ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1985

CONVEXITY AND CYLINDRICAL TWO-PIECE PROPERTIES

BY

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Let $f\colon M \to \mathbf{R}^n$ be a smooth immersion of a compact manifold. In particular we say that f is trivial if M is diffeomorphic to $\mathbf{S}^{n-1}$ and f embeds M as a round hypersphere.

The idea of k-cylindrical tautness and the related k-cylindrical weak and strong two-piece properties were discussed in [2]. It was shown that the weak $(n-2)$-cylindrical two-piece property is sufficient to imply that f is trivial. It was also shown that the weak 1-cylindrical two-piece property implies that f embeds $\mathbf{S}^{n-1}$ as a tight hypersphere and the comment was made that if f is 1-cylindrically taut then it is trivial. This fact is proved here.

We also consider the case $k = 2$. We show that the weak and strong versions of the two-piece property are distinct by giving an embedding of $\mathbf{S}^1 \times \mathbf{S}^{n-2}$ in $\mathbf{R}^n$ which has the weak 2-cylindrical two-piece property and by showing that if f has the strong version and $\dim M = n - 1$ then f embeds $\mathbf{S}^{n-1}$ as a tight hypersphere. We also prove that f is trivial if it is 2-cylindrically taut and $\dim M = n - 1$. There remains the possibility of nontrivial 2-cylindrically taut immersions of codimension 2 which must have very restrictive curvature properties.

To prove these results we need some theorems about convex sets which seem of interest in themselves.

1. Preliminary notations and results

Throughout this paper M will be a smooth, compact, connected m-dimensional manifold without boundary and $f\colon M \to \mathbf{R}^n$ will be a smooth immersion into n-dimensional Euclidean space. If $\Pi \subset \mathbf{R}^n$ is a k-plane, not necessarily through the origin, we define the solid k-cylinder with axis the k-plane Π and radius $r > 0$ to be the set $C = \{x \in \mathbf{R}^n\colon d(x, \Pi) \le r\}$ where $d(x, \Pi)$ is the Euclidean distance from x to Π. We write $\tilde{C}$ for the closure of $\mathbf{R}^n \setminus C$. Let us repeat for reference the definitions given in [2].

DEFINITION 1.1. The immersion $f\colon M \to \mathbf{R}^n$ is *k-cylindrically taut* if there exists some field $\mathbf{F}$ such that, for all solid k-cylinders C with axis Π, inclusion

Received September 14, 1982.

induces monomorphisms

$$H_i(f^{-1}(C), f^{-1}(\Pi)) \to H_i(M, f^{-1}(\Pi))$$

for all $i \in \mathbf{Z}^+$ where Čech homology is taken with coefficients in $\mathbf{F}$.

DEFINITION 1.2. We say that an immersion f: $M \to \mathbf{R}^n$ has the *k-cylindrical weak two-piece property* (k-cylindrical WTPP) if for all solid k-cylinders C with axis Π, $f^{-1}(\tilde{C})$ is connected, and, if $f^{-1}(\Pi) = \emptyset$, $f^{-1}(C)$ is connected.

We say it has the *strong two-piece property* (k-cylindrical STPP) if in addition when $f^{-1}(\Pi) \neq \emptyset$ every component of $f^{-1}(C)$ intersects $f^{-1}(\Pi)$.

In previous work it has been proved that k-cylindrically taut immersions are tight and satisfy the above two conditions [2].

Definitions (1.1) and (1.2) can be reinterpreted in terms of the number of critical points of cylindrical functions where the k-cylindrical function C_Π: $M \to \mathbf{R}^+$ is defined by setting $C_\Pi(p)$ to be the square of the Euclidean distance from $f(p)$ to Π. This interpretation is usually more convenient when discussing actual examples. In this paper we will only need the interpretation of the cylindrical 2-piece properties as discussed in [4]. First it is easy to see that the set of solid k-cylinders C with axis Π such that both ∂C and Π are transversal to f is an open dense set in the set of all k-cylinders (with the obvious topology). Further, if one of the conditions holds for all these k-cylinders then it holds for the rest of them. These cylinders correspond to k-cylindrical functions which have only a finite number of critical points outside $f^{-1}(\Pi)$. The k-cylindrical WTPP just says that such cylindrical functions have one maximum and, if $f^{-1}(\Pi)$ is empty, one minimum. The k-cylindrical STPP requires, in addition, that if $f^{-1}(\Pi)$ is not empty there are no minimum points outside it.

In the rest of this paper we will suppose that f is substantial, that is, $f(M)$ does not lie in any hyperplane of $\mathbf{R}^n$. The reason we make this assumption lies in the following proposition.

