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# THE CLASSIFYING SPACE OF SMOOTH FOLIATIONS

BY

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One of the most striking examples of a foliated manifold was constructed by Thurston [24] on the 3-sphere. He exhibited a family of codimension-one foliations on $S^3$ for which the Godbillon-Vey class takes on a continuous range of values. In subsequent explicit constructions of foliations with non-zero secondary classes, almost always a discrete group $\Gamma$ is given which acts on $S^q$ and the foliation is defined on the quotient $M^n = B \times_\Gamma S^q$, where $\Gamma$ acts freely on $B$ [1], [5], [6], [13], [19], [21]. The quotient space $M$ is very complicated as a manifold, and it is natural to ask whether there exist codimension $q$ foliations on spheres $S^n$ with non-vanishing secondary classes for $q > 1$ (Haefliger's problem 2, p. 241 of [22].) One of the aims of this paper is to answer this question affirmatively (Corollary 4.6). For $n = 2q + 1$, if $S^n$ admits a rank $q$ subbundle $Q \subseteq TS^n$ then there exists a family of codimension $q$ foliations on $S^n$ for which a set of secondary classes takes on a continuous range of values. Other values of $n > 2q$ also work, and $S^n$ can be replaced with any closed, oriented $n$-manifold $M$ which admits a $q$-frame field $Q \subseteq TM$. This is a consequence of Thurston's realization theorem [25], and the following two theorems which we will prove.

For each $q > 1$ a sequence of non-negative integers $\{v_{q,n}\}$ is defined in §2.8 with the properties:

(1) For $q = 2$, $\lim_{k \to \infty} v_{2,4k+1} = \infty$ and $v_{2,4k+1} > 0$ for all $k > 0$.
(2) For $q = 3$, $\lim_{k \to \infty} v_{3,3k+1} = \infty$ and $v_{3,3k+1} > 0$ for all $k > 1$.
(3) For $q > 3$, $\lim_{n \to \infty} v_{q,n} = \infty$.

We denote by $B\Gamma_q$ Haefliger's classifying space of codimension $q$ smooth foliations. The *integral* homotopy groups of $B\Gamma_q$ are denoted $\pi_*(B\Gamma_q)$.

THEOREM 1. *For each $q > 1$ and $n > 2q$ there is an epimorphism of abelian groups*

$$\pi_n(B\Gamma_q) \to \mathbf{R}^{v_{q,n}}.$$

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For $q$ odd this is precisely the content of [14, Proposition 6.17]. For $q$ even, this gives the first non-triviality and variation results for the homotopy groups of $B\Gamma_q$.

Let $B\overline{\Gamma}_q$ be the homotopy fiber of $\nu$: $B\Gamma_q \to BGL_q$. We also obtain variation results for the secondary classes in $H^*(B\overline{\Gamma}_q)$. In §2.6 we define a non-empty set $\overline{V}_q \subseteq H^*(W_q)$ such that:

THEOREM 2. *The characteristic map $k_*$: $H^*(W_q) \to H^*(B\overline{\Gamma}_q)$ is injective on the span of $\overline{V}_q$, and the image set $k_*(\overline{V}_q)$ is independently variable. That is, $k_*(\overline{V}_q)$ defines a surjection $H_*(B\overline{\Gamma}_q; Z) \to \mathbf{R}^{\#\overline{V}_q}$ of the integral homology groups.*

For all $q \geq 2$ this theorem gives further independence results for the secondary classes, in addition to those previously established by Baker [1], Heitsch [13] and Kamber-Tondeur [19]. For even $q$, the above yields the first variation results proved for $H^*(B\overline{\Gamma}_q)$.

In previous papers [14], [15] we constructed a set of independent rigid classes in $H^*(B\overline{\Gamma}_q)$ for even codimensions. We show in §3 that many of these cohomology classes are spherically supported (Theorem 3.2) and the corresponding spherical cycles generate a large free graded Lie subalgebra in $\pi_*(B\Gamma_q)$. From this we derive:

COROLLARY 4.8. *Let $q \geq 10$ be even and suppose $n \gg q$. If $S^n$ admits a rank $q$ subbundle $Q \subseteq TS^n$, then there exists an infinite set $\{\mathscr{F}_\alpha\}$ of codimension $q$ foliations on $S^n$ which are distinct up to homotopy and concordance, but whose tangential distributions $\{F_\alpha \subseteq TS^n\}$ are all homotopic as embedded subbundles.*

This answers problem 3 posed by Lawson in [20], and gives further examples of non-trivially foliated spheres.

Our analysis of $B\overline{\Gamma}_q$ for even codimensions is based on the following idea. The general technique for studying $H_*(B\Gamma_q)$ has been to construct a foliated manifold $M$ with non-trivial secondary classes. The classifying map $f$: $M \to B\Gamma_q$ then has non-trivial image $f_* \subseteq H_*(B\Gamma_q; Z)$. In most cases, the rational Pontrjagin classes vanish for the normal bundle $Q$ of the foliation on $M$, and we show that the map $f$: $M \to B\Gamma_q$ can then be modified via CW-space techniques to produce non-trivial spherical cycles in $H_*(B\Gamma_q; Z)$. The corresponding homotopy classes in $\pi_*(B\Gamma_q)$ generate a free graded Lie subalgebra whose elements come from $\pi_*(B\overline{\Gamma}_q)$. The images of some of these classes under $\pi_*(B\overline{\Gamma}_q) \to H_*(B\overline{\Gamma}_q; Z)$ are shown to be detected by the set $k_*(\overline{V}_q)$ of Theorem 2.

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We will use $H^*(X)$ to denote the singular cohomology of $X$ with real coefficients, and $H_*(X; R)$ denotes the singular homology with coefficients in a ring $R$. All spaces are connected and have a basepoint, and maps are always assumed to be continuous.

## 1. Homotopy properties of $B\Gamma_q$

Recall that $\nu\colon B\Gamma_q \to BGL_q$ classifies the universal normal bundle and $B\overline{\Gamma}_q$ denotes the homotopy fiber of $\nu$. The fundamental result about $B\overline{\Gamma}_q$ is:

THEOREM 1.1 (Haefliger-Mather-Thurston [26]). *The space $B\overline{\Gamma}_q$ is $(q+1)$-connected.*

The double cover $B\Gamma_q^+$ of $B\Gamma_q$ is thus simply connected and there is a fibration

$$B\overline{\Gamma}_q \to B\Gamma_q^+ \xrightarrow{\nu} BSO_q.$$

The space $B\Gamma_q^+$ classifies foliations with orientable normal bundles, and is technically easier to work with. We will need the following general result. For any simply connected CW complex $X$, the localization of $X$ at $\{0\}$ (or the *rationalization* of $X$) is denoted $X_{\mathbf{Q}}$. The basic property of $X_{\mathbf{Q}}$ is there exists an inclusion $X \subseteq X_{\mathbf{Q}}$ inducing an isomorphism

$$\pi_*(X) \otimes \mathbf{Q} \xrightarrow{\cong} \pi_*(X_{\mathbf{Q}}) \otimes \mathbf{Q} \cong \pi_*(X_{\mathbf{Q}}).$$

For details see [2], [23].

LEMMA 1.2. *Let $f\colon X \to Y$ be a map of CW complexes with both $X$ and $Y$ simply connected. Assume the rational space $Y_{\mathbf{Q}}$ is an $H$-space and that $f_*\colon H_n(X; \mathbf{Q}) \to H_n(Y; \mathbf{Q})$ is the trivial map for all $n \leq N$. Then there exists a CW complex $X'$ and map $g\colon X' \to X$ so that*:

(a) $g_*\colon H_*(X'; \mathbf{Q}) \cong H_*(X; \mathbf{Q})$.

(b) *For the $N$-skeleton $X'_N$ of $X'$, the restriction of the composition, $f \circ g|_{X'_N}\colon X'_N \to Y$ is homotopic to the constant map.*

In other words, if a map $f\colon X \to Y$ of simply connected spaces is rationally trivial up to dimension $N$, and $Y_{\mathbf{Q}}$ is an $H$-space then we can multiply $f$ on the cells of $X$ to make the new map trivial up to dimension $N$.

*Proof.* For a CW complex $C$, let $C_n$ denote the $n$-skeleton.

To prove 1.2, we inductively construct spaces $X_n'$ and maps $g_n$: $X_n' \to X$ which satisfy:

($a_n$) $(g_n)_*$: $H_m(X_n'; \mathbf{Q}) \to H_m(X; \mathbf{Q})$ is an isomorphism for $0 \leq m \leq n$.

($b_n$) If $n \leq N$ then $f \circ g_n$: $X_n' \to Y$ is homotopic to the constant map.

($c_n$) $X_n'$ is obtained from $X_{n-1}'$ by attaching $n$-cells.

We then set $X' = \bigcup_{n=1}^{\infty} X_n'$ and define $g = \lim g_n$: $X' \to X$; the map $g$ clearly satisfies (a) and (b) of the theorem.

By Milnor's Theorem (see [28, p. 196]) we can assume $X$ has a minimal CW complex structure. This guarantees the existence of a basis of $H_*(X; \mathbf{Q})$ represented by cells in the CW complex of $X$.

To proceed, set $X_1' = \{*\}$, the basepoint and let $g_1$: $X_1' \to X$ be the map onto the basepoint in $X$.

