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# CYCLIC INNER FUNCTIONS IN THE BERGMAN SPACES AND WEAK OUTER FUNCTIONS IN $H^p$, $0 < p < 1$

BY

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Let $X$ denote a topological vector space of analytic functions on the unit disk so that $H^\infty \subset X$ and convergence in $X$ implies uniform convergence on compact sets. If $f \in X$ then $[f]$ denotes the closure of $\{Pf: P$ is a polynomial$\}$; i.e., $[f]$ is the smallest invariant (under multiplication by $z$) closed subspace containing $f$. We say $f$ is *$X$-cyclic* if $[f] = X$. We shall be concerned with the case when the function is an inner function. If $q$ is an inner function we say that $q$ is *$X$-inner* if whenever $q_0$ is an inner function and $q_0 \in [q]$, then $q$ divides $q_0$. Initially, we shall consider a general class of Banach spaces which includes the Bergman spaces. Any of these spaces will be denoted by $B$. In Section 1 conditions on $B$ are obtained so that if $q$ is an inner function, then $q = q_1 q_2$ where $q_1$ is $B$-cyclic and $q_2$ is $B$-inner. In Section 2, with further conditions imposed on $B$ (the Bergman spaces still satisfy these conditions), we characterize the $B$-cyclic and $B$-inner functions. In Section 3 the case when $X = H^p$, $0 < p < 1$, with the weak topology is considered. In this setting $X$-cyclic inner functions are called *weak outer functions* and $X$-inner functions are called *weak inner functions.* Using the results from Section 2 we characterize the weak inner and weak outer functions in $H^p$, $0 < p < 1$. Also it is shown that for a large class of singular inner functions $S_\mu$, the quotient spaces $H^p/S_\mu H^p$ contain compact convex sets with no extreme points.

The author would like to thank J. H. Shapiro. Much of this paper was improved by reading [11] in which he "cleaned up" the author's rather crude first manuscript. Also it should be noted that P. Ahern has independently obtained an alternate proof of the factorization in the Bergman spaces using the characterization of the cyclic inner functions. B. Korenblum has (also independently) obtained a characterization of the cyclic inner functions in the Bergman spaces using results from [5].

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## 1. Factorization of inner functions

We shall let $D$ denote the unit disk, $T$ the unit circle and $H$ the space of analytic functions on the disk. We also let $M(T)$ denote the finite Borel measures on $T$ and we let $m$ denote normalized Lebesgue measure on $T$; i.e., $m(T) = 1$.

The set of probability measures will be called $P(T)$ and the set of finite measures singular with respect to Lebesgue measure will be called $S(T)$. We now consider a Banach space $(B, \|\cdot\|)$ of analytic functions on the disk so that convergence in $B$ implies uniform convergence on compact sets and so that $B$ satisfies the following conditions:

(B1) The polynomials are dense in $B$.

(B2) If $f \in B$ and $g \in H^\infty$, then $fg \in B$ and $\|fg\| \leq \|f\| \cdot \|g\|_\infty$.

(B3) If $\langle g_n \rangle$ is a uniformly bounded sequence in $H^\infty$ and $g_n \to 0$ pointwise in $D$, then $\|fg_n\| \to 0$ for all $f \in B$.

The Bergman spaces are examples of such spaces. The Bergman spaces will be of particular interest to us and we shall define them now. If $1 \leq p \leq \infty$ and $\alpha > -1$, define

$$\|f\|_{p,\alpha}^p = \iint_D |f(z)|^p (1 - |z|)^\alpha \, dx \, dy$$

for every measurable function $f$ on $D$ and let

$$A_\alpha^p = \{f \in H \colon \|f\|_{p,\alpha} < \infty\}$$

The space $A_\alpha^p$ is a Banach space and is called a *weighted Bergman Space*.

We return now to the space $B$. Note that since convergence in $B$ implies uniform convergence on compact sets, Blaschke products are $B$-inner. Thus we shall temporarily concentrate on singular inner functions. That is, if $\mu \in S(T)$ the *singular inner function* $S_\mu$ is defined for all $z \in D$ by

$$S_\mu(z) = \exp\left\{\int_T \frac{z + w}{z - w} \, d_\mu(w)\right\}.$$

LEMMA 1.1. (1) *If $S_\mu$ is B-inner and $\nu \leq \mu$, then $S_\nu$ is B-inner.*

(2) *If $\mu \in S(T)$ and $\mu$ is the least upper bound of a collection $A \subset S(T)$ such that $S_\nu$ is B-inner for each $\nu \in A$, then $S_\mu$ is B-inner.*

*Proof.* (1) Suppose $S_\nu$ is not $B$-inner. Then there exists $q$ such that $S_\nu$ does not divide $q$ but $q \in [S_\nu]$. Thus there exist polynomials $P_n$ so that $P_n S_\nu \to q$. By (B2), $P_n S_\mu \to q S_{\mu-\nu}$. But then $S_\mu$ does not divide $q S_{\mu-\nu}$ and $q S_{\mu-\nu} \in [S_\mu]$. This contradiction shows that $S_\nu$ is $B$-inner.