PROPOSITION 1.3. (a) *Let f: $M \to \mathbf{R}^n$ be a k-cylindrically taut immersion and suppose that there is some $(n-r)$-plane $H \subset \mathbf{R}^n$ such that $f(M) \subset H$ and $r \leq n-k$. Let ϕ: $H \to \mathbf{R}^{n-r}$ be an isometry. Then the immersion $\phi \circ f$: $M \to \mathbf{R}^{n-r}$ is l-cylindrically taut for any $l \geq 0$ such that $k-r \leq l \leq k$.*

(b) *The same result holds if cylindrical tautness is replaced throughout by the appropriate cylindrical WTPP or STPP.*

Proof. We only have to observe that with the dimensions in the theorem any solid l-cylinder in H can be represented as $C \cap H$ for some solid k-cylinder in $\mathbf{R}^n$.

The particular case we use in this paper is when $k = 2$ and $r = 1$. If f is not substantial and has one of the 2-cylindrical properties we obtain an immersion with the appropriate 1-cylindrical property which gives us much more information. In particular the 1-cylindrical WTPP is enough to show that the immersion is substantial. In fact we get:

COROLLARY 1.4. *Let $f\colon M \to \mathbf{R}^n$ have the 2-cylindrical WTPP and suppose f is not substantial. Then M is homeomorphic to $\mathbf{S}^{n-2}$ and f is tight.*

Proof. This follows directly from Proposition 1.3 above with $k = 2$, $r = 1$, $l = 1$ and Theorem 3.4 of [2].

Actually we will obtain a stronger result in this paper which will show that if f is 2-cylindrically taut but not substantial then f is a trivial embedding of $\mathbf{S}^{n-2}$ in a hyperplane.

From now on $f\colon M \to \mathbf{R}^n$ will always be a substantial immersion.

2. Convex hulls and convex envelopes

Let $A \subset \mathbf{R}^n$ be any bounded subset. Then $\mathscr{H}A$ will denote the closed convex hull of A. This may lie in a k-plane, say $\mathscr{H}A \subset \Pi \subset \mathbf{R}^n$; if k is minimal the convex envelope $\partial\mathscr{H}A$ is the boundary of $\mathscr{H}A$ as a subset of Π. We will need a few results about convex hulls and convex envelopes which we will collect together here.

THEOREM 2.1. *Let E be an open subset of $\mathbf{R}^n$ which is the disjoint union of the sets $\{E_\lambda\colon \lambda \in \Lambda\}$, $E_\lambda \cap E_\mu = \emptyset$ if $\lambda \neq \mu$. Suppose each E_λ is convex, has non-empty interior and is closed in E. Then E_λ is also open for each $\lambda \in \Lambda$.*

Proof. Take any $x \in E_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. We can find an open ball $B \subset E$ with centre at x. Take any straight line through x and let L be its intercept with B. Then L is homeomorphic to $\mathbf{R}$. For any $\lambda \in \Lambda$, $E_\lambda \cap L$ is convex and closed in L. Thus it is an interval, or a singleton set or is empty. In any case we can talk about the end-points of $E_\lambda \cap L$, it will have at most two. Let C be the collection of all end-points as λ runs over Λ. We will show that if C is not empty it is a perfect set. Suppose then that it is not empty.

First observe that $L \setminus C$ is the union of the interiors (as subsets of L) of the intervals $E_\lambda \cap L$, $\lambda \in \Lambda$. So C is closed in L. Now take any $u \in C$ then u is the end-point of some interval, $E_\lambda \cap L$, or it could be that $\{u\} = E_\lambda \cap L$. In any case, any open interval I about u contains points outside $E_\lambda \cap L$. But if $v \in I \cap E_\mu \cap L$, $\mu \neq \lambda$ then since $E_\mu \cap L$ is an interval and $u \notin E_\mu \cap L$, $v \in E_\mu \cap L$ we must have an end-point of $E_\mu \cap L$ between u and v. This end-point will lie in I since I is an interval, so $I \cap C \neq \{u\}$. That is, u is not an isolated point of C. Thus C is a perfect set and therefore it is uncountable. However since $\mathbf{R}^n$ and hence E is second-countable it only admits a countable number of disjoint open sets. Since each E_λ has non-empty interior this implies that Λ is countable. Since each E_λ contains at most two points in C this implies that C is countable. This is a contradiction.

We deduce that C is empty and thus since $x \in E_{\lambda_0} \cap L$ $L = E_{\lambda_0} \cap L$. Now if $x \in E_{\lambda_0}$ this means $L \subset E_{\lambda_0}$ and since λ_0 is fixed and L is arbitrary this means that $B \subset E_{\lambda_0}$ and hence E_{λ_0} is open.