Let $n > 1$ and assume that $g_{n-1}$: $X_{n-1}' \to X$ is given. Choose a set of $n$-cells $\{e_i^n | i \in I_n\}$ in the CW structure for $X$ which gives a basis for $H_n(X; \mathbf{Q})$. For each $i \in I_n$, let $\alpha_i$: $S^{n-1} = \partial e_i^n \to X_{n-1}$ be the attaching map. Our choice of $I_n$ is made so that the inclusion

$$X_{n-1} \bigcup_{I_n} e_i^n \subseteq X$$

induces an isomorphism on $H_n(\quad; \mathbf{Q})$. Each map $\alpha_i$ determines a class

$$[\alpha_i] \in \pi_{n-1}(X_{n-1});$$

by Mayer-Vietoris the image $[\alpha_i] \in H_{n-1}(X_{n-1}; \mathbf{Q})$ is zero.

The hypothesis that $X$ is simply connected implies $g_{n-1}$: $X_{n-1}' \to X_{n-1}$ is a rational $(n-1)$-equivalence, and

$$\pi_{n-1}(X_{n-1}') \otimes \mathbf{Q} \cong \pi_{n-1}(X_{n-1}) \otimes \mathbf{Q}.$$

Therefore, there exist maps $\gamma_i$: $S^{n-1} \to X_{n-1}'$ such that $|g_{n-1} \circ \gamma_i|$ is a multiple of $[\alpha_i]$ in $\pi_{n-1}(X_{n-1})$. Note that

$$[\gamma_i] = (g_{n-1})_*^{-1}([\alpha_i]) = 0 \quad \text{in } H_{n-1}(X_{n-1}'; \mathbf{Q}).$$

Now set

$$\tilde{X}_n = X_{n-1}' \bigcup_{\substack{\gamma_i, \\ i \in I_n}} e_i^n.$$

There is a natural extension of $g_{n-1}$ to a map $\tilde{g}_n$: $\tilde{X}_n \to X$, defined by mapping $\partial e_i^n$ into $X_{n-1}$ via the appropriate multiple of $\alpha_i$, and then we can extend across $e_i^n$. It is straightforward that

$$(\tilde{g}_n)_*: H_n(\tilde{X}_n; \mathbf{Q}) \to H_n(X; \mathbf{Q})$$

is an isomorphism. If $n > N$, we set $X'_n = \tilde{X}_n$ and $g_n = \tilde{g}_n$ to finish the inductive step.

For $n \leq N$, the inductive hypothesis implies that the composition

$$f \circ \tilde{g}_n \colon \tilde{X}_n \to Y$$

is homotopic to a constant when restricted to the $(n-1)$-skeleton $X'_{n-1}$. Let $W = \tilde{X}_n / X'_{n-1}$ be the quotient, with $c\colon \tilde{X}_n \to W \cong \bigvee_{I_n} S^n$ the collapse map. Then $f \circ \tilde{g}_n$ induces a map $h\colon W \to Y$ on the quotient.

Consider the commutative diagram

$$\begin{array}{ccccc}
H_n(\tilde{X}_n; \mathbf{Q}) & \xrightarrow{(f\circ \tilde{g}_n)_*} & \searrow & & \\
\Big\downarrow{\scriptstyle c_*} & & & & \\
H_n(W; \mathbf{Q}) & \xrightarrow{h_*} & H_n(Y; \mathbf{Q}) & \cong & H_n(Y_{\mathbf{Q}}; Z) \\
\Big\uparrow & & \Big\uparrow{\scriptstyle \mathscr{H}} & & \Big\uparrow \\
\pi_n(W) & \xrightarrow{h_\#} & \pi_n(Y) \otimes \mathbf{Q} & \cong & \pi_n(Y_{\mathbf{Q}})
\end{array}$$

The map $(f \circ \tilde{g}_n)_*$ is zero by hypothesis; by construction, $c_*$ is surjective and so $h_*$ is the zero map. The assumption $Y_{\mathbf{Q}}$ is an $H$-space implies the Hurewicz map $\mathscr{H}$ is injective. Therefore, $h_\#$ is rationally trivial in degree $n$.

For each $i \in I_n$, let $\beta_i\colon S^n \to W$ be the inclusion of the $i$-th factor of the wedge corresponding to the inclusion $e_i^n \subseteq \tilde{X}_n$. The above shows that $h_\#(\beta_i) \in \pi_n(Y)$ has finite order $n_i > 0$. Define a new space in place of $\tilde{X}_n$:

$$X'_n = X'_{n-1} \bigcup_{\substack{n_i \gamma_i, \\ i \in I_n}} e_i^n.$$

There is a natural map $\mu_n\colon X'_n \to \tilde{X}_n$ which is the identity on $X'_{n-1}$, and has degree $n_i$ on $e_i^n$. Set $g_n = \tilde{g}_n \circ \mu_n$. It remains to check that $g_n\colon X'_n \to X$ satisfies $(b_n)$. First note $f \circ g'_n = f \circ \tilde{g}_n$ on $X'_{n-1}$, so $f \circ g'_n$ is homotopic to the composition

$$X'_n \to X'_n / X'_{n-1} \xrightarrow{\bar{\mu}_n} \tilde{X}_n / X'_{n-1} \xrightarrow{h} Y.$$

But $h \circ \bar{\mu}_n$ is homotopic to a constant by choice of the $n_i$. ∎

**Proposition 1.3.** *Let $f\colon M \to B\Gamma_q^+$ be a continuous map, and assume that*

$$(\nu \circ f)^*\colon H^n(BSO_q) \to H^n(M)$$

*is zero in degrees $n \leq N \leq q + 1$. Then there exists an $N$-connected space $X$, a map $f_X$: $X \to B\Gamma_q^+$ and a map $\xi$ on cohomology so that*

$$\begin{array}{ccc} H^n(M) & \xleftarrow{\ f^*\ } & \\ \Big\downarrow \xi & & H^n(B\Gamma_q^+) \\ H^n(X) & \xleftarrow{\ f_X^*\ } & \end{array} \tag{1.4}$$

*commutes, and $\xi$ is an isomorphism for $n > N + 1$.*

*Proof.* We may assume that $M$ is a CW complex, and then set $\hat{M} = M/M_1$. Since $B\Gamma_q^+$ is simply connected we get an induced $\hat{f}$: $\hat{M} \to B\Gamma_q^+$. The space $\hat{M}$ is simply connected, so apply Lemma 1.2 to $\hat{M}$ and $\nu \circ \hat{f}$ to obtain a map $g$: $\hat{M}' \to \hat{M}$ which is rational homology equivalence, and $\nu \circ \hat{f} \circ g$: $\hat{M}'_N \to BSO_q$ is homotopic to a constant. Since $N \leq q + 1$ and $\nu$: $B\Gamma_q^+ \to BSO_q$ is $(q + 2)$-connected, the map $\hat{f} \circ g$: $\hat{M}'_N \to B\Gamma_q^+$ is also homotopic to a constant. We define $f_X$ to be the induced map on the identification space

$$f_X: X = \hat{M}'/\hat{M}'_N \to B\Gamma_q^+.$$

Note that $X$ is $N$-connected, and $\xi$ is defined by the isomorphisms (for $n > N + 1$)

$$\xi: H^n(M) \cong H^n(\hat{M}) \cong H^n(\hat{M}') \cong H^n(X). \qquad \blacksquare$$

DEFINITION 1.5. A class $c \in H_n(X; Z)$ is said to be spherical if there is a map $\alpha$: $S^n \to X$ representing the homology class of $c$. The spherical cycles for $X$ consists of the cycles in the image of the Hurewicz map $\pi_*(X) \to H_*(X; Z)$.

The spherical cycles for $B\Gamma_q^+$ play an important role in analyzing the homotopy type of this space. We come now to the main result of this section.

PROPOSITION 1.6. *Let $z \in H^{2q+1}(B\Gamma_q^+)$ be such that there is a cycle $c$ in the image of $f$: $M \to B\Gamma_q^+$ with $z(c) \neq 0$, and $(\nu \circ f)^*$ is trivial in degrees less than $q$. Then there is a spherical class in $H_{2q+1}(B\Gamma_q^+; Z)$ on which $z$ is non-trivial.*

COROLLARY 1.7. *Let $z \in H^{2q+1}(B\Gamma_q^+)$ satisfy (1.6) and let $i$: $B\overline{\Gamma}_q \to B\Gamma_q^+$ be the inclusion of the fiber. Then $i^*(z)$ is non-zero in $H^{2q+1}(B\overline{\Gamma}_q)$.*

*Proof.* Let $f$: $M \to B\Gamma_q^+$ be given. By Proposition 1.3 we replace $f$ with a map $f_X$: $X \to B\Gamma_q^+$ where $X$ is $(q - 1)$-connected and $z$ is non-zero on a cycle in the image of $f_{X*}\colon H_{2q+1}(X; Z) \to H_{2q+1}(B\Gamma_q^+; Z)$. The aim is to show that a multiple of each cycle in the image of $f_{X*}$ is spherical.

*Case* 1. $q$ is odd. Then $H^q(BSO_q) = \{0\}$ and in choosing $X$ we can apply 1.3 with $N = q$ so that $X$ is $q$-connected. By the rational Hurewicz theorem [2] the map $\pi_{2q+1}(X) \otimes \mathbf{Q} \to H_{2q+1}(X; \mathbf{Q})$ is onto, and so a multiple of each class in $H_{2q+1}(X; Z)$ is spherical. The claim follows.