(2) Suppose $BS_{\mu_0} \in [S_\mu]$ ($B$ is a Blaschke product) where $\mu = \sup A$. Since $[S_\mu] \subset \bigcap_{\nu \in A} [S_\nu]$, $S_\nu$ divides $BS_{\mu_0}$ for each $\nu \in A$; i.e., $\mu_0 \geq \nu$ for each $\nu \in A$. But then $\mu_0 \geq \mu$ so that $S_\mu$ divides $BS_{\mu_0}$.

We shall now show that with a certain condition every inner function is the product of a $B$-cyclic function and a $B$-inner function. First suppose that $\lambda \in B^*$. For $n = 0, 1, 2, \ldots$ let $a_n = \lambda(z^n)$. If $\langle b_n \rangle$ is a sequence in $l_1$, then $\sum_{n=0}^{\infty} b_n z^n$ converges to a function in $B$. Hence the series $\sum_{n=0}^{\infty} a_n b_n$ is convergent for every $\langle b_n \rangle \in l_1$ and consequently $\langle a_n \rangle \in l_\infty$. If we let $g(z) = \sum_{n=0}^{\infty} \bar{a}_n z^n$ then $g \in H$. Thus every $\lambda \in B^*$ can be identified with a unique (since the polynomials are dense) function $g \in H$. We shall simply think of $B^*$ as a subset of $H$ and if $g \in B^*$ we shall denote the linear functional by $\lambda_g$. Note that if $g \in B^* \cap H^\infty$ and $P$ is a polynomial, then

$$\lambda g(P) = \frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta.$$

The same will hold for any $f$ in $B$ which is a uniform limit of polynomials, i.e., $f \in A$. Suppose $f \in H^\infty$. Then by (B3), $f_r \to f$ in $B$ where $f_r(z) = f(rz)$. Hence

$$\lambda g(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta.$$

THEOREM 1. *Suppose that $B$ satisfies* (B1)–(B3) *and whenever $q$ is an inner function that is not $B$-cyclic there exists $g \in B^* \cap H^\infty$ $(g \neq 0)$ such that $g^n \in B^*$ for every integer $n$ and $\lambda_g([q]) = 0$. If $B_0 S_\mu$ is an inner function with $B_0$ a Blaschke product, then $\mu = \mu_1 + \mu_2$ where*

(1) $\mu_1 \perp \mu_2$,
(2) $S_{\mu_1}$ *is $B$-cyclic,*
(3) $B_0 S_{\mu_1}$ *is $B$-inner,*
(4) $[B_0 S_\mu] = [B_0 S_{\mu_2}]$.

*Proof.* First let $\mu \in S(T)$ and let $\mu_0 = \sup \{\nu \in M(T): \nu \leq \mu$ and $S_\nu$ is $B$-inner$\}$. By the Lebesgue Decomposition Theorem we may write $\mu = \mu_1 + \mu_2$ where $\mu_1 \perp \mu_0$ and $\mu_2 \ll \mu_0$. Now $\mu_0 \leq \mu_2$ since $\mu_0 \leq \mu$. By the above lemma, $S_{\mu_0}$ is $B$-inner. We intend to show that $\mu_2 = \mu_0$. Suppose $\mu_2 \neq \mu_0$. Then

$$\mu_3 = (\mu_2 - \mu_0) \wedge \mu_0$$

is a positive measure and since $\mu_3 \leq \mu_0$, $S_{\mu_3}$ is $B$-inner. Hence there exists $g \in B^* \cap H^\infty$ $(g \neq 0)$ such that $g^n \in B^*$ for every positive integer $n$ and $\lambda_g([S_{\mu_3}]) = 0$. Thus for $n = 0, 1, 2, \ldots,$

$$\int_0^{2\pi} e^{in\theta} S_{\mu_3}(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta = 0.$$

Consequently for some $f \in H$ with $f(0) = 0$, $S_{\mu_3} \bar{g} = f$ a.e. We may write $f =$

$FBS_\nu$ where $F$ is an outer function, $B$ is a Blaschke product and $S_\nu$ is singular inner. Let $\mu_4 = \mu_3 - (\mu_3 \wedge \nu)$ and $\nu_1 = \nu - (\mu_3 \wedge \nu)$ so that $\mu_4 \perp \nu_1$ and $S_{\mu_4}\bar{g} = FBS_{\nu_1}$.

Note that $\mu_4$ is still a positive measure since otherwise $g$ and $\bar{g}$ are in $H^\infty$ so that $g$ is then a constant and since $g(0) = 0$ we would have $g = 0$. Also $\mu_4 \leq \mu_3 \leq \mu_0$. Now let $N$ be the largest positive integer such that $(N-1)\mu_4 \leq \mu_0$. Since $\mu_4 \leq \mu_3 \leq \mu_2 - \mu_0$, $N\mu_4 \leq \mu_2$. We claim that $S_{N\mu_4}$ is $B$-inner. Let $q$ be an inner function and suppose $q \in [S_{N\mu_4}]$. Now $S_{N\mu_4}\bar{g}^N = f_1^N \in H^\infty$ where $f_1(0) = 0$. Also $g^N \in B^*$. Hence for $n = 0, 1, 2, \ldots,$

$$\int_0^{2\pi} e^{in\theta} S_{N\mu_4}(e^{i\theta})\overline{g(e^{i\theta})^N}\, d\theta = \int_0^{2\pi} e^{in\theta} f_1(e^{i\theta})^N\, d\theta = 0$$

