the rest of this section $f \in I(P, R)$ is called strictly upper triangular (SUT) in case $e_\alpha f e_\alpha = 0$ for every $\alpha \in \hat{P}$.

LEMMA 1.3. *Let P be a pre-ordered set and R a ring. Then $I(P, R)$ has no nonzero* SUT *idempotents.*

Proof. By [12, Proposition 4] any SUT $f \in I(P, R)$ is in the Jacobson radical of $I(P, R)$. If $f^2 = f$ then $f(1 - f) = 0$. Since $1 - f$ is invertible, $f = 0$.

THEOREM 1.4. *Let R be an indecomposable commutative ring and let P, Q be pre-ordered sets. If $I(P, R) \approx I(Q, R)$ then $P \approx Q$.*

Proof. Let φ: $I(P, R) \to I(Q, R)$ be an isomorphism. For $\alpha \in \hat{P}$, $\beta \in \hat{Q}$, (cf. [12]), let $f_\alpha = \varphi(e_\alpha)$ and let $h_\beta^\alpha = e_\beta f_\alpha e_\beta$. If $f, g \in I(P, R)$ then

$$\varphi(e_\alpha f g e_\alpha) = f_\alpha \varphi(f) \varphi(g) f_\alpha$$

and

$$\varphi(e_\alpha f e_\alpha g e_\alpha) = f_\alpha \varphi(f) f_\alpha \varphi(g) f_\alpha.$$

Since $e_\alpha f g e_\alpha = e_\alpha f e_\alpha g e_\alpha$, $f_\alpha \varphi(f) \varphi(g) f_\alpha = f_\alpha \varphi(f) f_\alpha \varphi(g) f_\alpha$. This says that h_β^α, when viewed as an element of $M_{|\beta|}(R)$, where $|\beta|$ denotes the cardinality of β, has the property that for any $A, B \in M_{|\beta|}(R)$, $h_\beta^\alpha A B h_\beta^\alpha = h_\beta^\alpha A h_\beta^\alpha B h_\beta^\alpha$. Also h_β^α is idempotent in $M_{|\beta|}(R)$ since f_α is idempotent in $I(Q, R)$. Therefore by Lemma 1.2, $h_\beta^\alpha = 0$ or 1. In other words, the diagonal blocks of f_α are 1 or 0.

Let $S_\alpha = \{\beta \in \hat{Q}\colon h_\beta^\alpha = 1\}$. If $\alpha_1 \neq \alpha_2$ then $e_{\alpha_1} e_{\alpha_2} = 0$ so $f_{\alpha_1} f_{\alpha_2} = 0$ and this implies that $S_{\alpha_1} \cap S_{\alpha_2} = 0$. Next we show that $\bigcup_{\alpha \in \hat{P}} S_\alpha = \hat{Q}$. Suppose $\beta \in \hat{Q}$ and $\beta \notin \bigcup S_\alpha$. Then there is an idempotent e in $I(P, R)$ such that $\varphi(e) = e_\beta$. By Lemma 1.3, e is not SUT. Hence there is an $\alpha \in \hat{P}$ such that $e_\alpha e e_\alpha = e_\alpha$. But then $\varphi(e_\alpha) = \varphi(e_\alpha e e_\alpha) = f_\alpha e_\beta f_\alpha$, which is SUT. This contradiction shows that $\beta \in \bigcup_{\alpha \in \hat{P}} S_\alpha$.

If $f \in I(P, R)$ is SUT then $e_\alpha f e_\alpha = 0$ for all $\alpha \in \hat{P}$. Therefore $f_\alpha \varphi(f) f_\alpha = 0$ for all $\alpha \in \hat{P}$. This implies that $\varphi(f)$ is SUT.