*Case* 2. $q$ is even. Then $\pi_{q+1}(BSO_q) \cong \pi_{q+1}(B\Gamma_q^+)$ is a finite group. We construct a new space $X'$ by attaching $(q + 2)$-cells to $X$ so that $\pi_{q+1}(X')$ is a torsion group, and $f_X$ extends to $f_X'\colon X' \to B\Gamma_q^+$. The space $X'$ is $(q - 1)$-connected, and $\pi_{q+1}(X') \otimes \mathbf{Q} = \{0\}$. Therefore, in the minimal model $\mathscr{M}_{X'}$ for $X'$, (see [2] or [23] for details) the decomposable cocycles occur in degrees $2q$ and $\geq 2q + 2$. In particular, $H^{2q+1}(X')$ is represented by closed indecomposable elements in $\mathscr{M}_{X'}$. By the fundamental theorem of minimal models [Theorem 10.1; 23] the map $\pi_{2q+1}(X') \otimes \mathbf{Q} \to H_{2q+1}(X'; \mathbf{Q})$ is onto. This completes the proof of (1.6).

To prove the corollary, let $\alpha\colon S^{2q+1} \to B\Gamma_q^+$ define a cycle for which $z(\alpha) \neq 0$. The group $\pi_{2q+1}(BSO_q)$ is always finite, so replacing $\alpha$ by a non-zero multiple if necessary we can assume $\nu \circ \alpha$ is homotopic to a constant. Thus, $\alpha$ factors as

$$S^{2q+1} \xrightarrow{\bar{\alpha}} B\overline{\Gamma}_q \xrightarrow{i} B\Gamma_q^+$$

and $i^*(z)(\bar{\alpha}) = z(\alpha) \neq 0$. ∎

*Remark* 1.8. It is important to note that if a fixed space $M$ is given with a family of maps $f_\lambda\colon M \to B\Gamma_q^+$, $\lambda \in \mathscr{A}$, so that a cocycle $z$ is non-trivial on the images of $f_{\lambda*}$, then the proof of (1.6) shows that we can construct a fixed space $X'$ with an induced family of maps $f_{\lambda, X'}\colon X' \to B\Gamma_q^+$ for which $H_{2q+1}(X'; \mathbf{Q})$ consists of spherical classes. That is, there is a fixed isomorphism $\xi\colon H^n(X') \cong H^n(M)$ making (1.4) commute for all $\lambda \in \mathscr{A}$.

## 2. Non-trivial secondary classes for framed foliations

The construction of the secondary classes of a smooth foliation is briefly recalled. For details, see Bott [3], [4] or Kamber-Tondeur [18]. Examples and results of Heitsch and Rasmussen are presented, which we use in conjunction with Corollary 1.7 to study $H^*(B\overline{\Gamma}_q)$.

Let $I(GL_q)$ denote the algebra of Ad-invariant polynomials on the Lie algebra $gl_q$. This has an algebra basis given by the Chern polynomials $\{c_1, \ldots, c_q\}$ where degree $c_j = z_{2j}$; a vector space basis of $I(GL_q)$ is given by the monomials $c_J = c_1^{j_1} \cdots c_q^{j_q}$, where $J = (j_1, \ldots, j_q)$ with $j_l \geq 0$. The truncated polynomial algebra $I_q$ is the quotient

$$I_q = I(GL_q)/\text{ideal}\{c_J | \deg c_J > 2q\}.$$

Define a differential graded algebra

$$W_q = \Lambda(y_1, \ldots, y_q) \otimes I_q$$

where $\Lambda(y_1, \ldots, y_q)$ is an exterior algebra on indeterminants $y_i$ with degree $y_i = 2i - 1$. The differential is determined by setting $dy_i = c_i$. Also, a differential subalgebra $WO_q \subseteq W_q$ is defined by

$$WO_q = \Lambda(y_1, y_3, \ldots, y_{q'}) \otimes I_q,$$

where $q'$ is the greatest odd integer $\leq q$.

Let $\mathcal{F}$ be a foliation of codimension $q$ on a manifold $M$, $Q$ is the normal bundle to $\mathcal{F}$ and $A(M)$ is the deRham algebra of $M$. The choice of a basic connection $\omega$ on $Q$ and a metric $g$ on $Q$ determines a differential algebra homomorphism $\Delta(\omega, g)$: $WO_q \to A(M)$, and it is a classical result that the induced map

$$\Delta_* = \Delta(\omega, g)_*\colon H^*(WO_q) \to H^*(M)$$

does not depend upon the choice of the connection $\omega$ or metric $g$, but depends only on the concordance class of $\mathcal{F}$.

If $Q$ is a trivial bundle, then the choice of a framing $s$ defines a map

$$\Delta(\omega, s)\colon W_q \to A(M),$$

and the induced map

$$\Delta^s_* = \Delta(\omega, s)_*\colon H^*(W_q) \to H^*(M)$$

depends only upon the concordance class of $\mathcal{F}$ and the homotopy class of $s$.

The construction of the homomorphisms $\Delta_*$ and $\Delta^s_*$ is functorial, and there are universal maps denoted by

$$\tilde{\Delta}_*\colon H^*(WO_q) \to H^*(B\Gamma_q) \quad \text{and} \quad k_*\colon H^*(W_q) \to H^*(B\overline{\Gamma}_q).$$

A basis for the space $H^*(W_q)$ is easily described. First, a basis for $W_q$ is given by the monomials of the form

$$y_I c_J = y_{i_1} \cdots y_{i_s} c_J,$$

where

$$I = (i_1, \ldots, i_s) \quad \text{with } 1 \leq i_1 < \cdots < i_s \leq q,$$
$$J = (j_1, \ldots, j_q) \quad \text{with } j_l \geq 0 \quad \text{and} \quad \deg c_J \leq 2q.$$

An element $y_I c_J$ is a cocycle if degree $y_{i_l} c_J > 2q$ for all $1 \leq l \leq s$. According to Vey [7], the following subset of cocycles is a basis of $H^*(W_q)$:

$$Z_q = \{ y_I c_J | 1 \leq i_1 < \cdots < i_s \leq q; \text{ degree } y_{i_1} c_J > 2q;$$
$$\deg c_J \leq 2q; \ l < i_1 \Rightarrow j_l = 0 \}.$$

The elements of $Z_q$ are called *admissible* cocycles.

An admissible cocycle $y_I c_J$ with $\deg y_{i_1} c_J > 2q + 2$ is said to be *rigid*. This terminology reflects the fact that these secondary classes are invariant under homotopy (a one-parameter deformation of the foliation [12].) An admissible cocycle which is not rigid is said to be *variable*, and has degree $y_{i_1} c_J = 2q + 1$.

A set of cocycles $ZO_q$ which is a basis for $H^*(WO_q)$ can also be exhibited. For our purposes, we only note that there is an inclusion

$$H^{2q+1}(WO_q) \subseteq H^{2q+1}(W_q)$$

whose image is spanned by the admissible cocycles $y_i c_J$ of degree $2q + 1$ such that either $i$ is odd, or $c_J$ contains a factor $c_l$ for some odd $l$. Hence, for $q$ even this inclusion is actually an isomorphism.

An explicit construction of foliated manifolds for which some secondary classes are independently variable has been given by Heitsch [13] and Rasmussen [21]. We recall their results in a form convenient to our purposes.

THEOREM 2.1 (Rasmussen). *There exists a compact, oriented 5-manifold $M$ with a family $\{\mathcal{F}_\alpha | \alpha \in \mathcal{A}\}$ of codimension 2 foliations for which the values of the secondary classes $\Delta_*\{y_1 c_1^2, y_1 c_2\} \subseteq H^5(M)$ vary independently with $\alpha$.*

The common normal bundle of the foliations $\{\mathcal{F}_\alpha\}$ is not trivial as it has non-zero Euler class.

For higher codimensions, there are the following two results, which are Theorems 6.2 and 6.3 of [13]. A subset $V_q \subseteq H^{2q+1}(W_q)$ will be defined in (2.4), with $V_q$ containing at least three elements for all $q \geq 3$. For $q$ even, recall that we identify $H^{2q+1}(WO_q) = H^{2q+1}(W_q)$.

THEOREM 2.2 (Heitsch). *Let $q \geq 4$ be even. There exists a compact, oriented manifold $M$ of dimension $2q + 1$ with a family of codimension $q$ foliations $\{\mathcal{F}_\lambda | \lambda \in \mathcal{A}\}$ such that the secondary classes $\Delta_*(V_q) \subseteq H^{2q+1}(M)$ vary independently with $\lambda$.*

THEOREM 2.3 (Heitsch). *Let $q \geq 3$ be odd. There exists a compact, oriented manifold $M$ with a family $\{(\mathscr{F}_\lambda, s) | \lambda \in \mathscr{A}\}$ of codimension $q$ foliations, and a framing $s$ of their common normal bundle $Q$, such that the secondary classes $\Delta^s_*(V_q) \subseteq H^{2q+1}(M)$ vary independently with $\lambda$.*

The foliations $\mathscr{F}_\lambda$ of (2.2) are obtained by starting with a flat $\mathbf{R}^{q+1}$-vector bundle $E$ over a compact manifold $X^{q+1}$. An auxiliary vector field $v_\lambda$ on $E$ is introduced which is non-zero off the zero section $E_0 \subseteq E$. By construction, the flow of the vector field $v_\lambda$ preserves the flat bundle structure on $E \to X$. Therefore, the span of $v_\lambda$ and the horizontal distribution on $E$ of the flat bundle together form a codimension $q$ foliation $\mathscr{F}'_\lambda$ on $E^* = E - E_0$. The manifold $M$ consists of the unit sphere bundle inside $E^*$, and $\mathscr{F}_\lambda$ is the restriction of $\mathscr{F}'_\lambda$. The reader is referred to Example 5.2 of [13] for complete details. The vector fields $v_\lambda$ used are all homotopic, so all normal bundles $Q'_\lambda$ of $\mathscr{F}'_\lambda$ are homotopic. We can identify the normal bundle $Q_\lambda$ of $\mathscr{F}_\lambda$ with a common bundle $Q \to M$. The bundle $Q$ is not trivial as there is an inclusion

$$\begin{array}{ccc} TS^q & \longrightarrow & Q \\ \downarrow & & \downarrow \\ S^q & \longrightarrow & M, \end{array}$$

so $Q$ has non-zero Euler class. But we note that the rational pontrjagin classes of $Q$ all vanish. To see this, observe that $Q'_\lambda \oplus \langle v_\lambda \rangle \cong p^*E$ where $p: E^* \to X$. The bundle $p^*E \to E^*$ is flat, so $Q \oplus \langle v_\lambda \rangle|_M$ is flat and hence has zero rational pontrjagin classes, which implies our claim.