Thus $\lambda_g N([S_{N\mu_4}]) = 0$. But then for $n = 0, 1, \ldots,$

$$\int_0^{2\pi} e^{in\theta} q(e^{i\theta})\overline{g(e^{i\theta})^N}\, d\theta = 0$$

so that $q\bar{g}^N \in H^\infty$. But

$$q\bar{g}^N = \frac{F^N B^N S_{N\nu_1} q}{S_{N\mu_4}}$$

Since $\mu_4 \perp \nu_1$, $S_{N\mu_4}$ divides $q$. Thus $S_{N\mu_4}$ is $B$-inner. But $N\mu_4 \leq \mu_2 \leq \mu$. By the definition of $\mu_0$, $N\mu_4 \leq \mu_0$. But this contradicts our choice of $N$. Hence $\mu_2 = \mu_0$ and $S_{\mu_2}$ is $B$-inner.

Our next claim is that $S_{\mu_1}$ is $B$-cyclic. Suppose $S_{\mu_1}$ is not $B$-cyclic. Then there exists $g \in B^* \cap H^\infty$ so that $g \neq 0$ and

$$\lambda_g([S_{\mu_1}]) = 0.$$

By an argument similar to the above $\bar{g}S_{\mu_1} = h_1 \in H^\infty$ where $h(0) = 0$. Also $S_{\mu_1}$ does not divide $h_1$ since $g \neq 0$. Thus

$$\bar{g} = \frac{h_1}{S_{\mu_1}} = \frac{h_2}{S_{\gamma_1}}$$

where $\gamma_1 \neq 0$, $\gamma_1 \leq \mu_1$, and $S_{\gamma_1}$ and $h_2$ have no common divisor. Suppose $S_\gamma \in [S_{\gamma_1}]$. Clearly $\lambda_g([S_{\gamma_1}]) = 0$ and consequently $\lambda_g([S_\gamma]) = 0$. Hence $\bar{g}S_\gamma = h_3 \in H^\infty$ and therefore $h_3/S_\gamma = h_2/S_{\gamma_1}$. Thus $\gamma \leq \gamma_1$ and it follows that $S_{\gamma_1}$ is $B$-inner. But this contradicts the choice of $\mu_0 = \mu_2$ since $\mu_1 \perp \mu_0$ implies $\gamma_1 \perp \mu_0$.

Now let $B_0 S_\mu$ be a singular inner function and let $B_0$ be a Blaschke product. Then $\mu = \mu_1 + \mu_2$ where $S_{\mu_1}$ is $B$-cyclic and $S_{\mu_2}$ is $B$-inner. Suppose $q = B_1 S_\nu$ is another inner function and $q \in [B_0 S_{\mu_2}]$. $q$ must have at least as many zeros (counting multiplicities) as $B_0$ so that $B_0$ divides $B_1$. Also $q \in [S_{\mu_2}]$ so that $\mu_2 \leq \mu_3$ since $S_{\mu_2}$ is $B$-inner. Thus $B_0 S_\mu$ is $B$-inner. Therefore (1), (2) and (3) hold. Since $S_{\mu_1}$ is $B$-cyclic there exist polynomials $P_n$ so that

$P_n S_{\mu_1} \to 1$ and therefore $P_n B_0 S_\mu \to B_0 S_{\mu_2}$. Hence $[B_0 S_{\mu_2}] \subset [B_0 S_{\mu_2}]$. But $[B_0 S_\mu] \subset [B_0 S_{\mu_2}]$. Therefore $[B_0 S_\mu] = [B_0 S_{\mu_2}]$.

*Remarks.* The same result will hold for weaker hypotheses than (B1)–(B3) and the assumption that $B$ is a Banach space. However, Theorem 1 will suffice in the present form because we are mainly interested in the Bergman spaces.

## 2. Factorization of inner functions in the Bergman spaces

We now impose further conditions on our space $B$ and subject to those conditions we shall obtain a specific factorization of inner functions. Henceforth, assume that $B$ also satisfies the following conditions:

(B4) There exists $\alpha > 0$ and $c_0 > 0$ such that for every $f \in B$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have $|a_n| \leq c_0 \|f\|(n+1)^\alpha$ for $n = 0, 1, 2, \dots$.

(B5) There exists $\beta > 0$ such that $\|z\| \leq n^{-\beta}$ for $n = 2, 3, \dots$

It is easily verified that the Bergman spaces satisfy conditions (B4) and (B5). Before stating the factorization first recall that a closed set $K$ in the unit circle $T$ is called a *Carleson set* if $m(K) = 0$ and if $T \sim K = \bigcup_{n=1}^{\infty} I_n$ is the canonical decomposition of $T \sim K$ into disjoint open arcs, then

$$\sum_{n=1}^{\infty} m(I_n) \log\left(\frac{1}{m(I_n)}\right) < \infty.$$

Now let $O(T)$ denote all measures $\mu \in S(T)$ such that $\mu(K) = 0$ for every Carleson set $K$ and let $I(T)$ denote all measures $\mu \in S(T)$ so that $\mu = \sum_{n=1}^{\infty} \mu_n$ with each $\mu_n$ supported on a Carleson set. Observe that if $\mu \in S(T)$ then $\mu$ can be uniquely written $\mu = \mu_1 + \mu_2$ where $\mu_1 \in O(T)$ and $\mu_2 \in I(T)$. Also $\mu_1 \perp \mu_2$. We now state the main result of this section.