Our next theorem depends on the idea of a *supporting flag*. This is essentially the same idea as that of a top^k-set introduced by Kuiper [3]. A flag is a sequence of planes $(H_{n-k} \subset \cdots \subset H_{n-2} \subset H_{n-1})$ such that for each $l = n-k, \ldots, n-2, n-1$, H_l is a hyperplane in H_{l+1}. In particular H_{n-1} is a hyperplane in $\mathbf{R}^n$. It supports A if for each l, H_l is a supporting hyperplane to $H_{l+1} \cap A$ in H_{l+1}. At the moment we only require the case $k = 2$.

THEOREM 2.2. *Let $A \subset \mathbf{R}^n$ be compact and suppose that for every flag $(H_{n-2} \subset H_{n-1})$ which supports A,*

$$H_{n-1} \cap A \subset H_{n-2} \Rightarrow H_{n-1} \cap A \text{ is convex.}$$

Then for any $x \in \mathscr{H}A \setminus A$ there is a unique hyperplane H through x which supports A. Further, $\partial\mathscr{H}(A \cap H)$ is homeomorphic to $\mathbf{S}^{n-2}$ and belongs to A.

Proof. First take any $x \in \partial\mathscr{H}A$ with $x \notin A$. Then there is a supporting hyperplane H_x to A at this point. $H_x \cap A$ cannot lie in any $(n-2)$-plane, otherwise we would be able to find a flag $(H' \subset H_x)$ which supports A with $H_x \cap A \subset H'$ and this would mean that

$$H' \cap A = \mathscr{H}(H' \cap A).$$

Since $H_x \cap \mathscr{H}A = \mathscr{H}(H_x \cap A)$ this would imply that

$$x \in H_x \cap \mathscr{H}A = \mathscr{H}(H' \cap A) = H' \cap A$$

which contradicts the choice of $x \notin A$. Also H_x must be unique otherwise we could again find a flag $(H_{n-2} \subset H_{n-1})$ supporting A with $H_{n-1} \cap A \subset H_{n-2}$ and $x \in H_{n-2}$. By the same argument this would contradict the choice of $x \notin A$. Thus for any $x \in \partial\mathscr{H}A$, $x \notin A$ there is a unique supporting hyperplane H_x and $H_x \cap \mathscr{H}A$ is closed, convex and has non-empty interior as a subset of H_x. We fix on some point, which we may as well take as the origin in $\mathbf{R}^n$, lying in the interior of $\mathscr{H}A$. We may assume that such a point exists otherwise A would lie in a hyperplane and no element in $\partial\mathscr{H}A$ would have a unique supporting hyperplane and hence, by the above, $\partial\mathscr{H}A \subset A$. We then let C_x denote the cone with vertex at the origin and with cross-section

$H_x \cap \mathscr{H}A$. That is

$$C_x = \{tz\colon z \in H_x \cap \mathscr{H}A,\ t \in \mathbf{R}^+\}.$$

Clearly such a cone is closed, convex and has non-empty interior in $\mathbf{R}^n$. Also if $y \in \partial\mathscr{H}A$, $y \notin A$, either $H_x = H_y$ and hence $C_x = C_y$ or $H_x \cap H_y \cap \mathscr{H}A \subset A$. So if we let

$$D = \{tz\colon z \in A \cap \partial\mathscr{H}A,\ t \in \mathbf{R}^+\}$$

either $C_x = C_y$ or $C_x \cap C_y \subset D$. Thus if we put $E = \mathbf{R}^n \setminus D$ and let $\{E_\lambda\colon \lambda \in \Lambda\}$ be the collection of sets $\{E \cap C_x\colon x \in \partial\mathscr{H}A \setminus A\}$ relabelled so that $E_\lambda \neq E_\mu$ if $\lambda \neq \mu$ we see that E is an open set in $\mathbf{R}^n$ which is the disjoint union of the sets $\{E_\lambda\colon \lambda \in \Lambda\}$ where each E_λ is closed, convex and has non-empty interior in E. Hence, from (2.1) every E_λ is also open.

Thus if H is a supporting hyperplane of A and $x \in \mathscr{H}A \cap H$, $x \notin A$, then $C_x \setminus D \in \{E_\lambda\colon \lambda \in \Lambda\}$ and so x lies in the interior of C_x. This implies that x lies in the interior of $\mathscr{H}A \cap H$ as a subset of H and, in particular, $x \notin \partial\mathscr{H}(A \cap H)$. This proves that $\partial\mathscr{H}(A \cap H) \subset A$.