The construction of $\{\mathscr{F}_\lambda\}$ for (2.3) proceeds as above, except we let $M$ be the $SO_q$-frame bundle of the normal bundle and lift the foliations to $M$. The normal bundle of the foliations of $M$ has a canonical trivialization $s$.

*Remark* 2.4. The set $V_q$ is defined as follows. First, let $q = 2k - 1$ for $k \geq 2$. Each Chern polynomial $c_j$ defined on $gl_q$ is the restriction of a Chern polynomial $\tilde{c}_j$ defined on $gl_{q+1}$. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ let $A_\lambda = \operatorname{diag}(\lambda_1, \lambda_1, \ldots, \lambda_k, \lambda_k) \in gl_{q+1}$. Define a rational function on $\mathbf{R}^k$:

$$\bar{c}_j(\lambda) = \frac{\tilde{c}_j(A_\lambda)}{\det A_\lambda}.$$

For the vector space spanned by the set of functions

$$\{\bar{c}_i \bar{c}_J | y_i c_J \in Z_q; \deg c_i c_J = 2q + 2\}$$

choose a basis $\{\bar{c}_{i_l} \bar{c}_{J_l} | 1 \leq l \leq d\}$. The set $V_q$ is the collection of admissible cocycles

$$V_q \equiv \{y_{i_l} c_{J_l} | 1 \leq l \leq d\}.$$

For $q = 2k$, we let $\lambda = (\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{R}^{k+1}$, set

$$A_\lambda = \operatorname{diag}(\lambda_1, \lambda_1, \ldots, \lambda_k, \lambda_k) \in gl_q,$$

and define

$$\bar{c}_j = \frac{c_j(A_\lambda)}{\det A_\lambda \cdot \lambda_{k+1}}.$$

For the span of the set

$$\{\bar{c}_i \bar{c}_J | y_i c_J \in Z_q;\ \deg c_i c_J = 2q + 2\},$$

choose a basis as before and set

$$V_q \equiv \{ y_{i_l} c_{J_l} | 1 \le l \le d \}.$$

We begin our study of $H^*(B\overline{\Gamma}_q)$ with:

THEOREM 2.5. *$k_*$: $H^5(W_2) \to H^5(B\overline{\Gamma}_2)$ is injective, and evaluation of*

$$\{k_*(y_1 c_1^2), k_*(y_1 c_2)\}$$

*defines an epimorphism of abelian groups*

$$\pi_5(B\overline{\Gamma}_2) \to \mathbf{R} \oplus \mathbf{R}.$$

*Proof.* Let $M$ and $f_\alpha$: $M \to B\Gamma_2^+$ be as in (2.1) so

$$f_\alpha^* \circ \tilde{\Delta}_*\{ y_1 c_1^2, y_1 c_2\} \subseteq H^5(M)$$

varies independently with $\alpha$. Since $H^1(BSO_2) = 0$ we can use Propositions 1.3 and 1.6 to construct a space $X'$, an isomorphism $\xi$: $H^5(X') \cong H^5(M)$ and, for all $\alpha \in \mathscr{A}$, maps $g_\alpha$: $X' \to B\Gamma_2^+$ so that

$$g_\alpha^* = \xi \circ f_\alpha^*: H^5(B\Gamma_2^+) \to H^5(X').$$

Further, $H_5(X'; Z) \cong Z$ contains a non-zero spherical class represented by $\beta$: $S^5 \to X'$. Because $\xi$ is fixed, it follows that

$$(g_\alpha \circ \beta)^* \circ \tilde{\Delta}_*\{ y_1 c_1^2, y_1 c_2\} \subseteq H^5(S^5)$$

varies independently with $\alpha$.

Now lift $g_\alpha \circ \beta$ to a map $h_\alpha\colon S^5 \to B\tilde{\Gamma}_2$, noting that $\pi_5(BSO_2) = 0$. From the diagram

$$\begin{array}{ccccc} H^5(WO_2) & \xrightarrow{\tilde{\Delta}_*} & H^5(B\Gamma_2) & \xrightarrow{(g_\alpha \circ \beta)^*} & \\ \downarrow \cong & & \downarrow i^* & & H^5(S^5) \\ H^5(W_2) & \xrightarrow{k_*} & H^5(B\bar{\Gamma}_2) & \xrightarrow{h_\alpha^*} & \end{array}$$

we see that $k_*\{y_1c_1^2, y_1c_2\}$ takes on all values in $\mathbb{R}^2$ when evaluated on the subgroup spanned by the classes $\{[h_\alpha] \mid \alpha \in \mathscr{A}\} \subseteq \pi_5(B\bar{\Gamma}_2)$. ■

The analysis of $B\bar{\Gamma}_q$ for $q > 2$ is based on the examples of Heitsch. For each $q > 2$ we defined a set $V_q \subseteq H^{2q+1}(W_q)$ in Remark 2.4. Now define two extensions of this set. For a number $x$, $[x]$ denotes the greatest integer $\leq x$.

DEFINITION 2.6.

$$\bar{V}_q \equiv \left\{ y_I c_J \in Z_q \mid y_{i_1} c_J \in V_q \text{ and } (i_2, i_3, \ldots, i_s) \subseteq \left(2, 4, \ldots, 2\left[\frac{q-1}{2}\right]\right)\right\},$$

$$\bar{V}_q^s \equiv \left\{ y_I c_J \in \bar{V}_q \mid i_s \leq \left[\frac{q+2}{2}\right]\right\}.$$

THEOREM 2.7. (a) $k_*\colon H^*(W_q) \to H^*(B\bar{\Gamma}_q)$ *is injective on the span of the set* $\bar{V}_q$ *and* $k_*(\bar{V}_q)$ *is a set of independently variable classes.*

(b) *Evaluation of the set* $k_*(\bar{V}_q^s)$ *on spherical classes defines an epimorphism of abelian groups* $\pi_*(B\bar{\Gamma}_q) \to \mathbf{R}^{\#\bar{V}_q^s}$.

If we set $\bar{V}_2 = \{y_1c_1^2, y_1c_2\}$ then (2.5) and (2.7) together imply Theorem 2 of the introduction.

*Remark* 2.8. For even codimensions this theorem gives the first results on the variation of secondary classes in $H^*(B\bar{\Gamma}_q)$. (The independence of certain secondary classes was previously shown in [1, 5, 13].) For example, when $q = 4$ the set $V_4$ is a basis of $H^9(W_4)$, and $\bar{V}_4^s = \bar{V}_4$ is the union of $V_4$ with a basis for $H^{12}(W_4)$, so (2.7) implies $k_*$ is injective in degrees 9 and 12.

*Remark* 2.9. For odd codimensions, (2.7) yields complementary results to those of Heitsch. For example, $V_3 = \{y_1c_1^3, y_1c_1c_2, y_2c_2\}$ is a basis for $H^7(W_3)$ and

$$\bar{V}_3 = \bar{V}_3^s = V_3 \cup \{y_1y_2c_1^3, y_1y_2c_1c_2\},$$

the latter set a basis for $H^{10}(W_3)$. Both of these classes in degree 10 vary by [Theorem 6.12; 13] but not necessarily independently. Using (2.7) and (6.12) we can say that the set

$$\{y_1c_1^3, y_1c_1c_2, y_2c_2, y_1y_2c_1^3, y_1y_2c_1c_2, y_1y_3c_1^3, y_1y_2y_3c_1^3\}$$

is mapped to an independently variable set by $k_*\colon H^*(W_3) \to H^*(B\overline{\Gamma}_3)$.

*Proof of* 2.7. For $q$ even, let $M$ and $f_\lambda\colon M \to B\Gamma_q^+$ be as in (2.2), so

$$f_\lambda^* \circ \tilde{\Delta}_*(V_q) \subseteq H^{2q+1}(M)$$

varies independently with $\lambda \in \mathscr{A}$. By the remark after (2.3), the map

$$(\nu \circ f_\lambda)^*\colon H^n(BSO_q) \to H^n(M)$$

is trivial for $n < q$, so we can use Propositions 1.3 and 1.6 to construct $X'$,

$$\xi\colon H^n(X') \cong H^n(M) \quad \text{for } n > q + 1$$

and maps $g_\lambda\colon X' \to B\Gamma_q^+$ so that $f_\lambda^* = \xi \circ g_\lambda^*$ in degrees $n > q + 1$. Further,

$$H_{2q+1}(X'; Z) \cong Z$$

contains a non-trivial spherical class represented by $\beta\colon S^{2q+1} \to X'$. The group $\pi_{2q+1}(BSO_q)$ is finite so replacing $\beta$ with a positive multiple if necessary, we can assume $\nu \circ g_\lambda \circ \beta$ is homotopic to a constant for all $\lambda$. Let

$$h_\lambda\colon S^{2q+1} \to B\overline{\Gamma}_q$$

denote a lift of $g_\lambda \circ \beta$. As in the proof of (2.5) we conclude that $k_*(V_q)$ is independently variable when evaluated on the subgroup spanned by the classes $\{[h_\lambda] | \lambda \in \mathscr{A}\} \subseteq \pi_{2q+1}(B\overline{\Gamma}_q)$.