THEOREM 2. *Suppose that $B$ satisfies conditions* (B1)–(B5). *If $B_0 S_{\mu_1} S_{\mu_2}$ is an inner function with $B_0$ a Blaschke product, $\mu_1 \in O(T)$ and $\mu_2 \in I(T)$ then $S_{\mu_1}$ is $B$-cyclic, $B_0 S_{\mu_2}$ is $B$-inner and $[B_0 S_{\mu_1} S_{\mu_2}] = [B S_{\mu_2}]$.*

We now give some machinery for the proof of Theorem 2. We first state a theorem due to H. S. Shapiro [8].

THEOREM 3. *If $\mu$ is a singular measure such that $\mu(K) > 0$ for some Carleson set $K$ and if $m$ is a positive integer, then there exists $g \in H^\infty$ with*

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

*so that* $|b_n| = O(n^{-m})$ *and*

$$\int_0^{2\pi} e^{in\theta} S_\mu(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta = 0$$

*for* $n = 0, 1, 2, \ldots$ .

Because of Theorem 3 we need only prove that if $\mu \in O(T)$, then $S_\mu$ is $B$-cyclic to obtain Theorem 2. To see this, suppose $S_\mu$ is not $B$-cyclic. Then $\mu \notin O(T)$ so that $\mu(K) > 0$ for some Carleson set $K$. If we take $m \geq \alpha + 2$ and choose $g$ according to Theorem 3 then $\lambda_g \in B^*$ by condition (B4) and, of course, $g^n \in B^*$ for every positive integer $n$. Hence Theorem 1 applies. If $S_\mu$ is $B$-cyclic then by Theorem 3, $\mu \in O(T)$. Thus $S_\mu$ is $B$-cyclic if and only if $\mu \in O(T)$ and consequently $S_\mu$ is $B$-inner if and only if $\mu \in I(T)$. We now proceed to prove that $S_\mu$ is $B$-cyclic if $\mu \in O(T)$. We begin with a few preliminary results. If $S_\mu$ is a singular inner function and $f \in B$, then we let $d(f, [S_\mu])$ denote the quotient (by $[S_\mu]$) pseudonorm of $f$; i.e.,

$$d(f, [S_\mu]) = \inf\{\|f - g\| : g \in [S_\mu]\}.$$

PROPOSITION 2.1. *Suppose* $\mu_n$ *is an increasing sequence of singular measures,* $\mu_n \to \mu$ *and* $f \in B$. *Then*

$$d(f, [S_{\mu_n}]) \to d(f, [S_\mu]).$$

*Proof.* First note that since $\mu_1 \leq \mu_2 \leq \cdots \leq \mu$, $[S_{\mu_1}] \supset [S_{\mu_2}] \supset \cdots \supset [S_\mu]$. Hence the sequence $d(f, [S_{\mu_n}])$ is increasing and bounded above by $d(f, [S_\mu])$. We may choose polynomials $P_n$ so that

$$\|f - P_n S_{\mu_n}\| - d(f, [S_{\mu_n}]) \to 0.$$

Hence

$$\begin{aligned}\|f - P_n S_\mu\| &\leq \|f - f S_{\mu-\mu_n}\| + \|S_{\mu-\mu_n}\|_\infty \|f - P_n S_{\mu_n}\| \\ &\leq \|f(1 - S_{\mu-\mu_n})\| + \|f - P_n S_{\mu_n}\|\end{aligned}$$

By (B3), $\|f(1 - S_{\mu-\mu_n})\| \to 0$ so that $d(f, [S_{\mu_n}]) \to d(f, [S_\mu])$.

If $\mu \in M(T)$ we define the *modulus of continuity of* $\mu$, $\omega_\mu$ by

$$\omega_\mu(\delta) = \sup\{\mu(I) : I \text{ is an arc in } T \text{ and } m(I) < \delta\}.$$

We now state a lemma which is essentially Theorem 2 and the following remark in [7].

LEMMA 2.2. *There exists a constant* $c_1 > 0$ *so that if* $0 < \delta \leq 3/4$ *and* $\mu \in S(T)$ *with* $\omega_\mu(\delta) \leq c(\delta \log 1/\delta)$ *then* $|S_\mu(z)| \geq (1 - |z|)^{cc_1}$ *if* $|z| \leq 1 - \delta$.

Before pressing on recall the statement of the Corona Theorem [1, p. 205].

THE CORONA THEOREM. *For every positive integer n, there exists a constant $K > 0$ such that whenever $f_1, \ldots, f_n \in H^\infty$ with $\|f_i\|_\infty \leq 1$, $1 \leq i \leq n$, and*

$$|f_1| + \cdots + |f_n| \geq \delta \quad \text{on } D,$$

*where $0 < \delta \leq 1/2$ then there exist $g_1, \ldots, g_n \in H^\infty$ with $\|g_i\|_\infty \leq \delta^{-K}$ so that*

$$f_1 g_1 + \cdots + f_n g_n = 1.$$

Let $K$ denote the constant from the Corona Theorem in the case $n = 2$. Let

$$c = \beta/3c_1K \quad \text{and} \quad N = \max\{2, 4^{1/cc_1}\}$$

($\beta$ is the constant from condition (B5) and $c_1$ is the constant from Lemma 2.2). Notice that $S_\mu$ is $B$-cyclic if and only if $1 \in [S_\mu]$. The following lemma provides an initial estimate of $d(1, [S_\mu])$.