3. 2-cylinder two-piece property

THEOREM 3.1. *Let $f\colon M \to \mathbf{R}^n$ be a substantial immersion with the 2-cylindrical WTPP, $n \geq 4$. Let $S = \partial\mathscr{H}f(M)$. Then either $S \subset f(M)$, so $\dim M = n - 1$, or, for every $x \in S \setminus f(M)$ there is a unique hyperplane H through x which supports S and $\partial(H \cap S) \subset f(M)$, where $\partial(H \cap S)$ is homeomorphic to $\mathbf{S}^{n-2}$, so $\dim M = n - 1$ or $n - 2$.*

Proof. Let $(H_{n-2} \subset H_{n-1})$ be any flag which supports S. Suppose

$$H_{n-1} \cap f(M) \subset H_{n-2}.$$

Then we claim that $H_{n-2} \cap f(M)$ must be convex because otherwise we could find a line in H_{n-2} which intersected $f(M)$ in a disconnected set. Then we could find a 2-plane $\Pi \subset H_{n-1}$ which intersected H_{n-2} in the line and hence also intersected $f(M)$ in a disconnected set. Then we could find a solid 2-cylinder C with axis parallel to Π, which did not intersect S, or $f(M)$ and such that $C \cap f(M) = \Pi \cap f(M)$ is not connected. This contradicts (1.2). Thus $f(M)$ satisfies the conditions in (2.2).

It is not difficult to prove that if $f(M)$ contains a homeomorphic image of $\mathbf{S}^r$ then $\dim M \geq r$. Thus, observing that since f is substantial S is a homeomorphic image of $\mathbf{S}^{n-1}$, and applying Theorem (2.2) with $f(M) = A$, $S = \mathscr{H}A$ we obtain the required conclusion.

THEOREM 3.2. *Let $f\colon M \to \mathbf{R}^n$ be an immersion with the 2-cylindrical STPP, $n \geq 4$ and $\dim M = n - 1$. Then M is diffeomorphic to $\mathbf{S}^{n-1}$ and f is a tight embedding.*

Proof. We can find a nondegenerate height function H_z. If H_z has only critical points of index 0 or $n-1$ then M is diffeomorphic to $\mathbf{S}^{n-1}$ and there is nothing more to prove. So suppose $p \in M$ is a critical point of H_z with index not 0 or $n-1$. Then $y_0 = f(p) \notin S$ and if N is the normal line corresponding to p and passing through $y_0 = f(p)$, we know that all the focal points of f with base point p lie on N (there are none at ∞). The point y_0 on N separates N into two parts and if $n \geq 4$ there must be at least one part which contains two focal points. Take a point y_1 on the other part so that there are no focal points between y_0 and y_1 and at least two on the other side of y_0 to y_1. These two focal points have corresponding directions of curvature which determine a 2-plane Π' in the tangent space to M at p. Let $\Pi \subset \mathbf{R}^n$ be a 2-plane through y_1 parallel to Π', or more exactly, parallel to $df(\Pi')$. Let C be the solid 2-cylinder with axis Π and radius $\|y_0 - y_1\|$. We claim that y_0 is an isolated point in $C \cap f(M)$ or, equivalently, p is a nondegenerate critical point of index 0 for the 2-cylindrical function C_Π.

In fact, we can choose co-ordinates in $\mathbf{R}^n$ so that y_0 becomes the origin and locally f is given by $u \to (u, g(u))$ where $u \in \mathbf{R}^{n-1}$, $g\colon \mathbf{R}^{n-1} \to \mathbf{R}$. We can take $dg_0 = 0$ and by choosing the directions of curvature to be along the axes arrange that the quadratic d^2g_0 at the origin is represented by the diagonal matrix

$$\operatorname{diag}(k_1, k_2, \ldots, k_{n-1})$$

where $k_1, k_2, \ldots, k_{n-1}$ are the curvatures corresponding to the directions of the axes. If Π is given by $x_3 = \cdots = x_{n-1} = 0$, $x_n = -d$ then by hypothesis k_1, k_2 are positive, all the curvatures are non-zero and greater than $-1/d$. The 2-cylindrical function C_Π is then given by $u_3^2 + \cdots + u_{n-1}^2 + (g(u) + d)^2$ and so the hessian of C_Π is represented by the diagonal matrix

$$2\operatorname{diag}(k_1 d, k_2 d, 1 + k_3 d, 1 + k_4 d, \ldots, 1 + k_{n-1} d).$$

All these diagonal terms are non-zero and positive, so p is a nondegenerate critical point of index 0.