Next we show that each S_α is singleton. Let $\beta \in S_\alpha$. Then there is an idempotent $e \in I(P, R)$ such that $\varphi(e) = e_\beta$. Let

$$S = \left\{\gamma \in \hat{P}\colon e_\gamma e e_\gamma = e_\gamma\right\}.$$

If $\alpha \notin S$ then $e_\alpha e e_\alpha = 0$ so $f_\alpha e_\beta f_\alpha = 0$, which is a contradiction. So $\alpha \in S$. Suppose $\delta \neq \alpha$ and $\delta \in S$. Then $e_\delta e e_\delta = e_\delta$ so $f_\delta = f_\delta e_\beta f_\delta$ is SUT. This shows that $\delta \notin S$. Thus $S = \{\alpha\}$. It follows that $e_\alpha - e$ is SUT so $f_\alpha - e_\beta$ is SUT and $S_\alpha = \{\beta\}$.

Define a bijection $U: \hat{P} \to \hat{Q}$ by $U(\alpha) = \beta$ where $\beta \in S_\alpha$. Then

$$f_\alpha = e_{U(\alpha)} + N_\alpha$$

where N_α is SUT. For $\alpha_1, \alpha_2 \in \hat{P}$, $\alpha_1 \leq \alpha_2$ if and only if $e_{\alpha_1} I(P, R) e_{\alpha_2} \neq 0$. This in turn occurs if and only if $f_{\alpha_1} I(Q, R) f_{\alpha_2} \neq 0$, which happens if and only if $e_{U(\alpha_1)} I(Q, R) e_{U(\alpha_2)} \neq 0$; by Stanley's lemma [11, Lemma]. This inequality is equivalent to the assertion that $U(\alpha_1) \leq U(\alpha_2)$. Thus U is an order isomorphism. To show that $P \approx Q$ it suffices to show that for each $\alpha \in \hat{P}$, $|\alpha| = |U(\alpha)|$. For a pre-ordered set Z, let

$$J_Z = \{f \in I(Z, R) : f \text{ is SUT}\}$$

and observe that J_Z is an ideal in $I(Z, R)$. We then have $\varphi(J_P) = J_Q$ by an earlier argument. This gives

$$\frac{e_\alpha I(P, R) e_\alpha + J_P}{J_P} \approx \frac{f_\alpha I(Q, R) f_\alpha + J_Q}{J_Q}.$$

An elementary computation shows that the left side is isomorphic to $M_{|\alpha|}(R)$ and the right side is isomorphic to $M_{|U(\alpha)|}(R)$. This implies that $|\alpha| = |U(\alpha)|$.

Section 2

In order to use Theorem 1.4 to solve the isomorphism problem for products of indecomposable commutative rings we shall need some mildly technical lemmas. In some of these, routine verification will be left to the reader.

LEMMA 2.1. *Let P be a pre-ordered set and $\{R_i\}$ a family of rings. Then $I(P, \times_i R_i) \approx \times_i I(P, R_i)$.*

Proof. Let π_i: $\times_j R_j \to R_i$ be the ith projection map. Given $f \in I(P, \times_i R_i)$ send f to (f_i) where $f_i(x, y) = \pi_i f(x, y)$. It is clear that this map is an isomorphism.

LEMMA 2.2. *Let $P = \dot{\bigcup} P_i$ and R a ring. Then*

$$I(P, R) = I(\dot{\bigcup} P_i, R) \approx \times_i I(P_i, R).$$

Proof. For $f \in I(\dot{\cup} P_i, R)$, send f to (f_i), where $f_i(x, y) = f(x, y)$ when $x, y \in P_i$. By definition fg maps to $((fg)_i)$ where

$$(fg)_i(x, y) = fg(x, y) = \sum_{x \le z \le y} f(x, z) g(z, y) \quad \text{for } x, y \in P_i.$$

Any z appearing in the sum is in P_i so

$$(fg)_i(x, y) = \sum_{x \le z \le y} f_i(x, z) g_i(z, y) = f_i g_i(x, y).$$

Thus $((fg)_i) = (f_i) \cdot (g_i)$. The map is clearly additive, 1-1, onto and so is an isomorphism.

The next lemma can be proven by appealing to [12, Theorem 4 and concluding remarks], but for the sake of clarity we give the following simple proof.