LEMMA 2.3. *Suppose n is a fixed positive integer and $n \geq N$. If $\mu \in S(T)$ with*

$$\omega_\mu(1/n) \leq \frac{c \log n}{n}$$

*then there exists $g \in H^\infty$ such that*

$$\|g\|_\infty \leq n^{\beta/3} \quad \textit{and} \quad \|1 - gS_\mu\| \leq n^{-2\beta/3}.$$

*Proof.* By Lemma 2.2,

$$|S_\mu(z)| \geq n^{-cc_1} \quad \text{for } |z| \leq 1 - 1/n.$$

For $1 - 1/n \leq |z| < 1$ we have, since $n \geq N$,

$$|z^n| \geq (1 - 1/n)^n \geq 1/4 \geq n^{-c_1c}.$$

Hence

$$|S_\mu(z)| + |z^n| \geq n^{-c_1c} \quad \text{for all } z \in D.$$

Applying the Corona Theorem with $\delta = n^{-c_1c}$ (note that $n^{-c_1c} \leq 1/4$ since $n \geq N$) there exist $g_1, g_2 \in H^\infty$ such that $\|g_i\|_\infty \leq n^{Kc_1c} = n^{\beta/3}$ for $i = 1, 2$ and

$$g_1 S_\mu + g_2 z^n = 1.$$

Hence

$$\|1 - g_1 S_\mu\| = \|z^n g_2\| \leq \|z^n\| \, \|g_2\|_\infty \leq n^{-\beta} n^{\beta/3} = n^{-2\beta/3}.$$

If $(n_i)$ is a finite or infinite sequence of positive integers we define

$$D[(n_i)] = \frac{1}{n_1^{2\beta/3}} + \sum_{i \geq 2} \left(\frac{n_1, \ldots, n_{i-1}}{n_i^2}\right)^{\beta/3}$$

LEMMA 2.4. *Suppose $\mu \in S(T)$ and $\mu$ can be written as a finite or infinite sum $\mu = \sum_i \mu_i$ where each $\mu_i \in S(T)$ such that*

$$\omega_{\mu_i}(1/n_i) \leq \frac{c \log n_i}{n_i}$$

*with each $n_i \geq N$. Then $d(1, [S_\mu]) \leq D[(n_i)]$.*

*Proof.* First suppose $\mu = \sum_{i=1}^m \mu_i$. We proceed by induction on $m$. The case $m = 1$ follows from Lemma 2.2. Suppose the result is true for $m \geq 1$. By Lemma 2.2 there exists $g_1 \in H^\infty$ with $\|g_1\|_\infty \leq n_1^{\beta/3}$ such that

$$\|1 - g_1 s_{\mu_1}\| \leq n_1^{-2\beta/3}.$$

By the induction assumption there exists $g \in H^\infty$ such that

$$\|gS_{\mu_2 + \cdots + \mu_{m+1}} - 1\| \leq D[(n_2, \ldots, n_{m+1})].$$

Hence

$$\begin{aligned} \|g_1 g S_\mu - 1\| &= \|(g_1 S_{\mu_1})(gS_{\mu_2+\cdots+\mu_{m+1}} - 1) + (g_1 S_{\mu_1} - 1)\| \\ &\leq \|g_1 S_{\mu_1}\|_\infty \|gS_{\mu_2+\cdots+\mu_{m+1}} - 1\| + \|g_1 S_{\mu_1} - 1\| \\ &\leq n_1^{\beta/3} D[(n_2, \ldots, n_{m+1})] + n_1^{-2\beta/3} \\ &= D[(n_1, \ldots, n_{m+1})]. \end{aligned}$$

The case when the sum $\mu = \sum_i \mu_i$ involves an infinite number of terms follows from Proposition 2.1.

DEFINITION. If $\mu \in S(T)$ and $\varepsilon > 0$, $\mu$ is *$\varepsilon$-decomposable* if there exist $\mu_i \in S(T)$ and $n_i \geq N$ such that $\mu = \sum_i \mu_i$ and

$$\omega_{\mu_i}(1/n_i) \leq \frac{c \log n_i}{n_i}, \tag{1}$$

$$D[(n_i)] < \varepsilon. \tag{2}$$

$\mu$ is *smoothly decomposable* if $\mu$ is $\varepsilon$-decomposable for every $\varepsilon > 0$.

*Note.* By Lemma 2.4 if $\mu$ is smoothly decomposable then $S_\mu$ is $B$-cyclic. We now give a procedure for obtaining from any $\mu \in S(T)$ a measure $\mu_0 \leq \mu$ so that $\mu_0$ is $\varepsilon$-decomposable.