For $q$ odd, let $M$ and $f_\lambda\colon M \to B\overline{\Gamma}_q$ be as in (2.3) so that

$$f_\lambda^* \circ k_*(V_q) \subseteq H^{2q+1}(M)$$

varies independently with $\lambda \in \mathscr{A}$. Let $X' = M/M_{q+1}$, the result of collapsing the $(q + 1)$-skeleton of $M$ to the basepoint. Each $f_\lambda$ induces a map $h_\lambda\colon X' \to B\overline{\Gamma}_q$. Noting that $H^{2q+1}(M) \cong H^{2q+1}(X')$, we see that $h_\lambda^* \circ k_*(V_q)$ varies independently with $\lambda$. Further, $X'$ is $q + 1$ connected, so by the rational Hurewicz theorem the cokernel of

$$\pi_{2q+1}(X') \to H_{2q+1}(X'; Z)$$

is finite. It follows that $k_*(V_q)$ is independently variable when evaluated on the subgroup spanned by the images for $\lambda \in \mathscr{A}$ of

$$(h_\lambda)_\#\colon \pi_{2q+1}(X') \to \pi_{2q+1}(B\overline{\Gamma}_q).$$

The above shows that for all $q$, $k_*(V_q) \subseteq H^{2q+1}(B\overline{\Gamma}_q)$ is independently variable when evaluated on spherical classes. To finish the proof, we invoke [15, Theorem 3.5] which implies that $k_*(\overline{V}_q)$ is an independently variable set. To see that $k_*(\overline{V}_q^s)$ is independently variable on spherical cycles, we note this follows from [14, Proposition 6.12] applied to the set $V_q$. ■

Theorem 2.7(b) yields much information about the homotopy groups of $B\overline{\Gamma}_q$. For each $q \geq 2$ let $\alpha_{q,n}$ be the number of elements in $\overline{V}_q^s$ of degree $n$. Define a wedge of spheres

$$Y_q = \bigvee_{n=2q+1}^{2q+q^2} \left( \bigvee_{j=1}^{\alpha_{q,n}} S^n \right).$$

Then

$$Y_2 = S^5 \vee S^5, \quad Y_3 = S^7 \vee S^7 \vee S^7 \vee S^{10} \vee S^{10}, \dots .$$

Theorems 2.5 and 2.7(b) imply that for each $\lambda \in \mathbf{R}^{\#\overline{V}_q^s}$ there is a map $F_\lambda\colon Y_q \to B\overline{\Gamma}_q$ so that $F_\lambda^* \circ k_*(\overline{V}_q^s)$ takes value $\lambda$ on a set of generators for $H_*(Y_q; Z)$. Therefore, for $\lambda \in \mathscr{A}$ the images of

$$(F_\lambda)_\#\colon H_*(Y_q; Z) \subseteq \pi_*(Y_q) \hookrightarrow \pi_*(B\overline{\Gamma}_q)$$

form a subgroup with uncountable basis.

DEFINITION 2.8. For $q \geq 2$, $n > 0$, define integers

$$v_{q,n} = \dim_{\mathbf{Q}} \pi_n(Y_q) \otimes \mathbf{Q}.$$

The properties of the $\{v_{q,n}\}$ given in the introduction result from calculating the rational homotopy groups of $Y_q$ as in [10, p. 518].

*Proof of Theorem* 1. The proof is a consequence of 2.7(b) and the theory of dual homotopy invariants in [14] to which the reader is referred for details. Briefly, there is a universal map

$$h^\#\colon \pi^*(I_q) \to \pi^*(B\Gamma_q^+) \equiv \operatorname{Hom}_Z(\pi_*(B\Gamma_q^+), \mathbf{R})$$

where $\pi^*(I_q)$ is the space of indecomposable elements of the minimal model for $I_q$. There is an exact sequence of graded vector spaces

$$0 \to \pi^*(W_q) \xrightarrow{\xi} \pi^*(I_q) \to \mathrm{Span}_{\mathbf{R}}\{c_1, \ldots, c_q\} \to 0$$

and $\pi^*(W_q)$ is infinite-dimensional for $q > 1$. In general, one knows [11] that the graded vector space $\pi^*(W_q)$ is the dual of the suspension of a free graded Lie algebra with basis corresponding to $Z_q$:

$$\pi^*(W_q) \cong \left(s\mathbf{L}s^{-1}Z_q^*\right)^*.$$

From this we obtain an inclusion

$$\zeta\colon H^*(W_q) \cong \mathrm{Span}_{\mathbf{R}} Z_q \subseteq \pi^*(W_q) \hookrightarrow \pi^*(I_q),$$

and in [14, Theorem 3.1] we show:

PROPOSITION 2.9. *There is a commutative diagram*:

$$\begin{array}{ccccc} H^*(W_q) & \xrightarrow{k_*} & H^*(B\overline{\Gamma}_q) & & \\ & & & \searrow^{\mathscr{H}^*} & \\ \Big\downarrow \zeta & & & & \pi^*(B\overline{\Gamma}_q) \\ & & & \nearrow_{i^\#} & \\ \pi(I_q) & \xrightarrow{h^\#} & \pi^*(B\Gamma_q^+). & & \end{array}$$

Given a set $V \subseteq Z_q$ we get a free Lie subalgebra $\mathscr{L} = \mathbf{L}s^{-1}V^* \subseteq \mathbf{L}s^{-1}Z_q^*$ which is an algebra summand, and hence there is an inclusion $\xi\colon (s\mathbf{L}s^{-1}V^*)^* \to \pi^*(I_q)$. For $V = \overline{V}_q^s$ we let $\mathscr{L}_q$ denote the resulting graded Lie algebra, and $\mathscr{V}_q$ be a basis of $\mathscr{L}_q$ over $\mathbf{R}$ chosen so that $\zeta(\overline{V}_q^s) \subseteq \xi(s\mathscr{L}_q^*)$.

When $q > 1$, $\mathscr{V}_q$ is an infinite set. In fact, it is easy to see that

$$\mathscr{L}_q \cong s^{-1}\pi_*(Y_q) \otimes \mathbf{R},$$

so the integer $v_{q,n}$ of (2.8) is just the number of elements in $\mathscr{V}_q$ of degree $(n-1)$.

Theorem 2.7(b) asserts that the set $\mathscr{H}^* \circ k_*(\overline{V}_q^s) \subseteq \pi^*(B\overline{\Gamma}_q)$ is independently variable when evaluated on the subgroup generated by the images of the maps $(F_\lambda)_\#\colon \pi_*(Y_q) \to \pi_*(B\overline{\Gamma}_q)$. By Proposition 2.9, this shows the set

$h^{\#} \circ \zeta(\overline{V}_q^s) \subseteq \pi^*(B\Gamma_q^+)$ is independently variable on the subgroup generated by the images $(i \circ F_\lambda)_\#\colon \pi_*(Y_q) \to \pi_*(B\Gamma_q^+)$. As a Lie algebra, $\mathscr{L}_q$ is generated by the set $s^{-1}(\overline{V}_q^s)^* \subseteq s^{-1}Z_q^*$. We can thus use [16, Proposition 5.13] to conclude that the entire set $h^{\#} \circ \hat{\zeta}(s\mathscr{V}_q^*)$ is independently variable when evaluated on the *Lie subalgebra* of $\pi_*(B\Gamma_q^+)$ generated by the images of the maps $(i \circ F_\lambda)_\#$. This is precisely the claim of Theorem 1. ■

## 3. Rigid classes

We show there are rigid secondary classes which are non-zero on spherical cycles and deduce some consequences about $B\Gamma_q^+$. There are just two basic types of rigid classes which have been shown to be non-trivial [15, Theorem 1] and then only for even codimension. In spite of this, one has enough data to construct an infinite family of dual homotopy classes in $\pi^*(B\Gamma_q^+)$ which are "rigid" for all even $q \geq 6$.

DEFINITION 3.1. For $q = 2k$ with $k$ even, set

$$R_q = \left\{ y_2 y_{2i_2} \cdots y_{2i_s} c_2^k \mid 1 < i_2 < \cdots < i_s \leq \left[\frac{k+1}{2}\right] \right\}.$$

For $q = 4m - 2$ with $m > 1$, set

$$R_q = \left\{ y_2 y_{2i_2} \cdots y_{2i_s} c_2^k \mid 1 < i_2 < \cdots < i_2 \leq m \right\} \cup \{ y_{2m} c_{2m} \}.$$

In either case, $R_q$ is a set of admissible cocycles which are rigid.