LEMMA 2.3. *Let P, Q be pre-ordered sets and suppose $0 < n < \infty$. If $nP \approx nQ$ then $P \approx Q$.*

Proof. Write $P \approx \dot{\cup} m_i K_i$, $Q \approx \dot{\cup} n_i K_i$. Then the isomorphism $nP \approx nQ$ implies that $n(\dot{\cup} m_i K_i) \approx n(\dot{\cup} n_i K_i)$. So $\dot{\cup} nm_i K_i \approx \dot{\cup} nn_i K_i$. From this it follows that $nm_i = nn_i$ as cardinal numbers. Since n is finite, $m_i = n_i$ for each i. Thus $P \approx Q$.

As an immediate corollary we have the following interesting result which will be used later in the section.

COROLLARY 2.4. *If R respects pre-order then for finite $n > 0$, R^n respects pre-order.*

Proof. Assume $I(P, R^n) \approx I(Q, R^n)$. Then $\times_i^n I(P, R) \approx \times_i^n I(Q, R)$, so $I(nP, R) \approx I(nQ, R)$ using Lemmas 2.1 and 2.2. Thus $nP \approx nQ$ and $P \approx Q$ by Lemma 2.3.

One can also easily show that if there exists a finite $n > 0$ such that R^n respects pre-order, then R respects preorder. We shall later construct a ring which respects pre-order but which has a factor that doesn't.

LEMMA 2.5. *Let P be a connected pre-ordered set. Then* $\mathrm{CEN}(I(P, R)) \approx \mathrm{CEN}(R)$ *where* CEN *denotes the center of the ring.*

Proof. The proof will generalize an argument in [2, Theorem 1]. Suppose $f \in \mathrm{CEN}(I(P, R))$. Then for $\alpha, \beta \in \hat{P}$, $\alpha \ne \beta$, we have $e_\alpha f e_\beta = e_\alpha e_\beta f = 0$.

This says that only the diagonal blocks of f are nonzero. Also for $\alpha \in \hat{P}$ we have $e_\alpha f e_\alpha \in \mathrm{CEN}(M_{|\alpha|}(R))$. Hence $e_\alpha f e_\alpha = r_\alpha \cdot 1$ where $r_\alpha \in \mathrm{CEN}(R)$ and 1 is the identity in $M_{|\alpha|}(R)$. For $x, y \in P$, $x \leq y$, define δ_{xy} by

$$\delta_{xy}(a,b) = \begin{cases} 1 & \text{if } a = x \text{ and } b = y \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \leq \beta$ and suppose $x \in \alpha$ and $y \in \beta$. Then the equation $f\delta_{xy} = \delta_{xy} f$ gives $r_\alpha \delta_{xy} = r_\beta \delta_{xy}$, so $r_\alpha = r_\beta$. If α, β are arbitrary elements of $\hat{P}$, then since P is connected, there is a sequence $\alpha_1, \ldots, \alpha_n$ in $\hat{P}$ such that $\alpha_1 = \alpha$, $\alpha_n = \beta$ and either $\alpha_i \leq \alpha_{i+1}$ or $\alpha_{i+1} \leq \alpha_i$ for $i = 1, 2, \ldots, n-1$. Hence $r_\alpha = r_{\alpha_1} = \cdots = r_{\alpha_n} = r_\beta$. Therefore $f = r\delta$ where $r \in \mathrm{CEN}(R)$ and δ in the identity in $I(P, R)$.

Lemma 2.6. *Let P be a connected pre-ordered set and let R be a ring. Then $I(P, R)$ is indecomposable if and only if R is.*

Proof. The proof is immediate from Lemma 2.5 and elementary ring theory [1, Corollary 7.7].