DEFINITION. Let $\mu \in S(T)$ and let $P = \{I_1, \ldots, I_n\}$ be a partition of $T$ into closed arcs $I_i$ such that $m(I_i) = 1/n$ for each $i$. We say that

$$I_i \text{ is } \textit{light} \text{ if } \quad \mu(I_i) \leq \frac{c}{2} \frac{\log n}{n}$$

and

$$I_i \text{ is } \textit{heavy} \text{ if } \quad \mu(I_i) > \frac{c}{2}\frac{\log n}{n}.$$

We define $\mu_1 \in S(T)$ for each Borel set $E$ in $I_i$ by

$$\mu_1(E) = \begin{cases} \mu(E) & \text{if } \quad I_i \text{ is light,} \\ \dfrac{\mu(E)}{\mu(I_i)}\dfrac{c}{2}\dfrac{\log n}{n} & \text{if } \quad I_i \text{ is heavy.} \end{cases}$$

The measure $\mu_1$ is called a *P-grating* of $\mu$.

*Note.* (1) $\mu_1 \leq \mu$ and the support of $\mu - \mu_1$ lies in the union of the heavy intervals.

(2) $$\mu_1(I_i) = \frac{c}{2}\frac{\log n}{n} \quad \text{if} \quad I_i \text{ is heavy.}$$

(3) $$\omega_{\mu_1}(1/n) \leq c(1/n \log n).$$

DEFINITION. Suppose $\mu \in S(T)$ and $(P_i)$ is a sequence of partitions of $T$ into $n_i$-many closed arcs each of equal length such that $n \geq N$ and each $P_{i+1}$ refines $P_i$. Let $\mu_1$ be the $P_1$-grating of $\mu$ and $\mu_{m+1}$ be the $P_{m+1}$-grating of $\mu - (\mu_1 + \cdots + \mu_m)$. The resulting measure $\sum_i \mu_i$ is called the *$(P_i)$-grating of $\mu$.*

*Proof of Theorem* 2. Suppose that $S_\mu$ is not cyclic. We shall produce a Carleson set $K_0$ so that $\mu(K_0) > 0$. Since $\mu$ is not smoothly decomposable there exists $\varepsilon > 0$ so that $\mu$ is not $\varepsilon$-decomposable. Let

$$n_i = 2^{[2(i_0+i)]} \quad \text{for} \quad i = 1, 2, \ldots$$

where $i_0$ is chosen suitably large so that $D[(n_i)] < \varepsilon$.

Since $n_i$ divides $n_{i+1}$ we may select partitions $P_i$ consisting of $n_i$-many closed arcs of equal length and so that $P_{i+1}$ refines $P_i$. Let $\nu = \sum \mu_i$ be the $(P_i)$-grating of $\mu$. By (3),

$$\omega_{\mu_i}(1/n_i) \leq \frac{c \log n_i}{n_i}.$$

Hence $\nu$ is $\varepsilon$-decomposable and consequently $\nu \neq \mu$. Now let $H_i$ denote the union of all the heavy intervals in $P_i$ (with respect to $\mu - (\mu_1 + \cdots + \mu_{i-1})$). Clearly $H_1 \supset H_2 \supset \cdots$. By (1), $\mu - (\mu_1 + \cdots + \mu_i)$ has its support in $H_i$.

Thus if we let $K = \cap H_i$, $\mu - \nu$ has its support in $K$. Consequently $\mu(K) > 0$. By (2),

$$\mu_i(I) = \frac{c}{2} m(I) \log n_i$$

if $I$ is a heavy arc in $P_i$. Hence

$$\mu(T) \geq \mu_i(T) \geq \mu_i(H_i) = \frac{c}{2} m(H_i) \log n_i \tag{2.1}$$

so that $\lim_{i \to \infty} m(H_i) = 0$; i.e., $m(K) = 0$. Now let $L_i$ denote the union of the interiors of those light intervals in $P_i$ which lie in $H_{i-1}$. Let $K_0 = T \sim \cup L_i$. Clearly $K_0$ is closed and $K \subset K_0$. A point lies in $K_0 \sim K$ only if it is an endpoint of two adjacent light intervals. Hence $K_0 \sim K$ is countable so that $m(K_0) = 0$. Since $\mu(K_0) > 0$ it suffices to show that $K_0$ is a Carleson set; i.e., we must show that

$$\sum_i m(L_i) \log n_i < \infty.$$

But by (2.1), and since $L_i \subset H_{i-1}$,

$$\sum_{i \geq 2} m(L_i) \log n_i \leq \sum_{i \geq 2} m(H_{i-1}) \log n_i$$
$$= 2 \sum_i m(H_i) \log n_i \leq \sum_i \mu_i(T) \leq \mu(T) < \infty.$$

The equality follows since $\log n_i / \log n_{i-1} = 2$. This completes the proof.