However this means that $f^{-1}(y_0)$ is a component of $f^{-1}(C)$ which doesn't intersect $f^{-1}(\Pi)$. Let us show that $f^{-1}(\Pi)$ is non-empty. Since y_0 belongs to the interior of $\mathscr{H}S$ certainly $\Pi \cap S$ is non-empty, but if $x \in \Pi \cap S$, $x \notin f(M)$, we can find a hyperplane H through x supporting S and $\partial(H \cap S) \subset f(M)$ so x belongs to the interior of the convex set $H \cap C$ as a subset of H. Then $\Pi \cap H$ is a line through x and this must intersect $\partial(H \cap S) \subset f(M)$. Hence Π intersects $f(M)$. Thus C is a solid 2-cylinder with axis Π which intersects

$f(M)$, yet $f^{-1}(C)$ has a component which does not intersect $f^{-1}(\Pi)$. This contradicts (1.2).

We deduce that $f(M) \subset S$. Since $\dim M = \dim S = n - 1$ we can then use the theorem on invariance of domain to deduce that $f(M)$ is open and closed in S. Hence $f(M) = S$ and S must be smooth. Since f is a local diffeomorphism and M is compact, f is a covering map and so, in fact, a diffeomorphism. Thus f is a tight embedding of a hypersphere.

The method of (3.2) shows that if $\dim M = n - 2$ and f has the 2-cylindrical STPP then any point $p \in M$ for which $f(p) \notin S$ must have the following property. If N is a line through $f(p)$ which is normal at p then there are at most two focal points on N, the others are at infinity. Further they have multiplicity 1 and are separated by $f(p)$. This is a very stringent restriction on the curvature. One can get restrictions on the curvature for points p with $f(p) \in S$ by the same method. However this method definitely requires the strong version of the two-piece property.

Let us give an example to show that (3.2) is no longer true if we only require that f has the 2-cylindrical WTPP. We first consider $\mathbf{S}^{n-2}$ embedded as a round sphere in a hyperplane of $\mathbf{R}^n$, $n \geq 5$. Let Π be any 2-plane. Then the critical points of the 2-cylindrical function C_Π are the end-points of mutual normals between $\mathbf{S}^{n-2}$ and Π. Let N be a line normal to $\mathbf{S}^{n-2}$ at p and normal to Π at x. Then if l is the axis of $\mathbf{S}^{n-2}$, that is, the line through the centre of $\mathbf{S}^{n-2}$ and perpendicular to the hyperplane containing $\mathbf{S}^{n-2}$, l and N must intersect at a focal point z of multiplicity $n - 2$. In this case p will be a critical point of C_Π with index $n - 2$ if x lies between p and z; index 1 or 2 if z lies between x and p, and index 0 if p lies between x and z.

We want to show that if Π does not contain l and does not touch S, then C_Π has only one critical point of index 0 and one of index $n - 2$ (plus some of index 1 or 2 maybe).

Suppose C_Π has two critical points p_1, p_2 either both of index 0 or both of index $n - 2$. Thus there are normal lines N_1, N_2 through p_1, p_2 intersecting l at z_1, z_2 and intersecting Π perpendicularly at x_1, x_2. We really only need to consider the 3-plane which contains l, N_1 and N_2. The line joining x_1 to x_2 lies in Π and so is perpendicular to both N_1 and N_2. Hence $\|x_1 - x_2\|$ is the distance between N_1 and N_2. Now it is easy to check that this is impossible if, either p_i lies between x_i and z_i for $i = 1, 2$ (in this case it is easy to see that $\|p_1 - p_2\| < \|x_1 - x_2\|$), or if x_i lies between p_i and z_i for $i = 1, 2$. In fact we can also see that if $x_1 = p_1$ then by the same argument $\|x_1 - x_2\|$ is the distance from $x_1 = p_1$ to N_2 and if p_2 lies between x_2 and z_2 this is impossible since $\|x_1 - x_2\| > \|p_1 - p_2\|$. Thus C_Π has only one maximum point and only one minimum point unless Π intersects $\mathbf{S}^{n-2}$ and in the last case all the minimum points lie on Π. In other words $\mathbf{S}^{n-2}$ has the 2-cylindrical STPP.