THEOREM 3.2. *Let $q \geq 4$ be even. Then $k_*$ is injective on the span of $R_q$, and the image $k_*(R_q) \subseteq H^*(B\overline{\Gamma}_q)$ takes independent values when evaluated on spherical classes.*

Let $\{z_1, \ldots, z_d\}$ be an enumeration of the elements of $R_q$, with $n_i = \deg z_i$. Define a space $T = \bigvee_{i=1}^d S^{n_i}$. A direct consequence of (3.2) is:

COROLLARY 3.3. *For all even $q \geq 4$, there exists a map $G\colon T \to B\overline{\Gamma}_q$ such that the composition*

$$\operatorname{Span}_{\mathbf{R}} R_q \xrightarrow{k_*} H^*(B\overline{\Gamma}_q) \xrightarrow{G^*} H^*(T)$$

*is an isomorphism in positive degrees.*

*Proof of* 3.2. Let $q = 2k \geq 4$. Recall the construction in [15] of a manifold $M$ and $f\colon M \to B\bar{\Gamma}_q$ with $f^* \circ k_*(y_2 c_2^k) \neq 0$: There is a map

$$\tilde{g}\colon S^2 \times S^2 \to BSO_2 \quad \text{with } \tilde{g}^*(p_1) \neq 0.$$

Since $B\bar{\Gamma}_2$ is 3-connected [26], we lift this to $g\colon S^2 \times S^2 \to B\Gamma_2^+$. Set

$$X = \overset{k}{\times} (S^2 \times S^2) \quad \text{and} \quad f = \overset{k}{\times} g\colon X \to B\Gamma_{2k}.$$

Then $f^*(p_1^k) \neq 0$. Define $M$ to be the principal $SO_q$-bundle over $X$ which is induced by $\nu \circ f$. There is a homotopy commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\bar{f}} & B\bar{\Gamma}_q & \longrightarrow & ESO_q \\ \downarrow & & \downarrow i & & \downarrow \\ X & \xrightarrow{f} & B\Gamma_q^+ & \xrightarrow{\nu} & BSO_q. \end{array}$$

The Hirsch Lemma yields $\bar{f}^* \circ k_*(y_2 c_2^k) \neq 0$.

The manifold $M$ is compact, so give $M$ a finite CW complex structure. Let $\hat{M} = M/M_{q+1}$ be the space obtained by collapsing the $(q+1)$-skeleton to the basepoint. By (1.1), there is an induced map

$$F\colon \hat{M} \to B\bar{\Gamma}_q \quad \text{with } F^* \circ k_*\big(y_2 c_2^k\big) \neq 0.$$

The space $\hat{M}$ is $(q+1)$-connected so the Rational Hurewicz theorem implies that

$$\pi_{2q+3}(\hat{M}) \otimes \mathbf{Q} \to H_{2q+3}(\hat{M}; \mathbf{Q})$$

is onto. Let $\alpha\colon S^{2q+3} \to M$ represent a spherical class such that $F^* \circ k_*(y_2 c_2^k)[\alpha] \neq 0$, so $k_*(y_2 c_2^k)$ is non-zero on the spherical class $[F \circ \alpha]$.

For $q = 4m - 2$ with $m > 1$, the class $k_*(y_{2m} c_{2m})$ was shown to be non-zero on a spherical class $\tilde{f}\colon S^{2q+3} \to B\bar{\Gamma}_q$ in [14, Corollary 6.4]. Further, it was remarked after (6.4) that $k_*(y_2 c_2^k)[\tilde{f}] = 0$. This shows that $k_*(y_2 c_2^k)$ and $k_*(y_{2m} c_{2m})$ are independent when evaluated on spherical cycles.

To finish the proof of (3.2) we invoke the homotopy permance results, Proposition 6.9 and Corollary 6.10 of [14]. In the notation of [14, p. 380], for $q = 2k$, $k$ even, set $\mathscr{L} = \{y_2 c_2^k\}$, and for $q = 4m - 2$ with $m > 1$ set $\mathscr{L} = \{y_2 c_2^{2m}, y_{2m} c_{2m}\}$. Then the conditions of (6.9) are satisfied. Observing that $R_q$ is the set $\mathscr{L}'$ of (6.9), Theorem 3.2 follows. ■

Let $\mathscr{F}$ be a codimension $q$ foliation on a manifold $M$. For each $l > 0$ the product $M \times \mathbf{R}^l$ has a codimension $q + l$ foliation whose tangential distribution is that of $\mathscr{F}$. On the level of classifying spaces, this produces a commutative diagram:

$$\begin{array}{ccc} B\overline{\Gamma}_q & \xrightarrow{\bar{j}_{q,l}} & B\overline{\Gamma}_{q+l} \\ \downarrow & & \downarrow \\ B\Gamma_q^+ & \xrightarrow{j_{q,l}} & B\Gamma_{q+l}^+ \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ BSO_q & \longrightarrow & BSO_{q+l}. \end{array}$$

We are interested in the map induced on homotopy groups $(\bar{j}_{q,l})_{\#}$. For each $n$, define a filtration of $\pi_n(B\overline{\Gamma}_q)$ by setting

$$K_{q,n}^l = \ker(\bar{j}_{q,l})_{\#}.$$

Then $0 = K_{q,n}^0 \subseteq K_{q,n}^1 \subseteq \cdots \subseteq K_{q,n}^{n-q-1} = \pi_n(B\overline{\Gamma}_q)$, and there is a simple geometric description of these subgroups: An element $\alpha$: $S^n \to B\overline{\Gamma}_q$ defines a smooth foliation $\mathscr{F}$ on $S^n \times \mathbf{R} \subseteq \mathbf{R}^{n+1}$, and $\alpha$ is in $K_{q,n}^l$ if the codimension $q + l$ foliation on $S^n \times \mathbf{R}^{l+1}$ is integrably homotopic to the product foliation on $S^n \times \mathbf{R}^{l+1} \subseteq \mathbf{R}^{n-q+1} \times \mathbf{R}^{q+l}$.

The group $K_{q,2q+1}^1$ is uncountably generated for $q \geq 3$. A real parameter family of maps $h_\lambda$: $S^{2q+1} \to B\overline{\Gamma}_q$ was constructed in §2 from the Heitsch foliations. All of the foliations of (2.2) and (2.3) yield the same codimension $(q + 1)$-foliation when extended by $\mathbf{R}$: the flat bundle on $E^*$ in §2. Thus, all of the maps $h_\lambda$ are sent to the same element by $(\bar{j}_{q,1})_{\#}$, hence $0 \neq [h_\lambda] - [h_{\lambda'}] \in K_{q,1}$ for $\lambda \neq \lambda'$.

Let $G$: $T \to B\overline{\Gamma}_{2k}$ be the map and space of Corollary 3.3. In contrast to the above we have:

PROPOSITION 3.4. *Let $q \geq 4$ be even. Then the composition*

$$\pi_*(T) \otimes \mathbf{Q} \xrightarrow{G_\#} \pi_*(B\overline{\Gamma}_q) \otimes \mathbf{Q} \xrightarrow{(\bar{j}_{q,1})_\#} \pi_*(B\overline{\Gamma}_{q+1}) \otimes \mathbf{Q}$$

*is monic.*

*Proof.* The rigid classes in $H^*(W_q)$ are precisely the image of $H^*(W_{q+1}) \to H^*(W_q)$, and we have a commutative diagram [18, Remark 4.75]:

$$\begin{array}{ccc} H^*(W_q) & \xrightarrow{k_*} & H^*(B\overline{\Gamma}_q) \\ \uparrow & & \uparrow (\bar{j}_{q,1})^* \\ H^*(W_{q+1}) & \xrightarrow{k_*} & H^*(B\overline{\Gamma}_{q+1}). \end{array}$$

Lift $R_q$ to a subset $\tilde{R}_q \subseteq H^*(W_{q+1})$. Then by (3.3) the composition

$$(\bar{j}_{q,1} \circ G)^* \circ k_*: H^*(W_{q+1}) \to H^*(T)$$

is injective on the span of $\tilde{R}_q$. Thus the set $k_*(\tilde{R}_q)$ restricts to a dual basis for image $(\bar{j}_{q,1} \circ G)_*$. Further, the product of any two cocycles representing the classes $k_*(\tilde{R}_q)$ vanish. By [11, Theorem §4], this implies that the rational map

$$(j_{q,1} \circ G)_{\#}: \pi_*(T) \otimes \mathbf{Q} \to \pi_*(B\overline{\Gamma}_{q+1}) \otimes \mathbf{Q}$$

is injective. ■

Recall that $R_q = \{z_1, \ldots, z_d\}$ where $\deg z_i = n_i$. The space $T$ is a bouquet of spheres, so

$$\pi_*(T) \otimes \mathbf{Q} \cong s\mathbf{L}_{\mathbf{Q}}\{s^{-1}z_1, \ldots, s^{-1}z_d\} \equiv s\mathscr{L}_R$$

where $\mathscr{L}_R$ is a free graded Lie algebra over $\mathbf{Q}$ with generators of degree $s^{-1}z_i = n_i - 1$. Define $\mathscr{L}_R(n)$ to be the subspace of homogeneous elements of degree $(n - 1)$.

DEFINITION 3.5. For even $q \geq 4$ and $n \geq 0$ set

$$r_{q,n} = \dim \pi_n(T) \otimes \mathbf{Q} = \dim \mathscr{L}_R(n).$$

For $q = 4$, $R_4 = \{y_2 c_2^2\}$ so $\mathscr{L}_R(11) \cong \mathbf{Q}$ and $T \cong S^{11}$.