## 3. Weak outer functions in $H^p$, $0 < p < 1$

We are now in a position to answer some questions posed by Duren, Romberg and Shields in [2]. If $0 < p < 1$, the spaces $H^p$ are not locally convex and, in fact, Duren, Romberg and Shields proved that there exist nontrivial singular inner functions $S_\mu$ so that $S_\mu H^p$ is weakly dense; i.e., every continuous linear functional annihilating $S_\mu H^p$ also annihilates $H^p$ (note that $S_\mu H^p$ is a closed and proper subspace). Recall that Beurling's Theorem still holds for $H^p$, $0 < p < 1$; i.e., if $X$ is a closed subspace of $H^p$ and $X$ is invariant under multiplication by $z$, then $X = qH^p$ for some inner function $q$. If $q$ is an inner function we let $[q]_W$ denote the weak closure of $qH^p$. We say $q$ is *weak outer* if $[q]_W = H^p$ and *weak inner* if $[q]_W = qH^p$ (note that $[q]_W$ is invariant under multiplication by $z$). Duren, Romberg and Shields asked for a characterization of the weak inner and weak outer functions and they asked whether every inner function is a product of a weak inner and a weak outer function. They also asked whether any of this depends on $p$. To answer these questions we use the fact that the *containing Banach space* of $H^p$ is the Bergman space $A^1_{1/p-2}$ which is also called $B^p$; i.e., $H^p \subset B^p$, convergence in $H^P$ implies convergence in $B^P$ and both spaces have the same dual (every continuous linear functional on $H^P$ has a unique extension to a continuous linear functional on $B^P$). This is proved in [2] and [9]. If $q$ is an inner function then $[q]_W$ is precisely the set of $f \in H^P$ annihilated by all continuous linear functionals that annihilate $[q]$ (the invariant subspace of $B^P$ generated by $q$); i.e., $[q]_W = [q] \cap H^P$. With this remark and Beurling's theorem applied to $[q]_W$ we can answer the above questions with the following theorem.

THEOREM 4. *If $B_0 S_{\mu_1} S_{\mu_2}$ is an inner function with $B$ a Blaschke product, $\mu_1 \in O(T)$, $\mu_2 \in I(T)$, then $S_{\mu_1}$ is weak outer, $B_0 S_{\mu_2}$ is weak inner and*

$$[B_0 S_{\mu_1} S_{\mu_2}]_W = B_0 S_{\mu_2} H^P.$$

Observe that if $S_\mu$ is a weak outer function the quotient space $H^P/S_\mu H^P$ has *trivial dual*; i.e., zero is the only continuous linear functional. These spaces have received a fair amount of attention recently. For instance, N. J. Kalton and J. H. Shapiro have shown in [4] and [10] that these spaces admit nontrivial compact operators to another space $X$. The classical $F$-spaces with trivial dual do not admit nontrivial compact operators and this was the first such space discovered. In [10], J. H. Shapiro asked whether these spaces contain compact convex sets with no extreme points and whether every trivial dual $F$-space contains a compact convex set with no extreme points. N. J. Kalton answered the more general question by showing that certain Orlicz spaces with trivial dual contain only compact convex sets with extreme points [3]. We shall partially answer Shapiro's first question by showing that for a large class of measures $\mu$ (large in the Baire category sense) the spaces $H^P/S_\mu H^P$ contain compact convex sets with no extreme points. The following lemma will prove useful.

LEMMA 3.1. *Let $\langle r_n \rangle$ be a sequence in $(0, 1)$ and let $\delta_n > 0$ such that $\lim_{n\to\infty} \delta_n = 0$. Then*

$$\left\{\mu \in P(T)\colon \mu \in S(T) \quad \text{and} \quad \inf_{0 \le \theta \le 2\pi} |S_\mu(r_n e^{i\theta})| > \delta_n \text{ for infinitely many } n\right\}$$

*is a weak * dense $G_\delta$-set in $P(T)$.*

*Note.* $P(T)$ is a weak * compact subset of $C(T)^*$.

*Proof.* We still let

$$S_\mu(z) = \exp\left\{\int \frac{z+w}{z-w}\, d_\mu(w)\right\}$$

even if $\mu$ is not singular. Note that for any $\lambda \ge 0$ and $z \in D$, $S_{\lambda m}(z) = e^{-\lambda}$, hence if $\mu \in P(T)$ and $\mu \le \lambda m$, then $|S_\mu(z)| \ge e^{-\lambda}$ for every $z \in D$. Let

$$A = \{\mu \in P(T)\colon \mu \le \lambda m \quad \text{for some } \lambda > 0\}.$$

It is easily seen that $A$ is weak* dense in $P(T)$. Now let

$$F_n = \left\{\mu \in P(T)\colon \inf_{0 \le \theta \le 2\pi} |S_\mu(r_n e^{i\theta})| \le \delta_n\right\}.$$

$P(T)$ is weak*-metrizable and if $\mu_n \to \mu$ weak* then $S_{\mu_n} \to S_\mu$ uniformly on compact sets. Hence each $F_n$ is weak* closed and therefore $E_n = \bigcap_{K=n}^{\infty} F_K$ is weak* closed. Since $E_n \cap A = \emptyset$, $E_n$ is nowhere dense. Now let

$$C_n = \{\mu \in P(T)\colon \text{there exists } \nu \in M(T) \text{ such that } \nu \le \mu,\ \nu \le m \text{ and } \nu(T) \ge 1/n\}.$$

Each $C_n$ is weak* closed and nowhere dense (no measure supported by a finite set is in $C_n$). Also $P(T) \sim S(T) = \bigcup_{n=1}^{\infty} C_n$. Thus since

$$P(T) \sim \left\{\mu \in S(T)\colon \inf_{0 \le \theta \le 2\pi} |S_\mu(r_n e^{i\theta})| > \delta_n \text{ for infinitely many } n\right\} = \bigcup_{n=1}^{\infty} E_n \cup \bigcup_{n=1}^{\infty} C_n,$$

the lemma is proved.