Theorem 2.7. *Let R be an indecomposable ring which respects pre-order and suppose $\{S_i\}$ is a family of indecomposable rings such that $\mathrm{CEN}\, R \not\approx \mathrm{CEN}\, S_i$ for all i. Then for all finite $n > 0$, $R^n \times \times_i S_i$ respects pre-order.*

Proof. Suppose $I(P, R^n \times \times_i S_i) \approx I(Q, R^n \times \times_i S_i)$. Then

$$I(P, R^n) \times I(P, \times_i S_i) \approx I(Q, R^n) \times I(Q, \times_i S_i).$$

Let φ be an isomorphism. The hypotheses imply that each of the factors present decomposes into a product of indecomposable rings. For example, if $Q = \dot{\bigcup} Q_j$ where the Q_j are the components of Q then

$$I(Q, \times_i S_i) \approx \times_j I(Q_j, \times_i S_i) \approx \times_j \times_i I(Q_j, S_i)$$

using Lemmas 2.1 and 2.2. Lemma 2.6 implies that $I(Q_j, S_i)$ is indecomposable, so the assertion is established. It now follows, as is well known, that the indecomposable factors of

$$I(P, R^n) \times I(P, \times_i S_i)$$

are in isomorphic correspondence with those of

$$I(Q, R^n) \times I(Q, \times_i S_i),$$

the correspondence being implemented by φ. An indecomposable factor of

$I(P, R^n)$ which will have the form $I(K, R)$, K a component of P, will correspond to an indecomposable factor of $I(Q, \times_i S_i)$ or of $I(Q, R^n)$. In the first case we would have an isomorphism between $I(K, R)$ and, say, $I(Q_j, S_i)$. Then

$$\mathrm{CEN}(I(K, R)) \approx \mathrm{CEN}\big(I(Q_j, S_i)\big).$$

So by Lemma 2.5, $\mathrm{CEN} R \approx \mathrm{CEN} S_i$, which is a contradiction. Thus an indecomposable factor of $I(P, R^n)$ must correspond to an indecomposable factor of $I(Q, R^n)$. In fact, we can conclude that the indecomposable factors of $I(P, R^n)$ are in isomorphic correspondence with the indecomposable factors of $I(Q, R^n)$. Thus $I(P, R^n) \approx I(Q, R^n)$. Since R^n respects pre-order by Corollary 2.4, $P \approx Q$.

COROLLARY 2.8. *Let $\{R_i\}$ be a family of indecomposable commutative rings such that $R_i \not\approx R_j$ for $i \neq j$, and let $\{e_i\}$ be a family of nonzero cardinal numbers. Then $\times_i R_i^{e_i}$ respects pre-order iff there exists an i such that e_i is finite.*

Proof. If e_i is finite we apply Theorem 1.4 and Theorem 2.7 to $R_i^{e_i} \times \times_{j \neq i} R_j^{e_j}$ to conclude that $\times_i R_i^{e_i}$ respects pre-order. Conversely, if each e_i is infinite, take any connected pre-ordered set P. Then

$$\begin{aligned} I(2P, \times_i R_i^{e_i}) &\approx I(P, \times_i R_i^{e_i}) \times I(P, \times_i R_i^{e_i}) \\ &\approx I(P, \times_i R_i^{e_i} \times \times_i R_i^{e_i}) \\ &\approx I(P, \times_i R_i^{e_i}). \end{aligned}$$

However, since P is connected, $2P \not\approx P$, so $\times_i R_i^{e_i}$ does not respect pre-order.

Remarks. Since any product of indecomposable commutative rings is isomorphic to a product $\times_i R_i^{e_i}$ as in Corollary 2.8, we can, in fact, tell when such a ring respects pre-order. Although it is still unknown to us whether a finite product of rings which respect pre-order (partial order) respects pre-order (partial order), cf. [12], Corollary 2.4 and Theorem 2.7 can be used to decide the issue in certain concrete cases, e.g., in the cases of certain artinian rings. We shall have more to say about this problem in Section 4.

Let K, F be fields with $K \not\approx F$. It follows from [12] that $I(N^\omega, K)$ is an indecomposable ring which doesn't respect pre-order. By Theorem 2.7, $F \times I(N^\omega, K)$ respects pre-order, hence there are rings which respect pre-order but which have factors that don't.