For even $q \geq 6$, $\mathscr{L}_R$ has at least 2 algebra generators so $\mathscr{L}_R(n) \neq 0$ for an infinite number of $n > 2q + 2$. When $q \geq 10$, we have in addition that $r_{q,n}$ is positive for all $n \gg q$.

## 4. Existence of foliations

The space $B\Gamma_q$ was introduced by Haefliger [8], [9] to give a homotopy theoretic solution to the problem of classifying foliations. For an open manifold $M$, the Gromov-Phillips theorem [9] shows the homotopy classification yields a geometric classification. For compact manifolds, the realization theorem of Thurston [25] shows that we also obtain a geometric classification. Consequently, the calculations in Sections 2 and 3 of the homotopy groups and other properties of $B\Gamma_q$ have applications to the existence of distinct foliations on a given manifold.

There are three notions of equivalence between foliations which are important [20]. Let $\mathscr{F}_0, \mathscr{F}_1$ be codimension $q$ foliations on $M$.

DEFINITION 4.1. $\mathscr{F}_0, \mathscr{F}_1$ are *integrably homotopic* if there is a codimension $q$ foliation $\mathscr{F}$ on $M \times I$ with $\mathscr{F}$ transverse to each inclusion $M \times \{t\} \subseteq M \times I$, and $\mathscr{F}|_{M\times\{t\}} = \mathscr{F}_t$ for $t = 0, 1$.

DEFINITION 4.2. $\mathscr{F}_0, \mathscr{F}_1$ are *concordant* if there is a codimension $q$ foliation $\mathscr{F}$ on $M \times I$ with $\mathscr{F}|_{M\times\{t\}} = \mathscr{F}_t$ for $t = 0, 1$.

DEFINITION 4.3. $\mathscr{F}_0, \mathscr{F}_1$ are *homotopic* if there is a smooth 1-parameter family $\{\mathscr{F}_t: 0 \leq t \leq 1\}$ of codimension $q$ foliations on $M$ from $\mathscr{F}_0$ to $\mathscr{F}_1$.

For a compact manifold, $\mathscr{F}_0$ is integrably homotopic to $\mathscr{F}_1$ implies they are isotopic, while concordance is a weaker equivalence. Homotopy is the weakest notion of equivalence. Note that if $\mathscr{F}_0$ and $\mathscr{F}_1$ are homotopic, then there exists a codimension $q + 1$ foliation $\mathscr{F}$ on $M \times I$ which restricts to $\mathscr{F}_t$ on $M \times \{t\}$ for $t = 0, 1$.

*Question* 4.4 [20, Problem 3]. Does there exist a pair of foliations $\mathscr{F}_0, \mathscr{F}_1$ with tangential distributions $F_0, F_1 \subseteq TM$ homotopic as embedded subbundles but $\mathscr{F}_0$ and $\mathscr{F}_1$ not homotopic?

For instance, if $\mathscr{F}_0$ and $\mathscr{F}_1$ have dimension one then $F_0 \simeq F_1 \Leftrightarrow \mathscr{F}_0 \simeq \mathscr{F}_1$. However, when the codimension $q \geq 4$ is even and dimension $\geq 2q + 3$, we show the answer to (4.4) is yes.

In the following, $M$ is either a compact connected manifold without boundary or an open connected manifold. Let $\mathscr{C}_q(M)$ denote the set of concordance classes of codimension $q$ foliations on $M$. Recall that $\{v_{q,n}\}$ is the set of integers defined in (2.8).

THEOREM 4.5. *Let $M$ be a connected manifold and $Q \subseteq TM$ a trivial subbundle of rank $q$. Assume that either*

(a) *$M$ is closed, orientable and of dimension $n$,*

*or*

(b) *$M$ is open, has the homotopy type of an $n$-dimensional CW complex and $H^n(M) \neq 0$.*

*Then there is a set $\{\mathscr{F}_\alpha | \alpha \in \mathbf{R}^{v_{q,n}}\}$ of codimension $q$ foliations on $M$ with each* *normal bundle* $Q_\alpha$ *trivial and for* $\alpha \neq \beta$, $\mathcal{F}_\alpha$ *is not concordant to* $\mathcal{F}_\beta$. *If* $M$ *is closed then* $\mathcal{F}_\alpha$ *is not concordant to any orientation-preserving diffeomorph of* $\mathcal{F}_\beta$.

*In particular, there is an inclusion* $\mathbf{R}^{v_{q,n}} \subseteq \mathcal{C}_q(M)$.

The sphere $S^{4k+3}$ always admits a 3-frame field $Q \subseteq TS^{4k+3}$, so it follows from the properties of $\{v_{3,n}\}$ that for all $k > 0$ with $2k+1$ divisible by 3 the sphere $S^{4k+3}$ admits an uncountable number of distinct concordance classes of foliations (e.g., $S^7, S^{19}, S^{31}, \dots$).

It is remarked in [25, Corollary 2] that if $S^n$ admits a rank $q$ subbundle $Q \subseteq TS^n$ for $q < n/2$ then $S^n$ admits a trivial rank $q$ subbundle.

COROLLARY 4.6. *If* $S^n$ *admits a rank* $q$ *subbundle then* $S^n$ *has at least* $\mathbf{R}^{v_{q,n}}$ *distinct concordance classes of foliations.*

Recall that for $n \gg q \geq 3$, $v_{q,n} \neq 0$. Therefore (4.6) applies to many spheres. Further, in the proof of Theorem 4.5 we see that these foliations are distinguished by having differing dual homotopy invariants, so by Proposition 2.9 they often have non-trivial secondary invariants as well.

More generally, it is easy to construct compact manifolds with a trivial $q$-subbundle to which (4.5) applies. If $X$ is compact and almost parallelizable, take $M = X \times S^1$. For any orientable closed manifold $N$, $M = N \times T^q$ will work.

With regards to the weaker equivalence of homotopy we have the following. Let $q \geq 4$ be even. Recall that $\{r_{q,n}\}$ is the set of integers defined in (3.5).

THEOREM 4.7. *Assume* $M$ *satisfies the conditions of* (4.5) *and suppose* $r_{q,n} > 0$. *Then* $M$ *admits an infinite set* $\{\mathcal{F}_\alpha | \alpha \in Z^{r_{q,n}}\}$ *of codimension* $q$ *foliations with each normal bundle* $Q_\alpha$ *trivial,* $Q_\alpha$ *and* $Q_\beta$ *are concordant, and for* $\alpha \neq \beta$, $\mathcal{F}_\alpha$ *is not homotopic to* $\mathcal{F}_\beta$. *If* $M$ *is closed then* $\mathcal{F}_\alpha$ *is not diffeomorphic to* $\mathcal{F}_\beta$.

COROLLARY 4.8. *Let* $q \geq 4$ *be even and suppose* $r_{q,n} > 0$. *If* $S^n$ *admits a rank* $q$ *subbundle of* $TS^n$, *then* $S^n$ *admits an infinite set* $\{\mathcal{F}_\alpha | \alpha \in \mathcal{A}\}$ *of codimension* $q$ *foliations which are distinct up to homotopy, but whose tangential distributions are all homotopic as embedded subbundles.*

*Remark* 4.9. The simplest case is for codimension 6 on $S^{15}$.

*Proof of* 4.5. Given a trivial subbundle $Q \subseteq TM$ it is shown in [9], [25] that there is an inclusion $[M, B\overline{\Gamma}_q] \subseteq \mathcal{C}_q(M)$, where $[X, Y]$ denotes the set of homotopy classes of continuous maps of $X$ to $Y$ (not necessarily base point preserving!) We analyze the set $[M, B\overline{\Gamma}_q]$ when $M$ satisfies either (4.5a) or (4.5b). For $M$ closed and orientable, give $M$ a finite CW structure with exactly one $n$-cell $\{e^n\}$ and set $M' = M_{n-1}$, the $(n-1)$-skeleton of $M$. For $M$ open, replace $M$ with its homotopy equivalent CW complex of dimension $n$, which is again denoted by $M$. Let $\{e^n\}$ be an $n$-cell in $M$ corresponding to a generator of $H_n(M; \mathbf{Q})$ and let $M' = M - \{e^n\}$. In either case we obtain a map $\rho$: $M \to M/M' \cong S^n$ which is essential in homology. Consider the induced map on sets:

$$\rho' \colon \pi_n(B\overline{\Gamma}_q) = [S^n, B\overline{\Gamma}_q] \to [M, B\overline{\Gamma}_q]$$

We study the image of $\rho'$ to prove (4.5).

The key to the proof follows a suggestion made by Joe Neisendorfer to use the Barratt-Puppe sequence of $\alpha$: $S^{n-1} \cong \partial e^n \to M'$, the attaching map of the cell $\{e^n\}$. This is a homotopy right-exact sequence [27, Theorem 6.11]

$$S^{n-1} \xrightarrow{\alpha} M' \to M \xrightarrow{\rho} S^n \xrightarrow{\beta} \Sigma M' \to \tag{4.9}$$

where we set $\beta = \Sigma\alpha$. Exactness of (4.9) yields the exact sequence of sets:

$$[\Sigma M', B\overline{\Gamma}_q] \xrightarrow{\beta'} \pi_n(B\overline{\Gamma}_q) \xrightarrow{\rho'} [M, B\overline{\Gamma}_q]. \tag{4.10}$$

Note the first two sets have a natural group structure, and $[M, B\overline{\Gamma}_q]$ has a $\pi_n(B\overline{\Gamma}_q)$-action compatible with $\rho'$.