Before proving the main theorem of this section let us digress momentarily. If $(X, \|\cdot\|)$ is an $F$-space, $\varepsilon > 0$, $x \in X$ and $F$ is a finite set in $X$, then $F$ is called an *$\varepsilon$-needle set about $x$* if:

(1) $y \in F$ implies $\|y\| < \varepsilon$.
(2) $x \in \operatorname{co} F$, the convex hull of $F$.
(3) If $y \in \operatorname{co} F$, then there exists $\alpha \in [0, 1]$ such that $\|y - \alpha x\| < \varepsilon$.

If $x$ has an $\varepsilon$-needle set for every $\varepsilon > 0$, $x$ is called a *needle point* and if every $x \in X$ is a needle point, then $X$ is called a *needle point space.* For example, the spaces $L_p$, $0 \le p < 1$, are needle point spaces. Also, every needle point space contains a compact convex set with no extreme points [6].

*Note.* If $X$ is a needle point space then $X$ must have trivial dual. Also, if $Y$ is a dense subspace of $X$ and $x \in Y$, then it is easily verified from the definition that $x$ possesses $\varepsilon$-needle sets in $Y$ for arbitrarily small $\varepsilon$.

THEOREM 5. *Let $0 < p < 1$ and let $N(T)$ denote the set of all $\mu \in P(T) \cap S(T)$ so that $H^p/S_\mu H^p$ contains a compact convex set with no extreme points. Then $N(T)$ contains a weak* dense $G_\delta$-set in $P(T)$.*

*Proof.* Let $\varepsilon > 0$. Notice that $\bigcup_{n=1}^{\infty} z^{-n}H^p$ is dense in $L_p$ and the constant function 1 is a needle point in $L_p$. Thus 1 has an $\varepsilon$-needle set in $\bigcup_{n=1}^{\infty} z^{-n}H^p$; i.e. for a positive integer $n$ chosen suitably large and $h_1, \ldots, h_K \in H^p$,

$$\{z^{-n}h_1, \ldots, z^{-n}h_K\}$$

is an $\varepsilon$-needle set about 1. But then $\{h_1, \ldots, h_K\}$ is an $\varepsilon$-needle set about $z^n$. From this it easily follows that one can choose a positive sequence $(\varepsilon_n)$ so that $\lim_{n\to\infty} \varepsilon_n = 0$ and each $z^n$ possesses an $\varepsilon_n$-needle set in $H^P$. As before let $K$ be the constant from the Corona Theorem in the case $n = 2$. Select $\beta_n > 0$ so that $\lim_{n\to\infty} \beta_n = \infty$ but $\lim_{n\to\infty} \beta_n\varepsilon_n = 0$. Let $r_n = \beta_n^{-1/Kn}$ and let $\delta_n = \beta_n^{-1/K}$. Note that $\lim_{n\to\infty} \delta_n = 0$. We claim that if $\mu \in P(T) \cap S(T)$ so that

$$\inf_{0\le\theta\le 2\pi} |S_\mu(r_n e^{i\theta})| > \delta_n$$

for infinitely many $n$, then $H^P/S_\mu H^P$ is a needle point space. Showing this will complete the proof. Suppose $n$ is one of the integers for which

$$\inf_{0\le\theta\le 2\pi} |S_\mu(r_n e^{i\theta})| > \delta_n.$$

If $|z| \ge r_n$, then $|z^n| \ge r_n^n = \beta_n^{-1/K} = \delta_n$. Hence for every $z \in D$,

$$|S_\mu(z)| + |z^n| \ge \delta_n.$$

By the Corona Theorem, there exist $f$, $g \in H^\infty$ such that $fS_\mu + gz^n = 1$ with $\|f\|_\infty, \|g\|_\infty \le \delta_n^{-K} = \beta_n$. Now let $h \in H^\infty$ with $\|h\|_\infty \le 1$. Then

$$hfS_\mu + hgz^n = h.$$

Let $\{h_1, \ldots, h_K\}$ be an $\varepsilon$-needle set for $z^n$. Then $\{hgh_1, \ldots, hgh_k\}$ is a $\beta_n\varepsilon_n$-needle set for $hgz^n = h - hfS_\mu$. If we let $\pi$ denote the quotient map from $H^P$ to $H^P/S_\mu H^P$, then $\{\pi(hgh_1), \ldots, \pi(hgh_K)\}$ is a $\beta_n\varepsilon_n$-needle set for $\pi(h - hfS_\mu) = \pi(h)$. Since $\lim_{n\to\infty} \beta_n \varepsilon_n = 0$ and since the above holds for infinitely many $n$, $\pi(h)$ is a needle point in $H^P/S_\mu H^P$. It is easily verified that a multiple of a needle point is a needle point and that the set of needle points is closed. Thus every point in $\pi(H^\infty)$ is a needle point and since $\pi(H^\infty)$ is dense in $H^P/S_\mu H^P$, $H^P/S_\mu H^P$ is a needle point space.

*Remarks.* As a consequence of the above theorem, for most $\mu \in P(T)$, $H^P/S_\mu H^P$ contains compact convex sets with no extreme points. However for any sequence $r_n \in (0, 1)$ with $\lim_n r_n = 1$ and $\delta_n > 0$ with $\lim_{n\to\infty} \delta_n = 0$ it is possible (but not trivial) to produce $\mu \in O(T)$ so that

$$\inf_{0\le\theta\le 2\pi} |S_\mu(r_n e^{i\theta})| < \delta_n.$$

Thus the above argument does not apply to all $\mu \in O(T)$.

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