Now let M be a round tube about $\mathbf{S}^{n-2}$. More specifically M is the boundary of an ε-neighbourhood of $\mathbf{S}^{n-2}$ for ε sufficiently small. Thus M is a smooth embedding of $\mathbf{S}^{n-2} \times \mathbf{S}^1$ as a hypersurface in $\mathbf{R}^n$. We consider a 2-plane Π but we want to consider both the corresponding cylindrical function C_Π on M and the corresponding cylindrical function on $\mathbf{S}^{n-2}$ which we will call C_Π^*. The critical points of C_Π and C_Π^* are closely related. In fact if q_1 is a critical point of C_Π then the normal to M through q_1 is a line N which intersects l at a point z, intersects $\mathbf{S}^{n-2}$ at a point p and is a normal there to $\mathbf{S}^{n-2}$, intersects M again normally at a point q_2 and intersects Π perpendicularly at a point x. We may suppose for definiteness that q_1 lies between p and z and, of course, p must lie between q_1 and q_2. Now z is a focal point of multiplicity $n - 2$ and p is a focal point of multiplicity 1 for both q_1 and q_2 in M. So q_1 can never be a maximum point for C_Π and q_2 will be a maximum point if both p and z lie between q_2 and x. In particular this means that p is a maximum point for C_Π^*. We have seen that there is only one such point for almost all Π. Now let us consider minimum points of C_Π when Π does not intersect M. This means that x cannot lie between q_1 and q_2. Now q_1 cannot be a minimum point C_Π because it is easy to see that x would have to lie between q_1 and z and then there would always be one direction in the tangent plane to M at q_1 which was a principal direction for the focal point z and was parallel to Π. In this direction C_Π would be decreasing. So the only possibility is that q_2 is a minimum point and this means q_2 lies between x and p so that p is a minimum point for C_Π^*. We have shown that there is only one such point so that again for almost all Π which do not intersect M, C_Π can have only one minimum point. Thus M has the 2-cylindrical WTPP. However theorem (3.2) shows that it cannot have the 2-cylindrical STPP.

Note that if we embedded $\mathbf{S}^{n-2} \times \mathbf{S}^1$ in $\mathbf{R}^n$ as a round tube about $\mathbf{S}^1$, lying in a 2-plane, then it would not even have the 2-cylindrical WTPP since it would not satisfy the conclusions of (3.1).

4. Cylindrically taut convex envelopes

Although we have so far only used cylindrical two-piece properties, we will now consider k-cylindrically taut embeddings. The final step in showing that there exist only trivial 1-cylindrically taut immersions is a consequence of a more general result. We first prove two results which link k-cylindrical tautness with $(k + 1)$-cylindrical tautness to some extent.

PROPOSITION 4.1. *Let $f\colon M \to \mathbf{R}^n$ be an immersion and let $k < n - 2$. If for every solid $(k + 1)$-cylinder C_*, $f^{-1}(\tilde{C}_*)$ is connected then for every solid k-cylinder C, $f^{-1}(\tilde{C})$ is connected.*

Proof. We will, in fact, prove that if $f^{-1}(\tilde{C})$ is not connected, then there is a solid $(k + 1)$-cylinder C_* with $f^{-1}(\tilde{C}_*)$ not connected. We let C have axis the k-plane Π, and let p, q be points which lie in different components of

$f^{-1}(\tilde{C})$. Let $x, y \in \Pi$ be the feet of the perpendiculars from p and q respectively. Then the vectors $p - x$ and $q - y$ are both perpendicular to Π. Now we can take a $(k+1)$-plane Π^* with $\Pi \subset \Pi^*$ such that $p - x$ and $q - y$ are both perpendicular to Π^* also. This is because $k < n - 2$. If $z \in \Pi^*$ is the unit vector perpendicular to Π then the height function H_z is given by $H_z(p) = \langle f(p) \cdot z \rangle$ and $C_{\Pi^*} = C_\Pi - (H_z - d)^2$ where d is a constant given by $H_z(p) = H_z(q) = d$. Thus $C_{\Pi^*}(p) = C_\Pi(p)$ and $C_{\Pi^*}(q) = C_\Pi(q)$. So if we let C_* be the solid $(k+1)$-cylinder with axis Π^* and the same radius as C clearly $p, q \in C$ and $f^{-1}(\tilde{C}_*) \subset F^{-1}(\tilde{C})$. Hence $f^{-1}(\tilde{C}_*)$ is not connected.

COROLLARY 4.2. *Let $f\colon M \to \mathbf{R}^n$ have the k-cylindrical WTPP then for every closed ball $B \subset \mathbf{R}^n$, letting $\tilde{B}$ be the closure of $\mathbf{R}^n \setminus B$, $f^{-1}(\tilde{B})$ is connected.*

Proof. Notice that a closed ball B is a solid 0-cylinder. The corollary follows by finite induction.