LEMMA 4.11. *The image of $\beta'$ is contained in the torsion subgroup of $\pi_n(B\overline{\Gamma}_q)$.*

*Proof.* It suffices to show that $\beta$: $S^n \to \Sigma M'$ is an element of finite order in $\pi_n(\Sigma M')$. For a suspension the rational Hurewicz map

$$\mathscr{H} \colon \pi_\alpha(\Sigma M) \otimes \mathbf{Q} \to H_*(\Sigma M; \mathbf{Q})$$

is onto. For the inclusion $i$: $\Sigma M \to (\Sigma M, \Sigma M')$, naturality of $\mathscr{H}$ gives a commutative diagram:

$$\begin{array}{ccc}
\pi_{n+1}(\Sigma M) \otimes \mathbf{Q} & \xrightarrow{\mathscr{H}} & H_{n+1}(\Sigma M; \mathbf{Q}) \to 0 \\
\Big\downarrow i_\# & & \Big\downarrow \\
\pi_{n+1}(\Sigma M, \Sigma M') \otimes \mathbf{Q} & \xrightarrow{\cong} & H_{n+1}(\Sigma M, \Sigma M'; \mathbf{Q}) \cong \mathbf{Q}.
\end{array}$$

Hence, $i_\#$ is onto. For the map of pairs $(a, \beta)$: $(e^{n+1}, S^n) \to (\Sigma M, \Sigma M')$ we have the diagram with exact rows:

$$\begin{array}{ccccc} 0 = \pi_{n+1}(e^{n+1}) & \longrightarrow & \pi_{n+1}(e^{n+1}, S^n) & \xrightarrow{\partial} & \pi_n(S^n) \\ \downarrow & & \downarrow & & \downarrow \beta_\# \\ \pi_{n+1}(\Sigma M) & \xrightarrow{i_\#} & \pi_{n+1}(\Sigma M, \Sigma M') & \xrightarrow{\partial} & \pi_n(\Sigma M'). \end{array}$$

Since $i_\#$ has finite cokernel, the composition $\beta_\# \circ \partial$ has image in a finite subgroup of $\pi_n(\Sigma M')$, and $\beta_\#(1) = [\beta]$ therefore has finite order. ■

By Theorem 1, there is a group epimorphism $h$: $\pi_n(B\overline{\Gamma}_q) \to \mathbf{R}^{v_{q,n}}$. The composition $h \circ \beta'$ is trivial as $\mathbf{R}$ is torsion-free, so $h$ defines a map on the image of $\rho'$:

$$\bar{h}\colon \{\text{image}\,\rho'\} \to \mathbf{R}^{v_{q,n}}.$$

This shows the set $[M, B\overline{\Gamma}_q]$ contains at least $\mathbf{R}^{v_{q,n}}$ distinct elements.

To finish the proof of (4.5), for each $x \in \mathbf{R}^{v_{q,n}}$ chose a map

$$\alpha_x\colon S^n \to B\overline{\Gamma}_q \quad \text{with } h(\alpha_x) = x.$$

Set $f_x = \alpha_x \circ \rho$: $M \to B\overline{\Gamma}_q$ and let $\mathscr{F}_x$ be the foliation on $M$ with normal bundle concordant to $Q$ and classified by $f_x$ which the various existence theorems construct. For $x \neq y$ we saw above that $f_x \neq f_y$, so the foliations $\mathscr{F}_x$ and $\mathscr{F}_y$ are not concordant.

Suppose $M$ is closed and $g$ is an orientation-preserving diffeomorphism of $M$ with $g^{-1}(\mathscr{F}_y)$ concordant to $\mathscr{F}_x$. Then $f_x$ and $f_y \circ g$ are homotopic. We can assume $g$ is a CW-map; the quotient map $\bar{g}$: $M/M' \to M/M'$ is homotopic to the identity since $\bar{g}$ is orientation-preserving. Therefore, $\rho \simeq \rho \circ g$ and

$$f_y \equiv \alpha_y \circ \rho \simeq \alpha_y \circ \rho \circ g = f_y \circ g \simeq f_x$$

so $x = y$. ■

*Proof of* 4.7. Given $M$ satisfying (4.5a, b) let $\rho$: $M \to S^n$ and $\beta$: $S^n \to \Sigma M'$ be as in the proof of (4.5). We again consider the map

$$\rho'\colon \pi_n(B\overline{\Gamma}_q) \to [M, B\overline{\Gamma}_q].$$

The point of (4.7) is to produce foliations on $M$ which are not homotopic and this is based on the following observation.

Let $\alpha_1, \alpha_2 \in \pi_n(B\overline{\Gamma}_q)$, set $f_i = \alpha_i \circ \rho$ and let $\mathscr{F}_i$ be the resulting foliation on $M$. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are homotopic, then there exists a codimension $q + 1$

foliation $\mathscr{F}$ on $M \times I$ which restricts to $\mathscr{F}_i$ on $M \times \{i\}$. Let

$$f\colon M \times I \to B\Gamma_{q+1}$$

classify $\mathscr{F}$. Then $f$ is a homotopy between the maps $j_{q,1} \circ f_i\colon M \to B\Gamma_{q+1}$ for $i = 1, 2$. For $\eta\colon [M, B\bar{\Gamma}_q] \to [M, B\Gamma_{q+1}]$, this implies

$$\eta \circ \rho'([\alpha_1] - [\alpha_2]) = 0.$$

From the commutative diagram

$$\begin{array}{ccc}
\pi_n(B\bar{\Gamma}_q) & \xrightarrow{\rho'} & [M, B\bar{\Gamma}_q] \\
\downarrow {\scriptstyle (\bar{j}_{q,1})_\#} & & \\
\pi_n(B\bar{\Gamma}_{q+1}) & & \downarrow {\scriptstyle \eta} \\
\downarrow & & \\
\pi_n(B\Gamma_{q+1}) & \xrightarrow{\rho''} & [M, B\Gamma_{q+1}],
\end{array}$$

we see that $\rho'' \circ (\bar{j}_{q,1})_\#([\alpha_1] - [\alpha_2]) = 0$. The kernel of $\rho''$ is a torsion subgroup by Lemma 4.10, hence

$$(\bar{j}_{q,1})_\#[\alpha_1] = (\bar{j}_{q,1})_\#[\alpha_2] \in \pi_n(B\bar{\Gamma}_{q+1}) \otimes \mathbf{Q} \cong \pi_n(B\Gamma_{q+1}) \otimes \mathbf{Q}.$$

Therefore, if $\alpha_1, \alpha_2$ are chosen so that $(\bar{j}_{q,1})_\#[\alpha_i]$ are rationally distinct then $\mathscr{F}_1$ and $\mathscr{F}_2$ are not homotopic.

With the above remarks, it is clear how to proceed. Let $G\colon T \to B\bar{\Gamma}_q$ be as in (3.4). Choose a set $\{\alpha_1, \dots, \alpha_r\} \subseteq \pi_n(T)$ which yields a $Q$-basis for $\pi_n(T) \otimes \mathbf{Q} \cong \mathscr{L}_q(n)$. (So $r = r_{q,n}$). Let $A_n \subseteq \pi_n(T)$ be the free abelian group generated by the $\alpha_i$. Proposition 3.4 implies that

$$(\bar{j}_{q,1})_\# \circ G_\#\colon \pi_n(T) \to \pi_n(B\Gamma_{q+1}) \otimes \mathbf{Q}$$

is monic on the set $A_n$. For each $\alpha \in A_n$ let $\mathscr{F}_\alpha$ be the foliation on $M$ classified by $G \circ \alpha \circ \rho\colon M \to B\bar{\Gamma}_q$. The set $\{\mathscr{F}_\alpha | \alpha \in A_n\}$ is then seen to satisfy the conditions of (4.7). ∎

*Proof of* 4.8. For $M = S^n$, let $\{\mathscr{F}_\alpha | \alpha \in A_n\}$ be the set of codimension-$q$ foliations constructed above. Note that for all $\alpha \in A_n$ the normal bundle $Q_\alpha$ of $\mathscr{F}_\alpha$ is concordant to the given embedding $Q \subseteq TS^n$ of a trivial bundle [25, p. 217]. This implies $Q_\alpha$ and $Q_\beta$ are concordant for all $\alpha, \beta \in A_n$. Let

$$G^\alpha = (g_1^\alpha, g_2^\alpha)\colon S^n \to BO_q \times BO_{n-q}$$

classify the splitting $TS^n \cong Q_\alpha \oplus F_\alpha$ where $F_\alpha$ is the tangential distribution of $\mathscr{F}_\alpha$. For all $\alpha_0, \alpha_1 \in A_n$ we then obtain a commutative diagram:

$$\begin{array}{ccc} & & S^{n-q} \\ & & \downarrow \\ S^n \times \{0,1\} & \xrightarrow{G^0 \cup G^1} & BO_q \times BO_{n-q} \\ \downarrow & & \downarrow \\ S^n \times I & \longrightarrow & BO_q \times BO_{n-q+1}. \end{array}$$

Therefore, the homotopy classes of $G^0$ and $G^1$ differ by an element of $\pi_n(S^{n-q})$. This is a finite group for $q$ even, so we can find an infinite subset $\mathscr{A} \subseteq A_n$ so that $\alpha, \beta \in \mathscr{A}$ implies $G^\alpha \simeq G^\beta$. But this means $F_\alpha$ and $F_\beta$ are homotopic as embedded subbundles of $TS^n$, while $\mathscr{F}_\alpha$ and $\mathscr{F}_\beta$ are not homotopic for $\alpha \neq \beta$. ∎

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