We now proceed to solve the isomorphism problem for incidence rings over products of indecomposable commutative rings. Let $P = \dot\bigcup m_i K_i = Q = \dot\bigcup n_i K_i$ and let $R = \times_i R_i^{e_i}$ be a product of indecomposable rings as in Corollary 2.8.

Reasoning as in the proof of Theorem 2.7 we see that $I(P, R) \approx I(Q, R)$ if and only if for each i, $I(P, R_i^{e_i}) \approx I(Q, R_i^{e_i})$. So $I(P, R) \approx I(Q, R)$ if and only if for each i, $I(e_iP, R_i) \approx I(e_iQ, R_i)$. Since R_i respects pre-order, $I(P, R) \approx I(Q, R)$ if and only if for each i, $e_iP \approx e_iQ$, and this occurs if and only if for each i, $e_im_j = e_nn_j$ for all j. In summary we have the following.

THEOREM 2.9. *Let $R = \times_i R_i^{e_i}$ be a product of indecomposable commutative rings as in Corollary 2.8. Write $P = \dot{\bigcup} m_iK_i$ and $Q = \dot{\bigcup} n_iK_i$. Then $I(P, R) \approx I(Q, R)$ if and only if either*

(i) *there is an i such that e_i is finite and $P \approx Q$ or*

(ii) *each e_i is infinite and for any i, $e_im_j = e_in_j$ for any j.*

COROLLARY 2.10. *Commutative Noetherian rings respect pre-order.*

Section 3

It follows from Theorem 2.9 that if P, Q are connected pre-ordered sets, if R is a product of indecomposable commutative rings, and if $I(P, R) \approx I(Q, R)$ then $P \approx Q$. It is therefore plausible to conjecture that commutative rings respect order in the class of connected pre-ordered sets. In this section we will show that commutative rings respect order in the class of finite, connected, pre-ordered sets.

LEMMA 3.1. *Suppose that R is a commutative ring and that P, Q are connected pre-ordered sets. If $I(P, R)$ and $I(Q, R)$ are isomorphic as rings then they are isomorphic as R-algebras.*

Proof. Let φ: $I(P, R) \to I(Q, R)$ be a ring isomorphism. If $r \in R$ then $\varphi(r\delta) \in \mathrm{CEN}(I(Q, R))$, where δ is the identity in $I(P, R)$. Using Lemma 2.5 we obtain a ring isomorphism α: $R \to R$ such that $\varphi(r\delta) = \alpha(r)\delta$. Define ψ: $I(Q, R) \to I(Q, R)$ by $(\psi f)(x, y) = \alpha^{-1}(f(x, y))$. It is clear that ψ is a ring isomorphism and that $\psi \circ \varphi$ is a ring isomorphism. If $r \in R$ and $f \in I(P, R)$, then

$$\psi \circ \varphi(rf) = \psi \circ \varphi((r\delta)f) = \psi(\alpha(r)\delta \cdot \varphi(f)) = (r\delta)\psi \circ \varphi(f) = r\psi \circ \varphi(f).$$

So $\psi \circ \varphi$ is an R-algebra isomorphism.

THEOREM 3.2. *Let R be a commutative ring and let P, Q be finite, connected, pre-ordered sets. If $I(P, R) \approx I(Q, R)$ then $P \approx Q$.*