THEOREM 4.3. *Let $f\colon \mathbf{S}^{n-1} \to \mathbf{R}^n$ be a k-cylindrically taut immersion, $k \leq n - 2$. Then for every closed ball $B \subset \mathbf{R}^n$, $f^{-1}(B)$ is connected.*

Proof. We will, in fact, prove that if B is a closed ball with $f^{-1}(B)$ not connected then there exists a solid k-cylinder C with axis Π such that inclusion does not induce a monomorphism

$$H_k\big(f^{-1}(C), f^{-1}(\Pi)\big) \to H_k\big(S^{n-1}, f^{-1}(\Pi)\big).$$

Since f must be tight [2], it is in fact a diffeomorphism onto S where S is a convex envelope, that is, $S = \partial \mathscr{H}S$. To simplify notation we will ignore f and replace it by the inclusion $S \subset \mathbf{R}^n$.

Suppose then that B is a closed ball such that $S \cap B$ is not connected. We claim that there exists a ball B' with centre in $\operatorname{Int} \mathscr{H}S$ such that $S \cap B'$ is disconnected. To see this observe that $B \setminus S$ consists of at least three connected components, one of which is $B \cap \operatorname{Int} \mathscr{H}S$. If the centre of B is x and $x \notin \operatorname{Int} \mathscr{H}S$, then suppose $x \in \overline{U}$ where U is a component of $B \setminus \mathscr{H}S$. We choose y in another component of this set and observe that the segment $\widehat{xy}$ must intersect $\operatorname{Int} \mathscr{H}S$. Take the centre of B' to be on $\widehat{xy} \cap \operatorname{Int} \mathscr{H}S$ and choose its radius so that $\{x, y\} \subset B' \subset B$. Then B' intersects three distinct components of $B \setminus S$ and hence $S \cap B'$ is not connected.

By replacing B by B' if necessary we can suppose that we have a closed ball B with centre $x \in \operatorname{Int} \mathscr{H}S$ and $S \cap B$ disconnected. Let L_x be the usual distance function defined by $L_x(y) = \|y - x\|^2$ for $y \in S$. We can take two points p and q which lie in different components of $S \cap B$ and give an absolute minimum value to L_x on their respective components. Then the lines joining p to x and q to x are normal to S at p and q.

Take an $(n-k)$-plane H through p, q and x and let Π be a k-plane which is complementary to H and intersects Π at x. Then for a point $u \in S \cap H$ and $v \in S \cap \Pi$ there is a unique 2-plane Λ through u, v and x which intersects S in a simple, closed, convex curve. The line $\Pi \cap \Lambda$ will intersect this curve in v and another point, say v'. We define arc(uv) to be the arc of the curve $S \cap \Pi$ with end-points u, v which doesn't contain v'. In this way we get for any subset $E \subset H \cap S$, a set $\{\text{arc}(uv);\ u \in E,\ v \in \Pi \cap S\}$ which is homeomorphic to the join $(\Pi \cap S) * E$. Since $\Pi \cap S$ is homeomorphic to $\mathbf{S}^{k-1}$ we will identify this with the k-th suspension and call it $\Sigma^k E$. Thus S itself will also be called $\Sigma^k(H \cap S)$ and $\Pi \cap S$ will be called $\Sigma^k\emptyset$.

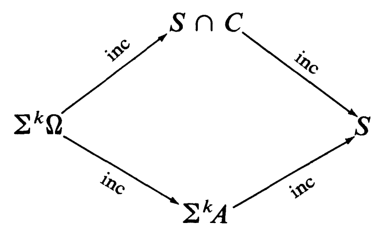
Let $\Omega = \{p, q\}$ and let C be the solid k-cylinder with axis Π such that $\Omega \subset C \cap H = B \cap H$. Taking $u \in H \cap S$, $v \in \Pi \cap S$ and the 2-plane Λ as above we see that $\Lambda \cap C$ is a band lying between two lines parallel to the line $\Pi \cap \Lambda$ which passes through v and x. The line $H \cap \Lambda$ goes through u and is perpendicular to $\Pi \cap \Lambda$. Thus if the line $H \cap \Lambda$ is normal to the curve $S \cap \Lambda$ at u then the whole convex curve must lie in this band; $S \cap \Lambda \subset C$. Thus taking $u = p$ and $u = q$ we see that $\Sigma^k\Omega \subset C$.

Let $A = B \cap H \cap S$ so that p, q lie in different components of A.

We now apply the relative version of the suspension theorem successively k-times to obtain a commutative diagram:

$$\begin{array}{ccccc} H_k(\Sigma^k\Omega, \Sigma^k\emptyset) & \to & H_k(\Sigma^k A, \Sigma^k\emptyset) & \to & H_k\big(\Sigma^k(H \cap S), \Sigma^k\emptyset\big) \\ \wr\Vert & & \wr\Vert & & \wr\Vert \\ H_0(\Omega) & \longrightarrow & H_0(A) & \longrightarrow & H_0(H \cap S). \end{array}$$

Then $H_0(\Omega)$ has two independent generators which map into two independent generators of $H_0(A)$ whereas $H_0(H \cap S)$ has only one independent generator. Thus there is an element $\alpha \in H_k(\Sigma^k\Omega, \Sigma^k\emptyset)$ which maps into a non-zero element in $H_k(\Sigma^k A, \Sigma^k\emptyset)$ but maps into the zero element in $H_k(S, \Pi \cap S)$. The object is to show that α also represents a non-zero element in $H_k(C \cap S, \Pi \cap S)$ under the inclusion $\Sigma^k\Omega \subset C \cap S$ but that this element maps into the zero element in $H_k(S, \Pi \cap S)$. This is done by describing a continuous map $\lambda\colon S \cap C \to \Sigma^k A$ such that the diagram



is homotopy commutative.

Since $S \cap C \cap H \subset \operatorname{Int} A \subset A \subset S \cap H$, as subsets of H, we can find a continuous function μ: $S \cap H \to \mathbf{I}$, where $\mathbf{I} = [0,1]$ is the unit interval, such that $S \cap C \cap H \subset \mu^{-1}(1)$ and $(S \cap H) \setminus A \subset \mu^{-1}(0)$. Observe that for any $u \in (S \cap H) \setminus C$, $v \in S \cap \Pi$, $C \cap \operatorname{arc}(uv)$ is a connected arc with one end at v but not containing u. We define

$$\lambda\colon C \cap \operatorname{arc}(uv) \to C \cap \operatorname{arc}(uv)$$

by keeping v fixed and multiplying the arc-length by $\mu(0)$. We can define λ to be the identity on $C \cap \operatorname{arc}(uv)$ if $u \in C \cap H \cap S$ and $v \in S \cap \Pi$ since in this case $\mu(u) = 1$. This defines a continuous map λ: $S \cap C \to S \cap C$. But the image is in fact in $\Sigma^k A$ because if $u \notin A$, $\mu(u) = 0$ so $\lambda\{C \cap \operatorname{arc}(uv)\} = \{v\} \subset \Sigma^k A$ and otherwise $\operatorname{arc}(uv) \subset \Sigma^k A$. So we have in fact defined a continuous map λ: $C \cap S \to \Sigma^k A$. It is easy to obtain a homotopy between the composition

$$\operatorname{inc} \circ \lambda\colon C \cap S \to \Sigma^k A \to S$$

and the inclusion $C \cap S \subset S$ by simply replacing μ by $t\mu + (1 - t)\mu_0$ where $t \in \mathbf{I}$ and μ_0: $S \cap H \to \mathbf{I}$ is the constant map with image 1. Since λ restricted to $\Sigma^k \Omega$ is the identity we have defined λ so that the diagram above is homotopy commutative. It is easy to deduce that the image of $\alpha \in H_k(\Sigma^k \Omega, \Sigma^k \emptyset)$ in $H_k(C \cap S, \Pi \cap S)$ is a non-zero element of the kernel of the homomorphism

$$H_k(C \cap S, \Pi \cap S) \to H_k(S, \Pi \cap S)$$

induced by inclusion. This contradicts the condition that $S \subset \mathbf{R}^n$ is k-cylindrically taut and so proves the theorem.

THEOREM 4.4. *Let f: $\mathbf{S}^{n-1} \to \mathbf{R}^n$ be k-cylindrically taut, $k \leq n - 2$, then f is taut.*

Proof. We know that f is tight so the image of f is a convex hypersphere and f is an embedding. So we can apply (4.3) and (4.2) to show that f has the spherical two-piece property, that is, it is taut so f embeds $\mathbf{S}^{n-1}$ as a round hypersphere [1].

THEOREM 4.5. *Let f: $M \to \mathbf{R}^n$ be a* 1*-cylindrically taut immersion. Then M is diffeomorphic to $\mathbf{S}^{n-1}$ and f embeds $\mathbf{S}^{n-1}$ as a round hypersphere.*

Proof. We know that f has the 1-cylindrical WTPP and so by (3.4) of [2], M is diffeomorphic to $\mathbf{S}^{n-1}$. The result then follows from (4.4).

THEOREM 4.6. *Let $f: M \to \mathbf{R}^n$ be a 2-cylindrically taut immersion with* $\dim M = n - 1$. *Then M is diffeomorphic to $\mathbf{S}^{n-1}$ and f embeds $\mathbf{S}^{n-1}$ as a round hypersphere.*

Proof. We know that f has the 2-cylindrical STPP and so by (3.2), M is diffeomorphic to $\mathbf{S}^{n-1}$. The result then follows from (4.4).

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