Proof. Let φ: $I(P, R) \to I(Q, R)$ be a ring isomorphism. By Lemma 3.1 we may assume that φ is an isomorphism of R-algebras. Since R is commutative it has a maximal ideal M. Let

$$I(P, M) = \{f \in I(P, R)\colon f(x, y) \in M \text{ for all } x, y \in P\}.$$

We shall show that

$$\varphi(I(P, M)) = I(Q, M).$$

Let $f \in I(P, M)$. Then $f = \sum_{i \le j} m_{ij}\delta_{ij}$ (cf. Lemma 2.5) where $m_{ij} \in M$, and so $\varphi(f) = \varphi(\sum_{i \le j} m_{ij}\delta_{ij}) = \sum_{i \le j} m_{ij}\varphi(\delta_{ij}) \in I(Q, M)$. Thus

$$\varphi(I(P, M)) \subseteq I(Q, M).$$

Using φ^{-1} we can conclude that

$$\varphi^{-1}(I(Q, M)) \subseteq I(P, M),$$

and so $\varphi(I(P, M)) = I(Q, M)$. Next consider the epimorphism

$$\pi\colon I(P, R) \to I(Q, R/M)$$

defined by the formula

$$(\pi f)(x, y) = [f(x, y)] \quad \text{in } R/M.$$

Since $\ker \pi = I(P, M)$, $I(P, R)/I(P, M) \approx I(Q, R/M)$. Therefore, from $I(P, R)/I(P, M) \approx I(P, R/M)$, we have $I(P, R/M) \approx I(Q, R/M)$. Since R/M is a field we find that $P \approx Q$.

Section 4

Throughout the section, C will denote the class of connected pre-ordered sets. We wish to describe how the methods of the previous section can be used to show that many familiar commutative rings respect order in C.

THEOREM 4.1. *Let R be a commutative ring and A a finitely generated ideal in R such that R/A respects order in C. Then R respects order in C.*

Proof. Let $A = (x_1, \ldots, x_n)$ and let $\varphi\colon I(P, R) \to I(Q, R)$ be an R-algebra isomorphism. If $f \in I(P, A)$ then f can be written as $\sum_{i=1}^n x_i f_i$ where $f_i \in I(P, R)$. So $\varphi(f) = \sum_{i=1}^n x_i\varphi(f_i) \in I(Q, A)$, and consequently, $\varphi(I(P, A)) = I(Q, A)$. The epimorphism $\pi\colon I(P, R) \to I(Q, R/Q)$ defined as in Theorem 3.2 has kernel $I(P, A)$. Hence $I(P, R/A) \approx I(Q, R/A)$, so $P \approx Q$.

COROLLARY 4.2. *If R is commutative and has a finitely generated maximal ideal then R respects order in C.*

COROLLARY 4.3 (D. D. Anderson). *If R is commutative and respects order in C then $R[x]$ and $R[[x]]$ respect order in C.*

Proof. Set $A = (x)$ and use the isomorphisms $R[x]/(x) \cong R[[x]]/(x) \cong R$.

The next somewhat surprising corollary settles a question in [12, concluding remarks] for commutative rings and connected pre-ordered sets.

COROLLARY 4.4. *Let R be a commutative ring which respects order in C and let S be any commutative ring. Then $R \times S$ respects order in C.*

Proof. Take $A = O \times S$ and use the isomorphism $R \times S/A \approx R$.

We would like to conclude by giving two interesting applications of the above theory. Let $R = \{f \in \times_{n=1}^{\infty} Z_2\colon f \text{ is eventually constant}\}$. R is not a product of indecomposable commutative rings (use the fact that an indecomposable Boolean ring must be Z_2 and a cardinality argument) and from $R \times R \approx R$ we see that R doesn't respect pre-order (cf. Corollary 2.8). However since $R \approx Z_2 \times R$, R respects order in C by Corollary 4.4.

For the second application let $R = \{f\colon \mathbf{Q} \to \mathbf{Q} \mid f \text{ is continuous at all but a finite number of rationals}\}$ (here $\mathbf{Q}$ has the relative topology induced by reals). R is not a product of indecomposable rings and R doesn't respect pre-order since $R \times R \approx R$. Let $f\colon \mathbf{Q} \to \mathbf{Q}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $f \in R$ and (f) is a maximal ideal. Therefore R respects order in C by Corollary 4.2. If however we set

$$R = \{f\colon \mathbf{Q} \to \mathbf{Q} \mid f \text{ is continuous}\}$$

then it is unknown whether R respects order in C.

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